STUDENT NUMBER/NAME: ....

Student Number

St Catherine's School

Waverley

2012

HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK 1 – 15%
MONDAY 20th FEBRUARY 2012

Mathematics

General Instructions

- Working time – 55 minutes
- Write using black or blue pen
- Board approved calculators may be used
- All necessary working should be shown
- Start each question in a new booklet

Total marks - 48

Attempt questions 1-3
All questions are of equal value

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a>0, \ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x>0$

Total marks – 48
Attempt Questions 1-3
All questions are of equal value

Begin each question in a **NEW** booklet. Extra writing booklets are available.

Question 1 (16 Marks) **Start a NEW booklet.** **Marks**

(a) Differentiate with respect to x:

(i) $3x^2 - 7 + \frac{5}{x}$ 2

(ii) $(4-3x)^5$ 2

(iii) $(x+1)\sqrt{x}$ 2

(b) (i) Find $\frac{ds}{dt}$ if $s = \frac{t^2+1}{t^2-1}$. 2

(ii) Hence, evaluate $\frac{ds}{dt}$ when $t = 2$. 1

(c) Find the equation of the normal to the curve $y = \sqrt{2x+1}$ at the point (4, 3). Leave your answer in general form. 4

(d) Find the value of p if the function $f(x) = 2x^3 - px^2 + 1$ has $f'(2) = 8$. 3

Question 2 (16 Marks) **Start a NEW booklet.** **Marks**

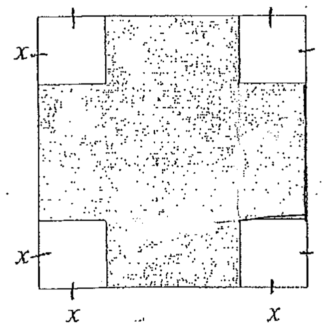
(a) The function $g(x)$ is given by $g(x) = x(x+3)^2$.

(i) Find the coordinates of the points at which the graph of $y = g(x)$ meets the x-axis. 1

(ii) Find the coordinates of the turning points of $g(x)$ and state whether they are maxima or minima. 3

(iii) Draw a sketch of $y = g(x)$ in the domain $-4 \le x \le 1$, showing clearly all essential features. 2

(b) An open rectangular box is to be made by cutting square corners out of a square piece of cardboard 60 cm x 60 cm and folding up the sides, as shown in the diagram.



Let V be the volume of the box, and let x be the side lengths of the square corners which are cut out.

(i) Show that $V = 3600x - 240x^2 + 4x^3$. 2

(ii) Find the maximum volume of the box. 3

Question 2 continued on next page

Question 2 continued

(c) The curve $y = ax^2 + bx + c$ passes through the points (1, 4) and (-1, 6) and obtains its maximum value when $x = -\frac{1}{2}$.

(i) Show that $a + b + c = 4$, $a - b + c = 6$ and $-a + b = 0$. **2**

(ii) Hence find the values of a, b and c. **3**

Question 3 (16 Marks) **Start a NEW booklet.** **Marks**

(a) Write the following series in sigma notation, starting the sum from $n = 1$. (Do not evaluate).

$$3 + 4 + 5 + \dots\dots + 15.$$ **1**

(b) Find the 40th term of the series,

$$44 + 38 + 32 + \dots\dots$$ **2**

(c) (i) Explain why the series,

$16\sqrt{3} + 4\sqrt{3} + \sqrt{3} + \dots\dots$ has a limiting sum. **1**

(ii) Hence find the limiting sum. **1**

(d) For the series,

$$-2\frac{1}{3} - \frac{2}{3} + 1 + 2\frac{2}{3} + \dots\dots\dots + 12\frac{2}{3}.$$

(i) Find the number of terms. **2**

(ii) Hence, find the sum of the series. **2**

(e) How many terms of the series $2 + 8 + 32 + \dots\dots$ must be taken for the sum to exceed ten million? **3**

Question 3 continued on next page

Question 3 continued

(f) (i) Express the following series,

$$6+14+34+\dots\dots+\left(3^n+(2n+1)\right)+\dots\dots\dots$$

as a sum of an arithmetic and geometric series. **1**

(ii) Hence find the sum of the first 15 terms. **3**

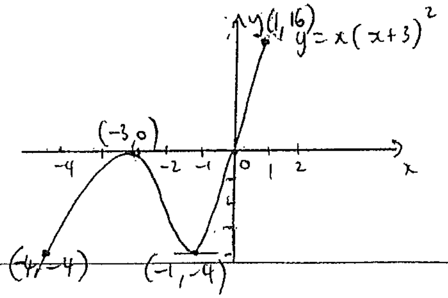
End of paper

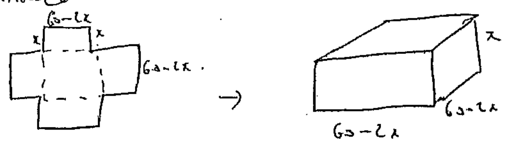
HSC ASSESSMENT TASK 1 · MATHEMATICS 2012

Qn	Solutions	Marks	Comments: Criteria
1 (a)	(i) $\frac{d}{dx}\left(3x^2-7+\frac{5}{x}\right) = 6x-5x^{-2}$ $= 6x - \frac{5}{x^2}$	2	
	(ii) $\frac{d}{dx}(4-3x)^5 = 5(4-3x)^4 \cdot -3$ $= -15(4-3x)^4$	2	
	(iii) $\frac{d}{dx}(x+1)\sqrt{x} = \frac{d}{dx}\left(x^{3/2}+x^{\frac{1}{2}}\right)$ $= \frac{3}{2}x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}$ $= \frac{3}{2}\sqrt{x} + \frac{1}{2\sqrt{x}}$	2	If used product rule, correct answer of $\frac{3x+1}{2\sqrt{x}}$ also accepted (1 mark for correct simplification).
(b)	(i) $s = \frac{t^2+1}{t^2-1}$ $\frac{ds}{dt} = \frac{(t^2-1).2t-(t^2+1).2t}{(t^2-1)^2}$ $= \frac{\cancel{2t^3}-2t-\cancel{2t^3}-2t}{(t^2-1)^2}$ $= \frac{-4t}{(t^2-1)^2}$	2	
	(ii) when $t=2$, $\frac{ds}{dt} = \frac{-8}{(4-1)^2}$ $= \frac{-8}{9}$	1	

P.T.O.

Qn	Solutions	Marks	Comments: Criteria
1	continued		
(c)	$y = (2x+1)^{\frac{1}{2}}$ $y' = \frac{1}{2}(2x+1)^{-\frac{1}{2}}.2$ $= (2x+1)^{-\frac{1}{2}}$ ∴ Gradient of tangent at $x=4$ is $(9)^{-\frac{1}{2}} = \frac{1}{3}$ ∴ Gradient of normal at $x=4$ is -3. ∴ Equation of normal is: $y-3 = -3(x-4)$ $y-3 = -3x+12$ $\therefore 3x+y-15=0$	4	$-\frac{1}{2}$ if not in general form.
(d)	$f(x) = 2x^3-px^2+1$ $f'(x) = 6x^2-2px$ Since $f'(2)=8$, then; $8 = 6(2)^2-2p(2)$ $8 = 24-4p$ $-16 = -4p$ $\therefore p = 4$	3.	

Qn	Solutions	Marks	Comments: Criteria
2 (a)	$g(x) = x(x+3)^2$		
(i)	Graph meets the x-axis when $g(x)=0$. i.e. $x(x+3)^2=0$ $\therefore x=0,\ -3$ $\therefore$ Co-ordinates of points are $(0,0)$ and $(-3,0)$	1	
(ii)	$g'(x) = x\cdot 2(x+3)+1(x+3)^2$ $= 2x^2+6x+(x^2+6x+9)$ $= 3x^2+12x+9$ $g''(x) = 6x+12$ For stationary points let $g'(x)=0$. i.e. $3x^2+12x+9=0$ $\div 3$ $\quad x^2+4x+3=0$ $(x+3)(x+1)=0$ $\therefore x=-3,\ -1$ $y=0,\ -4$. $\therefore$ Stationary points exist at $(-3,0)$ and $(-1,-4)$ For $(-3,0)$, $y''=6(-3)+12 < 0$ $\therefore (-3,0)$ is a maximum turning point For $(-1,-4)$, $y''=6(-1)+12 > 0$ $\therefore (-1,-4)$ is a minimum turning point.	3	
(iii).		2	For full marks, shape of graph, intercepts, stationary points and endpoints need to be shown.

Qn	Solutions	Marks	Comments: Criteria
2 (b) (i)	continued.  $V = x(60-2x)(60-2x)$ $= x(3600-240x+4x^2)$ $= 3600x-240x^2+4x^3$, as required.	2	
(ii)	$V = 3600x-240x^2+4x^3$ $\frac{dV}{dx} = 3600-480x+12x^2$ $\frac{d^2V}{dx^2} = -480+24x$ Let $\frac{dV}{dx}=0$. ie $12x^2-480x+3600=0$ $\therefore x^2-40x+300=0$ $(x-30)(x-10)=0$ $\therefore x=30,\ 10$ But when $x=30$, side length $(60-2x)=0$ $\therefore x=10$ only. When $x=10$, $\frac{d^2V}{dx^2} = -480+24(10) < 0$ $\therefore$ Maximum volume. Max. volume $= 10(60-20)(60-20)$ $= 16000\text{ cm}^3$ P.T.O.	3	

Qn	Solutions	Marks	Comments: Criteria
2	continued.		
(c) (i)	$y = ax^2 + bx + c$ Sub $(1,4)$: $4 = a+b+c$ ① Sub $(-1,6)$: $6 = a-b+c$ ② Max. value occurs when $y' = 0$ ie. $2ax + b = 0$ Sub $x = -\frac{1}{2}$ into y': $-a+b=0$. ③ $\therefore$ Three equations shown.	2	
(ii)	$-a+b=0$ $\therefore a = b$ ① Sub $a=b$ into $4 = a+b+c$ $\therefore 4 = 2a+c$ ② Sub $a=b$ into $6 = +\cancel{a} - \cancel{a} + c$ $\therefore c = 6$. ③ If $c=6$, then ② becomes $4 = 2a+6$ $\therefore a = -1$ $\therefore b = -1$ $\therefore a=-1,\ b=-1,\ c=6$.	3	

Qn	Solutions	Marks	Comments: Criteria
3	(a) $\sum_{n=1}^{13} (n+2)$	1	
	(b) $44 + 38 + 32 + \dots$ AP $a=44$, $d=-6$, $n=40$. $\therefore T_n = a+(n-1)d$ $\therefore T_{40} = 44 + (39).-6$ $= 44 - 234$ $\therefore T_{40} = -190$	2	
	(c) (i) $16\sqrt{3} + 4\sqrt{3} + \sqrt{3} + \dots$ $r = \frac{1}{4}$ $\therefore$ Since $-1 < r < 1$ (ie. $\lvert r \rvert < 1$), then a limiting sum exists.	1	
	(ii) $S_\infty = \dfrac{a}{1-r}$ $= \dfrac{16\sqrt{3}}{1-\frac{1}{4}}$ $= \dfrac{16\sqrt{3}}{\frac{3}{4}}$ $\therefore S_\infty = \dfrac{64\sqrt{3}}{3}$	1	
	(d) (i) $-2\frac{1}{3} - \frac{2}{3} + 1 + 2\frac{2}{3} + \dots + 12\frac{2}{3}$. AP. $a = -2\frac{1}{3}$, $d = \frac{5}{3}$, $T_n = 12\frac{2}{3}$ $T_n = a+(n-1)d$ $\therefore -2\frac{1}{3} + (n-1)\frac{5}{3} = 12\frac{2}{3}$ $-2\frac{1}{3} + \frac{5}{3}n - \frac{5}{3} = 12\frac{2}{3}$ $\frac{5}{3}n - 4 = 12\frac{2}{3}$ $\frac{5}{3}n = 16\frac{2}{3}$ $\therefore n = 10$. $\therefore$ There are 10 terms in the series.	2	

P.T.O

Qn	Solutions	Marks	Comments: Criteria
3 continued.			
(d)(ii)	$S_n = \frac{n}{2}(a+l)$ $\therefore S_{10} = \frac{10}{2}\left(-2\frac{1}{3} + 12\frac{2}{3}\right)$ $= 5\left(\frac{31}{3}\right)$ $\therefore S_{10} = 51\frac{2}{3}$	2	
(e)	$2 + 8 + 32 + \dots$ GP $a = 2$ $\quad S_n = \frac{a(r^n-1)}{r-1}$ $r = 4$ $n = ?$ $\quad S_n > 10\,000\,000$ $\therefore \frac{2(4^n-1)}{3} > 10\,000\,000$ $2(4^n-1) > 30\,000\,000$ $4^n - 1 > 15\,000\,000$ $4^n > 15\,000\,001$ $\log 4^n > \log 15\,000\,001$ $n \log 4 > \log 15\,000\,001$ $\therefore n > \frac{\log 15\,000\,001}{\log 4}$ $n > 11.91$ $\therefore n = 12$ $\therefore$ 12 terms must be taken for the sum to exceed 10 000 000.	3	
	P.T.O.		

Qn	Solutions	Marks	Comments: Criteria
3 continued			
(f)	(i) $6 + 14 + 34 + \dots + \left(3^n + (2n+1)\right) + \dots$ $3^1 + 3 \quad 3^2 + 5 \quad 3^3 + 7$ $\therefore$ Series can be written as: $\underbrace{(3^1 + 3^2 + 3^3 + \dots + 3^n)}_{GP} + \underbrace{(3 + 5 + 7 + \dots + (2n+1))}_{AP}$	1	
	(ii) First 15 terms can be written as: $(3^1 + 3^2 + 3^3 + \dots + 3^{15}) + (3 + 5 + 7 + \dots + 31)$ $S_{15} = \frac{3(3^{15}-1)}{2}$ $\qquad S_{15} = \frac{15}{2}(3+31)$ $= 255$ $\therefore$ Total of both series $= \frac{3(3^{15}-1)}{2} + 255$ $= 21523359 + 255$ $= 21523614$.	3	

— END OF PAPER —