Student Number:

St Catherine's School

Waverley

Year 12 Mathematics Extension 1

Task 2

March 2015

Time allowed: **75 minutes plus 5 minutes reading time**

Total marks: **45 marks**

Weighting: **20%**

INSTRUCTIONS

- There are 2 sections each of different value.
- Start each section in a new booklet. Section 1 consists of 5 multiple Choice questions and Questions 6 and 7 (25 marks).
  Section 2 consists of Questions 8 and 9 (20 marks).
- Start each question on a new page.
- All necessary working should be shown.
- Approved scientific calculators may be used.
- Marks may be deducted for careless or badly arranged work.
- A table of standard integrals may be found on the back of this paper.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\quad x > 0$

SECTION 1 START A NEW BOOKLET

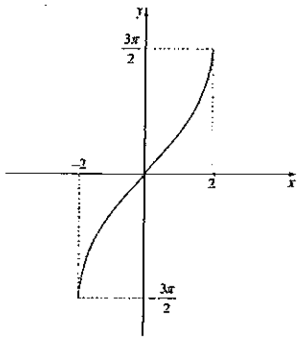
Multiple Choice **(5 marks)**

Answer in your writing booklet.

Choose the correct answer.

QUESTION 1

Which function best describes the following graph ?



(A) $\frac{3}{2}sin^{-1}2x$ (B) $3sin^{-1}\frac{x}{2}$ (C) $3sin^{-1}2x$ (D) $\frac{3}{2}sin^{-1}\frac{x}{2}$

QUESTION 2

A pair of parametric equations which could represent $x^2 + y^2 = 9$ is ?

(A) $x = 3cos\theta$ and $y = sin\theta$ (B) $x = 3t$ and $y = -3t$

(C) $x = 3cos\theta$ and $y = 3sin\theta$ (D) $x = \frac{t}{3}$ and $y = 3t$

Multiple Choice Questions continued

QUESTION 3

If $f(x) = e^{x+2}$, then the inverse function $f^{-1}(x)$ is ?

(A) $lnx - 2$ (B) $lnx + 2$ (C) $ln(x+2)$ (D) $ln(x-2)$

QUESTION 4

$\int \frac{6}{\sqrt{9-4x^2}}\,dx =$

(A) $3sin^{-1}\frac{3x}{2} + c$ (B) $sin^{-1}\frac{2x}{3} + c$

(C) $\frac{2}{3}sin^{-1}\frac{2x}{3} + c$ (D) $3sin^{-1}\frac{2x}{3} + c$

QUESTION 5

Which of the following is **not** a property of the parabola $x^2 = \frac{1}{16}y$?

(A) the latus rectum has a length of $\frac{1}{16}$ units

(B) the co-ordinates of the focus are $(0, \frac{1}{64})$

(C) the equation of the axis is $y = 0$

(D) the equation of the directrix is $y = -\frac{1}{64}$

End of multiple choice

QUESTION 6 **START A NEW PAGE** **(10 marks)**

(a) Differentiate:

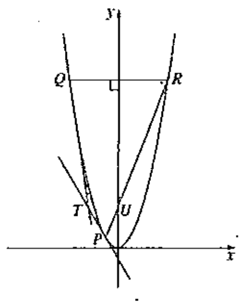
(i) $tan^{-1}\frac{x}{6}$ **1**

(ii) $(sin^{-1}4x)^3$ **1**

(b) (i) Show that $\frac{d}{dx}(xcos^{-1}x - \sqrt{1-x^2}) = cos^{-1}x.$ **2**

(ii) Hence, find the exact area bounded by the curve $y = cos^{-1}x$, the x axis and the lines $x = 0$ and $x = \frac{1}{\sqrt{2}}$. **2**

(c)



The points $P(2ap, ap^2)$, $Q(2aq, aq^2)$ and $R(2ar, ar^2)$ lie on the parabola $x^2 = 4ay$.

The chord QR is perpendicular to the axis of the parabola. The chord PR meets the axis of the parabola at U.

The equation of the chord PR is $y = \frac{1}{2}(p+r)x - apr$. (Do NOT prove this)

The equation of the tangent at P is $y = px - ap^2$. (Do NOT prove this)

(i) Find the co-ordinates of U. **1**

(ii) The tangents at the point P and Q meet at the point T.

Show that the co-ordinates of T are $(a(p+q), apq)$ **2**

(iii) Show that TU is perpendicular to the axis of the parabola. **1**

QUESTION 7 **START A NEW PAGE** **(10 marks)**

(a) Evaluate $\cos\left(sin^{-1}\left(\frac{-5}{13}\right)\right)$ without the use of a calculator. **3**

(b) Prove by mathematical induction that:

(i) $4^n - 1$ is divisible by 3 for all $n \geq 1$. **3**

(ii) $\sum_{r=1}^{a}\frac{1}{(3r-2)(3r+1)} = \frac{a}{3a+1}$ **4**

END OF SECTION 1

SECTION 2 START A NEW BOOKLET

QUESTION 8 START A NEW PAGE **(10 marks)**

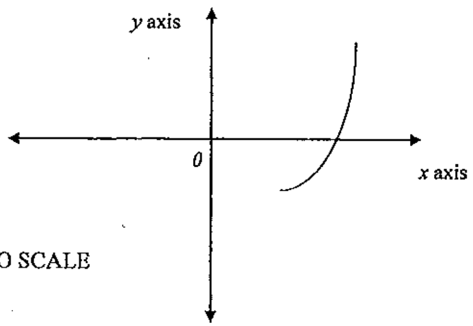
(a) For the function $f(x) = x^2 - 8x$,

(i) find the domain over which $f(x)$ is montonic increasing. **1**

(ii) find the equation of this inverse function, $f^{-1}(x)$ over this restricted domain. **2**

(iii) state the domain and range of the inverse function $f^{-1}(x)$. **2**

(iv) part of the graph of $f(x) = x^2 - 8x$ has been drawn below.



NOT TO SCALE

Copy this diagram into your booklet and on the same number plane sketch the graph of $y = f^{-1}(x)$. **1**

(v) find the point of intersection of $f(x)$ and $f^{-1}(x)$. **1**

Question 8 continues

(b) Write $\sin\left(2\cos^{-1}\left(\frac{2}{3}\right)\right)$ in the form $a\sqrt{b}$, where a and b are rational. **3**

QUESTION 9 **START A NEW PAGE** **(10 marks)**

$P(2ap, ap^2)$ is a point on the parabola $x^2 = 4ay$.

(i) Show that the equation of the normal to the parabola at the point P is $x + py = 2ap + ap^3$. **2**

(ii) If the normal at P cuts the y-axis at Q, show that the co-ordinates of Q are $(0, 2a + ap^2)$. **1**

(iii) Briefly explain why Q does not lie on the directrix. **1**

(iv) Show that the co-ordinates of R which divides the interval PQ externally in the ratio 2:1 are $(-2ap, 4a + ap^2)$. **2**

(v) Find the Cartesian equation of the locus of R. **2**

(vi) Show that if the normal at P passes through a given point (h, k) then p must be a root of the equation $ap^3 + (2a - k)p - h = 0$. **1**

(vii) Hence state the maximum number of normals to the parabola $x^2 = 4ay$ which can pass through any given point. **1**

END OF SECTION 2

END OF ASSESSMENT TASK

YEAR 12 MATHEMATICS EXT 1 TASK 2 2015 SOLUTIONS

Qn	Solutions	Marks	Comments: Criteria

SECTION 1

Q1 $y = 3\sin^{-1}\frac{x}{2}$ B

Q2 Sub $x = 3\cos\theta$ and $y = 3\sin\theta$ into $x^2+y^2=9$.   C

$$\begin{aligned} LHS &= 9\cos^2\theta + 9\sin^2\theta \\ &= 9(\cos^2\theta+\sin^2\theta) \\ &= 9 \\ &= RHS. \end{aligned}$$

Q3 $f(x) = e^{x+2}$ A

f^{-1}: $x = e^{y+2}$

$$\begin{aligned} \ln x &= \ln e^{y+2} \\ \ln x &= y+2 \\ \therefore y &= \ln x - 2 \end{aligned}$$

Q4   D

$$\begin{aligned} \int \frac{6}{\sqrt{9-4x^2}}\,dx &= \int \frac{6}{\sqrt{4(\frac{9}{4}-x^2)}}\,dx \\ &= \frac{6}{2}\int \frac{1}{\sqrt{\frac{9}{4}-x^2}}\,dx \\ &= 3\sin^{-1}\frac{2x}{3} + c \end{aligned}$$

Q5 $x^2 = \frac{1}{16}y$ C

$4a = \frac{1}{16}$

$\therefore a = \frac{1}{64}$

Equation of axis is $x=0$.

$\therefore$ (C) False

Qn	Solutions	Marks	Comments: Criteria

Q6 (a) (i) $\frac{d}{dx}\tan^{-1}\frac{x}{6} = \frac{6}{36+x^2}$ 1

(ii) $\frac{d}{dx}\left(\sin^{-1}4x\right)^3 = 3\left(\sin^{-1}4x\right)^2 \cdot \frac{4}{\sqrt{1-(4x)^2}}$ $\frac{1}{2}$ 1

$$= \frac{12\left(\sin^{-1}4x\right)^2}{\sqrt{1-16x^2}} \quad \frac{1}{2}$$

(b) (i) Show that: $\frac{d}{dx}\left(x\cos^{-1}x - \sqrt{1-x^2}\right) = \cos^{-1}x$ 2

$$\begin{aligned} LHS &= -x\cdot\frac{1}{\sqrt{1-x^2}} + 1\cdot\cos^{-1}x - \frac{1}{\cancel{2}}(1-x^2)^{-\frac{1}{2}}\cdot -\cancel{2}x \\ &= \frac{-x}{\sqrt{1-x^2}} + \cos^{-1}x + \frac{x}{\sqrt{1-x^2}} \\ &= \cos^{-1}x \\ &= RHS \end{aligned}$$

$LHS = RHS$ $\therefore$ Statement proven true.

(ii) $\therefore \int_0^{\frac{1}{\sqrt{2}}} \cos^{-1}x\,dx = \left[x\cos^{-1}x - \sqrt{1-x^2}\right]_0^{\frac{1}{\sqrt{2}}}$ 2

$$\begin{aligned} &= \left[\left(\frac{1}{\sqrt{2}}\cos^{-1}\frac{1}{\sqrt{2}} - \sqrt{\frac{1}{2}}\right) - (0-1)\right] \\ &= \left[\frac{1}{\sqrt{2}}\cdot\frac{\pi}{4} - \frac{1}{\sqrt{2}} + 1\right] \\ &= \frac{1}{\sqrt{2}}\left(\frac{\pi}{4} - 1 + \sqrt{2}\right)\ u^2 \end{aligned}$$

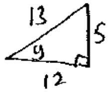
Qn	Solutions	Marks	Comments: Criteria
	Q6 continued		
(c)	(i) let $x=0$ and substitute into PR:		
	$\therefore y = \frac{1}{2}\cancel{(p+r)}0 - apr$		
	$\therefore y = -apr$	1	
	$\therefore U = (0, -apr)$		
	(ii) Equation of tangent at P is $y = px - ap^2$ ①		
	" " " Q is $y = qx - aq^2$ ②		
	Solving simultaneously:		
	$px - ap^2 = qx - aq^2$		
	$x(p-q) = a(p^2-q^2)$		
	$x = \dfrac{a(p+q)\cancel{(p-q)}}{\cancel{(p-q)}}$	2	
	$\therefore x = a(p+q)$		
	$\therefore y = p\left(a(p+q)\right) - ap^2$		
	$= \cancel{ap^2} + apq - \cancel{ap^2}$		
	$= apq$		
	$\therefore T = [a(p+q), apq]$, as required		
	Alternate Solution:		
	Sub $T = [a(p+q), apq]$ into the equations of tangents, $y = px - ap^2$ and $y = qx - aq^2$ and show that both equations are satisfied.		

Qn	Solutions	Marks	Comments: Criteria
	Q6 continued		
	(c)(iii) $T = [a(p+q), apq]$, $U = (0, -apr)$		
	QR is $\perp$ to axis of parabola.		
	$\therefore aq^2 = ar^2$ and by symmetry $-2aq = 2ar$	1	
	$\therefore q = -r$		
	$\therefore$ If $q = -r$, then		
	$T = [a(p+q), -apr]$ and $U = (0, -apr)$		
	$\therefore$ Since both T and U have the same y-ordinate, then TU is $\perp$ to the axis of the parabola $x=0$.		

Qn	Solutions	Marks	Comments: Criteria
Q7	(a) $\cos\left(\sin^{-1}\left(-\frac{5}{13}\right)\right)$		

let $y = \sin^{-1}\left(-\frac{5}{13}\right)$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\therefore \sin y = -\frac{5}{13}$



$\therefore \cos y = \frac{12}{13}$

$\therefore \cos\left(\sin^{-1}\left(-\frac{5}{13}\right)\right) = \frac{12}{13}$

Marks: 3

(b) (i) $4^n - 1$ is $\div$ by 3 for $n \geq 1$

Prove true for $n=1$

i.e. $4^1 - 1 = 3$, which is divisible by 3

Marks: 3

Assume true for $n=k$

i.e. $4^k - 1 = 3M$, where M is an integer

$\therefore 4^k = 3M+1$

Prove true for $n=k+1$

i.e. $4^{k+1} - 1 = 3N$, where N is an integer

$$\begin{aligned} LHS &= 4^{k+1} - 1 \\ &= 4^k . 4 - 1 \\ &= (3M+1)4 - 1 \quad \text{(by assumption)} \\ &= 12M + 4 - 1 \\ &= 12M + 3 \\ &= 3(4M+1) \text{, which is divisible by 3.} \end{aligned}$$

$\therefore$ If the statement is true for $n=k$ and it is true for $n=1$ and $n=k+1$, then it is true for $n=2, 3$, etc. $\therefore 4^n - 1$ is divisible by 3 for all $n \geq 1$

Qn	Solutions	Marks	Comments: Criteria
	Q7 continued		

(ii) $\sum_{r=1}^{a} \frac{1}{(3r-2)(3r+1)} = \frac{a}{3a+1}$

i.e. $\frac{1}{(1)(4)} + \frac{1}{(4)(4)} + \dots + \frac{1}{(3a-2)(3a+1)} = \frac{a}{3a+1}$

Prove true for $a=1$

$$\begin{aligned} LHS &= \frac{1}{(1)(4)} \\ &= \frac{1}{4} \\ RHS &= \frac{1}{3+1} \\ &= \frac{1}{4} \end{aligned}$$

$\therefore$ True for $a=1$

Marks: 4

Assume true for $a=k$

i.e. $\frac{1}{4} + \frac{1}{16} + \dots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1}$

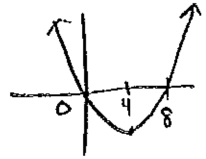
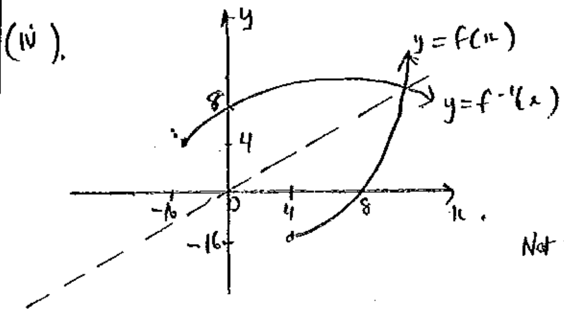
Prove true for $a=k+1$

ie. $\frac{1}{4} + \frac{1}{16} + \dots + \frac{1}{(3k+1)(3k+4)} = \frac{k+1}{3k+4}$

Using $S_{k+1} = S_k + T_{k+1}$

$$\begin{aligned} &= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)} \\ &= \frac{k(3k+4)+1}{(3k+1)(3k+4)} \\ &= \frac{3k^2+4k+1}{(3k+1)(3k+4)} \\ &= \frac{\cancel{(3k+1)}(k+1)}{\cancel{(3k+1)}(3k+4)} \end{aligned}$$

Qn	Solutions	Marks	Comments: Criteria
	$= \dfrac{k+1}{(3k+4)}$ $= RHS$ $\therefore LHS = RHS$ $\therefore$ Statement proven true. $\therefore$ If true for $n=k$, and is true for $n=1$ and $n=k+1$, then true for $n=2, 3, \ldots$ etc... $\therefore$ Statement proven true for all $n \geq 1$.		

Qn	Solutions	Marks	Comments: Criteria
Q8	(a) $f(x) = x^2 - 8x$.		
	(i)  Monotonic increasing for $x > 4$	1	
	(ii) f^{-1}: $x = y^2 - 8y$ $x + 16 = y^2 - 8y + 16$ $x + 16 = (y-4)^2$ $\therefore y - 4 = \pm\sqrt{x+16}$ $\therefore y = 4 \pm \sqrt{x+16}$ But $y > 4$ $\therefore y = 4 + \sqrt{x+16}$ is the equation of the inverse function.	2	
	(iii) When $x = 4$, $y = 16 - 32$ $y = -16$ $\therefore$ For $y = f^{-1}(x)$: Domain is $x > -16$ Range is $y > 4$	2	
	(iv)  Not to scale	1	

Qn	Solutions	Marks	Comments: Criteria

(v) Solve $y = x^2 - 8x$ with $y = x$

$\therefore x = x^2 - 8x$

$x^2 - 9x = 0$

$x(x-9) = 0$

$\therefore x = 0, 9$

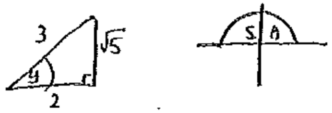
$\therefore$ Point of intersection is $(9, 9)$.

Marks: 1

(b) $\sin\left(2\cos^{-1}\left(\frac{2}{3}\right)\right)$

let $y = \cos^{-1}\frac{2}{3}$, $0 \le y \le \pi$

$\therefore \cos y = \frac{2}{3}$



$\therefore \sin\left(2\cos^{-1}\left(\frac{2}{3}\right)\right) = \sin 2y$

$= 2\sin y\cos y$

$= 2\left(\frac{\sqrt{5}}{3}\right)\left(\frac{2}{3}\right)$

$= \frac{4\sqrt{5}}{9}$

Marks: 3

Qn	Solutions	Marks	Comments: Criteria

Q9 (i) $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$y' = \frac{x}{2a}$

At $P(2ap, ap^2)$, $y' = \frac{2ap}{2a}$

$y' = p$

$\therefore m_\perp = -\frac{1}{p}$

$\therefore$ Equation of normal at P is:

$y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$\therefore x + py = 2ap + ap^3$, as required.

Marks: 2

(ii) If normal cuts y-axis at Q, then $x = 0$.

$\therefore 0 + py = 2ap + ap^3$

$\therefore y = 2a + ap^2$

$\therefore Q = (0, 2a + ap^2)$, as required.

Marks: 1

(iii) Directrix for $x^2 = 4ay$ has equation $y = -a$.

Q has y-ordinate of $2a + ap^2$.

$\therefore Q$ does not lie on the directrix.

Marks: 1

(iv). $P = (2ap, ap^2)$ $\quad Q = (0, 2a + ap^2)$

$\quad\; x_1 \;\; y_1 \qquad\qquad x_2 \;\; y_2$

$2 : 1$

$m \;\; n$

$P(x, y) = \left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}\right)$

$= \left(\frac{2(0) - 1(2ap)}{2 - 1}, \frac{2(2a + ap^2) - (ap^2)}{2 - 1}\right)$

P.T.O.

Qn	Solutions	Marks	Comments: Criteria
	Q9 continued $P(x,y) = \left(\frac{-2ap}{1}, \frac{4a+2ap^2-ap^2}{1} \right)$ $= (-2ap, 4a+ap^2)$, as required.	2	
	(v) $x = -2ap$ $\quad y = 4a+ap^2$ (2) $\therefore \frac{x}{-2a} = p$ (1) Sub (1) into (2): $y = 4a + a\left(\frac{x}{-2a}\right)^2$ $y = 4a + a\left(\frac{x^2}{4a^2}\right)$ $y = 4a + \frac{x^2}{4a}$, is the equation of locus of R.	2	
	(vi) $x + py = 2ap + ap^3$ is the equation of the normal. Sub in (h,k): $\therefore h + pk = 2ap + ap^3$ $\therefore ap^3 + 2ap - pk - h = 0$ $ap^3 + (2a-k)p - h = 0$, as required	1	
	(vii) Maximum number of normals will be $\underline{\underline{3}}$ (For a cubic)	1	
	END OF ASSESSMENT TASK		