Student Number:______________________

Mathematics Extension 2

St. Catherine's School Waverley

2015

ASSESSMENT TASK 2
18 March

Assessment weight 20%

Total marks – 41

• Attempt all questions. Write the answers
To the multiple choice questions on the booklet

TEACHER'S USE ONLY

Multiple Choice	/4
Question ~~4~~ 5	/18
Question ~~5~~ 6	/19
Total	/41

General Instructions
- *Reading time - 5minutes*
- Working time – 75 minutes
- Start each question in a new answer booklet.
- Write using black or blue pen only.
- Board-approved calculators may be used.
- All necessary working must be shown.
- Marks may be deducted for careless or badly arranged work.

STANDARD INTEGRALS

$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$

$\int \frac{1}{x}\,dx = \ln x,\ x > 0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$

$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE: $\ln x = \log_e x,\ x > 0$

Write the right choice in your answer booklet.

1 The remainder when the polynomial $10x^4 - 7x + 4$, when divided by $x + i$ is

A. $14 + 7i$

B. $14 - 7i$

C. $-6 + 7i$

D. $-6 - 7i$

2 Given that the eccentricity of the hyperbola $H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *is* e, the eccentricity of the ellipse $\frac{x^2}{a^2+b^2} + \frac{y^2}{b^2} = 1$ is

A. e^2

B. $-e$

C. $\frac{1}{e}$

D. $\sqrt{e}$

3 An ellipse has directrices $x = \pm 4$ and foci with coordinates $(\pm 2,0)$. The equation of the ellipse is

A. $2x^2 + y^2 = 8$

B. $x^2 + 2y^2 = 8$

C. $2x^2 + y^2 = 2$

D. $x^2 + 2y^2 = 1$

4 If the roots of the polynomial equation $x^3 + x + 1 = 0$ are α, β, γ. The equation with roots $\frac{1}{\alpha\beta}, \frac{1}{\beta\gamma}$ *and* $\frac{1}{\gamma\alpha}$ is

A. $x^3 + x - 1 = 0$

B. $x^3 + x^2 + 1 = 0$

C. $2x^3 + 2x + 1 = 0$

D. $x^3 - x^2 - 1 = 0$

Question 5 18 marks

a) (i) Sketch the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, showing the intercepts on both the axes, the coordinates of the foci and the equations of the directrices. 4

(ii) If P is any point on this ellipse, show that $PS + PS' = 8$. 2

(iii) Indicate clearly the point represented by K: $(4\cos\theta, 3\sin\theta)$, where θ is in the first quadrant. 1

b) The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $a > b > 0$, has eccentricity e.

(i) Show that the equation of the line passing through the focus $F: (ae, 0)$ and perpendicular to the asymptote $y = \frac{b}{a}x$ is given by $ax + by - a^2e = 0$ 2

(ii) Show that this line meets the same asymptote on the corresponding directrix 2

c) Decompose $\frac{5x^2-3x+13}{(x^2+4)(x-1)}$ into sum of its partial fractions 3

d) (i) Show that the equation of the tangent to the hyperbola $xy = 4$ at a point P (x_1, y_1) is given by the equation $xy_1 + yx_1 = 8$ 2

(ii) Hence show that the equation of the chord of contact from the point $(2, -1)$ is given by the equation $x - 2y + 8 = 0$ 2

Question 6 19 marks Start a new booklet.

a) The polynomial equation $P(x) = x^3 - 4x^2 + 9x - 10 = 0$ has roots α, β and γ.

(i) Show that the equation with roots $\alpha - 1,\ \beta - 1,\ \gamma - 1$ is 2

$F(x): x^3 - x^2 + 4x - 4 = 0$

(ii) Solve $F(x) = 0$ and hence find the roots of $P(x) = 0$ 2

(iii) Write $P(x)$ as a product of linear factors. 1

b) A sequence of numbers is given by $T_1 = 6, T_2 = 27$ and 4

$T_n = 6T_{n-1} - 9T_{n-2},\ for\ n \geq 3$. Show using mathematical induction that

$T_n = (n+1)3^n$

c) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b > 0$ cuts the x axis at the points M and N.
The ellipse has eccentricity e and $S: (ae, 0)$ is one focus of this ellipse.
P is a point $(a\cos\theta, b\sin\theta)$ on the ellipse.
The focal chord PQ is perpendicular to the x axis.

(i) Show that $\cos\theta = e$ 1

(ii) Show that $\frac{1}{MS} + \frac{1}{NS} = \frac{4}{PQ}$ 4

d) $P: \left(3p, \frac{3}{p}\right)$ and $Q\ \left(3q, \frac{3}{q}\right)$ are two points on different braches of the hyperbola $xy = 9$.
P and Q move so that the gradient of PQ is 2.

(i) Show that $pq = -\frac{1}{2}$ 1

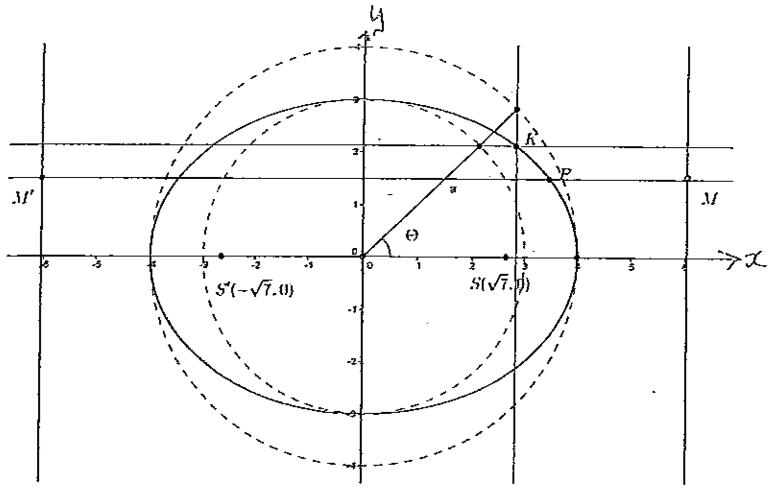
(i) Show that the point of intersection R of the tangents is $\left(\frac{6pq}{p+q}, \frac{6}{p+q}\right)$. 2

You are given that the equation of the tangent P is $x + p^2y = 6p$
Do not prove this.

(ii) Find the equation of the locus of R. 2

END of TASK

Qn	Solutions	Marks	Comment: Criteria
1	$P(-i) = 10(-i)^4 - 7(-i) + 4$ $= 14 + 7i$ (A)		
2	$H: b^2 = a^2(e^2-1)$ $1 + \frac{b^2}{a^2} = e^2$ ie $e^2 = \frac{a^2+b^2}{a^2}$ $E: b^2 = (a^2+b^2)(1-e_1^2)$ $\frac{b^2}{a^2+b^2} = 1 - e_1^2$ $e_1^2 = 1 - \frac{b^2}{a^2+b^2}$ $e_1^2 = \frac{a^2}{a^2+b^2}$ $\therefore e_1^2 = \frac{1}{e^2}$ so $e_1 = \frac{1}{e}$ (C)		
3	$ae = 2, \ \frac{a}{e} = 4$ $a = \frac{2}{e}$ $\frac{2}{e^2} = 4$ $e^2 = \frac{1}{2}$ $= \frac{1}{\sqrt{2}}$ $\therefore a = 2\sqrt{2} \Leftrightarrow a^2 = 8$ $b^2 = a^2(1-e^2)$ $= 8(1-\frac{1}{2})$ $= 4$ $\therefore \frac{x^2}{8} + \frac{y^2}{4} = 1$ OR $x^2 + 2y^2 = 8$ (B)		
4	$x^3 + x + 1 = 0$ NOTE $\alpha\beta\gamma = -1$ NEW ROOTS ARE $\frac{\alpha}{\alpha\beta\gamma}, \frac{\beta}{\alpha\beta\gamma}, \frac{\gamma}{\alpha\beta\gamma}$ ie $-\alpha, -\beta, -\gamma$ NEW EQUATION IS $(-x)^3 + (-x) + 1 = 0$ $-x^3 - x + 1 = 0$ OR $x^3 + x - 1 = 0$ (A)		



a) ii) $PS = ePM$, (See diagram for points P, S, S', M, M')
$PS' = ePM'$

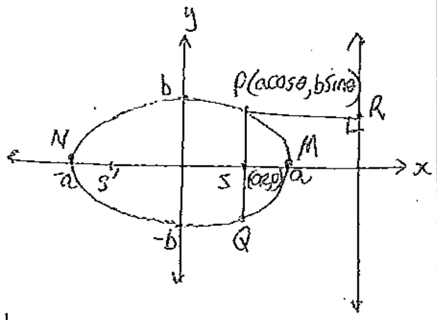
$PS + PS' = e(PM + PM')$
$= e\left(\frac{2a}{e}\right)$
$= 2a$
$= 2 \times 4$
$= 8.$

iii) See diagram (need to use auxiliary circles)

b) i) $m = -\frac{a}{b}$
$y - 0 = -\frac{a}{b}(x - ae)$
$by = -ax + a^2 e$
$ax + by - a^2 e = 0$

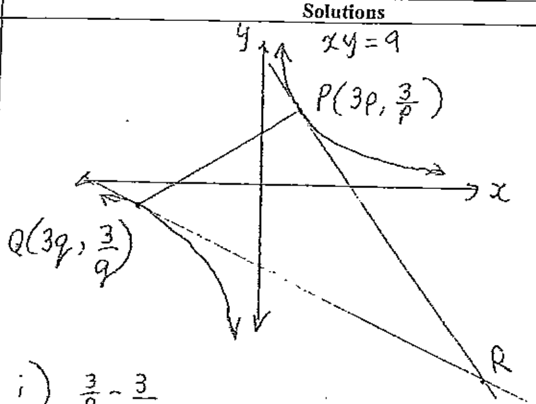
Qn	Solutions	Marks	Comment: Criteria
5 b) ii)	SOLVE $ax+by-a^2e=0$ AND $y=\frac{bx}{a}$ SIMULTANEOUSLY $ax+\frac{b^2x}{a}-a^2e=0$ $x\left(a+\frac{b^2}{a}\right)=a^2e$ $x=\frac{a^3e}{a^2+b^2}$ BUT $a^2e^2=a^2+b^2$ $=\frac{a^3e}{a^2e^2}$ $=\frac{a}{e}$ $\therefore$ THIS POINT LIES ON THE DIRECTRIX $x=\frac{a}{e}$		
c)	$\frac{5x^2-3x+13}{(x^2+4)(x-1)}=\frac{ax+b}{x^2+4}+\frac{c}{x-1}$ $5x^2-3x+13=(ax+b)(x-1)+c(x^2-4)$ LET $x=1$: $15=5c$, $c=3$ LET $x=0$: $13=-b+4c$, $b=12-13$, $=-1$ FOR COEFF OF x^2: $5=a+c$, $a=5-3$, $=2$ $\therefore \frac{5x^2-3x+13}{(x^2+4)(x-1)}=\frac{2x-1}{x^2+4}+\frac{3}{x-1}$		
d)	$xy=4$ $y+xy'=0$ $y'=\frac{-y}{x}$ AT (x_1,y_1) $m=\frac{-y_1}{x_1}$ $y-y_1=\frac{-y_1}{x_1}(x-x_1)$ $x_1y-x_1y_1=-y_1x+x_1y_1$ $y_1x+x_1y=2x_1y_1$ ($x_1y_1=4$ AS (x_1,y_1) LIES ON $xy=4$) $y_1x+x_1y=8$		
5 d) ii)	THE TANGENTS TO $xy=4$ AT (x_1,y_1) AND (x_2,y_2) WOULD SATISFY $xy_1+yx_1=8$ AND $xy_2+yx_2=8$ SINCE $(2,-1)$ LIES ON BOTH $2y_1-x_1=8$ $2y_2-x_2=8$ WHICH IMPLIES (x_1,y_1) AND (x_2,y_2) LIE ON $2y-x=8$		
6 a) i)	$P(x)=x^3-4x^2+9x-10=0$ $P(x+1)=(x+1)^3-4(x+1)^2+9(x+1)-10$ $=x^3+3x^2+3x+1-4x^2-8x-4+9x+9-10$ $=x^3-x^2+4x-4$ $\therefore F(x)=x^3-x^2+4x-4=0$ HAS ROOTS $\alpha-1,\beta-1,\gamma-1$		
ii)	$F(x)=x^2(x-1)+4(x-1)=0$ $(x^2+4)(x-1)=0$ $x=\pm 2i, 1$		
iii)	$P(x)=(x-2)(x-1-2i)(x-1+2i)$ (SINCE ZEROS ARE $x=2, 1\pm 2i$)		

Qn	Solutions	Marks	Comment: Criteria
6 b)	$T_1 = (2)3^1$ $= 6$ $T_2 = (3)3^2$ $= 27$ $\therefore$ TRUE FOR $n=1, n=2$ ASSUME TRUE FOR $n=k, n=k-1$ $T_k = (k+1)3^k$ $T_{k-1} = k.3^{k-1}$ PROVE TRUE FOR $n=k+1$ $T_{k+1} = (k+2)3^{k+1}$ $LHS = 6T_k - 9T_{k-1}$ $= 6(k+1)3^k - 9k.3^{k-1}$ $= 2(k+1)3^{k+1} - k.3^{k+1}$ $= (k+2)3^{k+1}$ $= RHS$ If true for $n=k$ and $n=k-1$, then proved true for $n=k+1$. But proved true for $n=1, 2$, $\therefore$ by induction true for all $n \geq 3$.		

Qn	Solutions	Marks	Comment: Criteria
6 c)	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  i) P LIES ON $x = ae$ $\therefore a\cos\theta = ae$ $\cos\theta = e$ ii) $LHS = \frac{1}{MS} + \frac{1}{NS}$ $= \frac{1}{a-ae} + \frac{1}{a+ae}$ $= \frac{1+e+1-e}{a(1+e)(1-e)}$ $= \frac{2}{a(1-e^2)}$ $RHS = \frac{4}{PQ}$ $= \frac{4}{2PS}$ $= \frac{2}{ePR}$ $= \frac{2}{e(\frac{a}{e} - ae)}$ $= \frac{2}{a(1-e^2)}$ $= LHS$		

Qn	Solutions	Marks	Comment: Criteria

(d)



i) $\dfrac{\frac{3}{p} - \frac{3}{q}}{3p - 3q} = 2$

$\dfrac{3q - 3p}{3pq(p-q)} = 2$

$6pq(p-q) = -3(p-q)$

$\div(p-q)$ since $p \neq q$

$-2pq = 1$

$pq = -\frac{1}{2}$

ii) $x + p^2 y = 6p$ (1)

$x + q^2 y = 6q$ (2)

$(p^2 - q^2)y = 6(p-q)$ (1) − (2) $(p \neq q)$

$(p+q)y = 6$

$y = \dfrac{6}{p+q}$

$x = 6p - p^2 \cdot \dfrac{6}{p+q}$

$x = \dfrac{6p^2 + 6pq - 6p^2}{p+q}$

$= \dfrac{6pq}{p+q}$

Qn	Solutions	Marks	Comment: Criteria

ii) Since $pq = -\frac{1}{2}$

$x = \dfrac{6pq}{p+q}$

$= -\dfrac{3}{p+q}$

$y = \dfrac{6}{p+q}$

$= -2\left(\dfrac{3}{p+q}\right)$

$\therefore y = -2x$ is the locus of R.