Student Number: ______________________

St Catherine's School

Waverley

YEAR 12 MATHEMATICS

HIGHER SCHOOL CERTIFICATE

ASSESSMENT TASK 2

17th March 2015

General Instructions

- Reading Time – 5 minutes
- Working Time – 75 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- Marks may be deducted for careless or badly arranged work
- Task Weighting – 25%
- Total Marks – 49

SECTION I **23 marks**

- Attempt Questions 1 – 4 in one booklet
- Show all necessary working

SECTION II **26 marks**

- Attempt Questions 5 – 6
- Answer each question in a separate booklet
- Show all necessary working

Question 1 – 3	/3
Question 4	/20
Question 5	/14
Question 6	/12
TOTAL	/49

STANDARD INTEGRALS

$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \; n \neq -1; \; x \neq 0, \text{ if } n < 0$

$\int \frac{1}{x}\,dx = \ln x, \; x > 0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \; a \neq 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax, \; a \neq 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \; a \neq 0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \; a \neq 0$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \; a \neq 0$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \; a \neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \; a > 0, \; -a < x < a$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \; x > a > 0$

$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE: $\ln x = \log_e x, \; x > 0$

SECTION I **START A NEW BOOKLET**
Total Marks 23
Attempt Questions 1 – 4

Answer either A, B, C or D.

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1. For what values of x is $f(x) = 3 + 4x^3 - x^4$ concave up?

(A) $x < 0$ or $x > 3$ (B) $0 < x < 3$

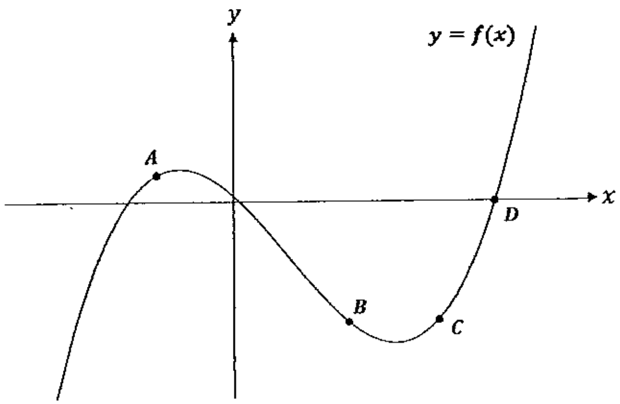
(C) $x < 0$ or $x > 2$ (D) $0 < x < 2$

2. Find $\int \frac{dx}{\sqrt{3x-2}}$

(A) $-\frac{3}{2}(3x-2)^{-\frac{3}{2}} + c$ (B) $\frac{3}{2}(3x-2)^{\frac{1}{2}} + c$

(C) $\frac{2}{3}(3x-2)^{\frac{1}{2}} + c$ (D) $6(3x-2)^{\frac{1}{2}} + c$

3.



The above curve has equation $y = f(x)$. Which point satisfies $f'(x) < 0$ and $f''(x) > 0$?

(A) point A (B) point B

(C) point C (D) point D

Question 4 **(20 marks)** **Marks**

(a) Find

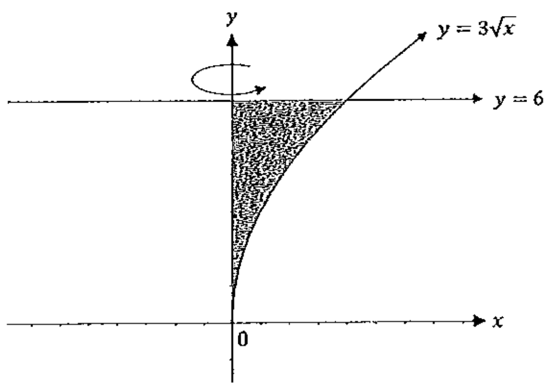
(i) $\int \frac{x^4 + x^3}{x}\, dx$ 2

(ii) $\int_1^3 (2x+5)(x-1)\, dx$ 3

(b) The gradient of a curve is given by $f'(x) = 3x^2 - 4x - 1$. The curve passes through the point $(1, 0)$.

What is the equation of the curve? 2

(c)

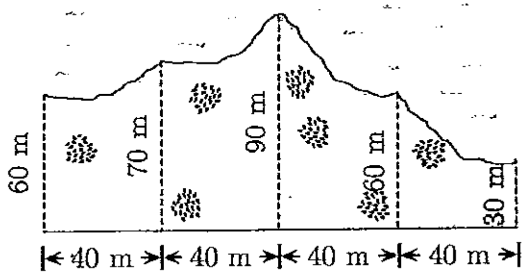


The shaded region in the diagram is bounded by the curve $y = 3\sqrt{x}$, the y-axis and the line $y = 6$.

Find the exact volume of the solid of revolution formed when the shaded region is rotated about the y -axis. 3

(d) Explain why the curve $f(x) = \frac{1}{1+x}$ is always decreasing for all values of x. 2

(e) The diagram shows the land that Alex bought near a lake.



(i) Copy and complete the table of values in your writing booklet. 1

Length of the land (in metres)		40	80		
Width of the land (in metres)	60		90		

(ii) Use Simpson's Rule with 5 functions values to find an estimate for the area of the land. Give your answer correct to 2 decimal places. 2

(f) The shaded region $OABC$ is bounded by the lines $x = 0$, $x = 5$, the curve $y = 3x^2$, the line $y = 4 - x$ and the x-axis, as in the diagram.

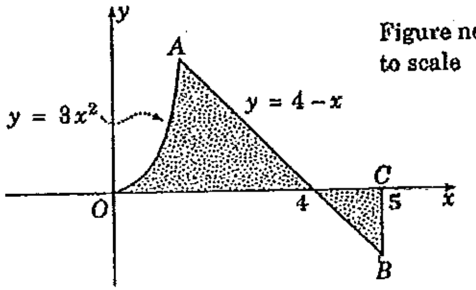


Figure not to scale

(i) Show that A has coordinates $(1, 3)$. 2

(ii) What is the area of the shaded region $OABC$? 3

End of Section I

SECTION II
Total Marks 26

Question 5 (14 marks) START A NEW BOOKLET **Marks**

(a) Consider the function defined by $f(x) = x(x-3)^2$

(i) Find the x-intercept(s). 1

(ii) Show that $f'(x) = 3(x-3)(x-1)$ and $f''(x) = 6(x-2)$. 3

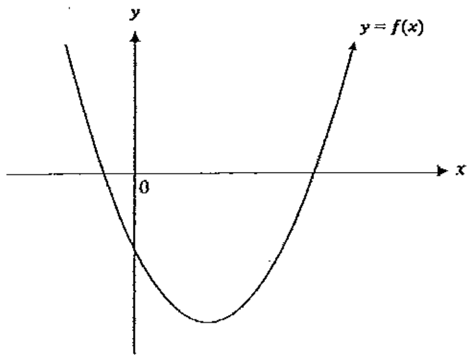
(iii) Find the coordinates of all stationary points and determine their nature. 3

(iv) Find the coordinates of any point(s) of inflexion. 2

(v) What is the minimum value of the function $f(x)$ in $-1 \leq x \leq 4$? 2

(vi) Sketch the curve $y = f(x)$ for $-1 \leq x \leq 4$ showing the stationary points, point(s) of inflexion, x-intercepts and coordinates of endpoints. 2

(b) The diagram below shows the graph of a certain function $y = f(x)$.

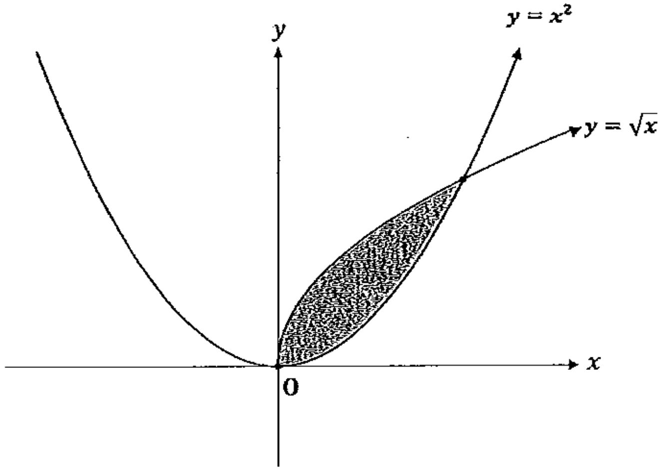


(i) Copy or trace the diagram into your writing booklet.

(ii) On the same set of axes, draw a sketch of the derivative $f'(x)$ of the function. 1

Question 6 (12 marks) START A NEW BOOKLET **Marks**

(a) The curves $y = x^2$ and $y = \sqrt{x}$ are sketched below.



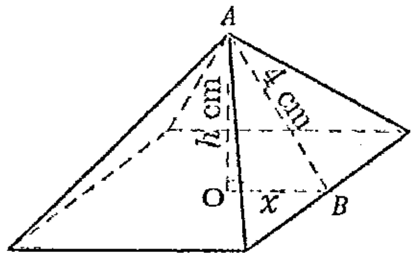
(i) Show that the points of intersection of the two curves are $(0, 0)$ and $(1, 1)$. **2**

(ii) Find the area of the shaded region bounded by $y = x^2$ and $y = \sqrt{x}$. **3**

(b) Given that $\int_1^k x\sqrt{x}\ dx = \frac{62}{5}$ and k is a constant, find the value of k. **2**

Question 6 continued **Marks**

(c) A square pyramid is to be formed with a slant height of 4 cm, a perpendicular height of h cm, and $OB = x$ as shown in the diagram.



(i) Express x^2 in terms of h. **1**

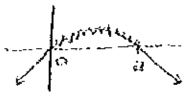
(ii) Show that the volume of the square pyramid is given by **1**

$$V = \frac{4h}{3}(16 - h^2)$$

(iii) Find the greatest possible volume of the square pyramid. Leave your answer in exact form. **3**

End of Section II

END OF TASK

Qn	Solutions	Marks	Comment: Criteria
	Year 12 Mathematics Higher School Certificate 2015 Assessment Task 2		
	Section 1		
(1)	$f'(x) = 12x^2 - 4x^3$ $f''(x) = 24x - 12x^2$ concave up when $f''(x) > 0$ $24x - 12x^2 > 0$ $12x(2-x) > 0$  $\therefore$ when $0 < x < 2$, $f(x)$ is concave up. (D)		
(2)	$\int \frac{dx}{\sqrt{3x-2}} = \int (3x-2)^{-\frac{1}{2}}\,dx$ $= \frac{2(3x-2)^{\frac{1}{2}}}{3} + c$ (C)		
(3)	$f'(x) < 0$ decreasing function ↘ $f''(x) > 0$ concave up ∪ combining $f'(x) < 0$ and $f''(x) > 0 \Rightarrow$ ↘ (B)		

Qn	Solutions	Marks	Comment: Criteria
(4) (a)	(i) $\int \frac{x^4 + x^3}{x}\,dx = \int \frac{x^4}{x} + \frac{x^3}{x}\,dx$	2	• $-\frac{1}{2}$ for not adding c
	$= \int x^3 + x^2\,dx$ ✓	1	
	$= \frac{1}{4}x^4 + \frac{1}{3}x^3 + c$ ✓	1	
	(ii) $\int_1^3 (2x+5)(x-1)\,dx$	3	
	$= \int_1^3 2x^2 - 2x + 5x - 5\,dx$		
	$= \int_1^3 2x^2 + 3x - 5\,dx$ ✓	1	
	$= \left[\frac{2x^3}{3} + \frac{3x^2}{2} - 5x\right]_1^3$ ✓	1	
	$= \frac{2(3)^3}{3} + \frac{3(3)^2}{2} - 5(3) - \left[\frac{2(1)^3}{3} + \frac{3(1)^2}{2} - 5(1)\right]$ ✓	1	
	$= 18 + \frac{27}{2} - 15 - \left(\frac{2}{3} + \frac{3}{2} - 5\right)$		
	$= \frac{58}{3}$ or $19\frac{1}{3}$		
(b)	$f(x) = \int f'(x)\,dx$	2	
	$= \int 3x^2 - 4x - 1\,dx$		
	$= x^3 - 2x^2 - x + c$ ✓	1	
	substitute $x = 1$ and $y = f(1) = 0$ $0 = 1^3 - 2(1)^2 - 1 + c$ $0 = -2 + c$ $c = 2$ ✓	1	
	$\therefore f(x) = x^3 - 2x^2 - x + 2$ is the equation of the curve.		

Qn	Solutions	Marks	Comment: Criteria

(c) (3 marks)

$y = 3\sqrt{x}$

$\frac{y}{3} = \sqrt{x}$

$\frac{y^2}{9} = x$ ✓ — 1

rotated about y-axis,

hence $V = \pi \int_{y_1}^{y_2} x^2 \, dy$

$= \pi \int_0^6 \left(\frac{y^2}{9}\right)^2 dy$

$= \pi \int_0^6 \frac{y^4}{81} \, dy$ ✓ — $\frac{1}{2}$

$= \frac{\pi}{81} \int_0^6 y^4 \, dy$

$= \frac{\pi}{81} \times \frac{1}{5} \left[y^5\right]_0^6$ OR $\frac{\pi}{405} \left[y^5\right]_0^6$ ✓ — 1

$= \frac{\pi}{405} \left(6^5 - 0^5\right)$ ✓ — $\frac{1}{2}$

$\therefore \; V = \frac{96\pi}{5}$ units^3

(d) (2 marks)

$f(x) = \frac{1}{1+x} = (1+x)^{-1}$

$f'(x) = -(1+x)^{-2}$

$= -\frac{1}{(1+x)^2}$ ✓ — 1

since $(1+x)^2 > 0$ for $x \in \mathbb{R}$ ✓

$\Rightarrow \frac{1}{(1+x)^2} > 0$

$\Rightarrow -\frac{1}{(1+x)^2} < 0$ ✓ — $\frac{1}{2}$

$\therefore \; f'(x) < 0$ for $x \in \mathbb{R}$ ✓ — $\frac{1}{2}$

$\therefore$ $f(x)$ is a decreasing function for $x \in \mathbb{R}$.

Qn	Solutions	Marks	Comment: Criteria

(e) (i) (1 mark)

length	0	40	80	120	160
width	60	70	90	60	30
	y_0	y_1	y_2	y_3	y_n

(ii) Simpson's Rule (2 marks)

$A \doteqdot \frac{h}{3}\left[y_0 + 4(\text{odd}) + 2(\text{even}) + y_n\right]$

where h = width of subinterval

$\therefore \; A \doteqdot \frac{40}{3}\left[60 + 4(70+60) + 2(90) + 30\right]$ ✓ ✓

$\doteqdot 10533.333\ldots$

$\doteqdot 10533.33 \text{ m}^2$ (2 decimal places)

Comment: 1 correct h; 1 correct [ ]

(f) (i) (2 marks)

A is the intersection point of $y = 3x^2$ & $y = 4 - x$

then solving simultaneously:

$3x^2 = 4 - x$

$0 = 3x^2 + x - 4$

$0 = (3x+4)(x-1)$

then $x = -\frac{4}{3}$ or $x = 1$ ✓ — 1

but A is in 1st quadrant so $x > 0$ ✓ — $\frac{1}{2}$

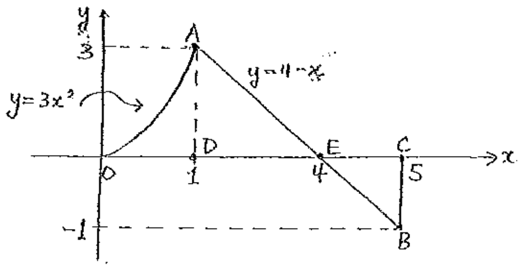
$\therefore$ A has x-coordinate $x = 1$

when $x = 1$, $y = 4 - 1$

$= 3$ ✓ — $\frac{1}{2}$

$\therefore \; A(1,3)$ as required.

(ii) (3 marks)



Qn	Solutions	Marks	Comment: Criteria

when $x = 5$, $y = 4 - 5$
$= -1$

then $B(5, -1)$

Area OABC $= \int_0^1 3x^2\,dx$ + Area ΔADE + Area ΔECB

$= \left[x^3\right]_0^1 + \frac{1}{2} \times DE \times AD + \frac{1}{2} \times EC \times BC$ — $1\frac{1}{2}$

$= 1^3 - 0^3 + \frac{1}{2} \times 3 \times 3 + \frac{1}{2} \times 1 \times 1$ — $1\frac{1}{2}$

$= 1 + \frac{9}{2} + \frac{1}{2}$

$= 6$ units^2

Section 2

(1)

(a) $f(x) = x(x-3)^2$ — 3

(ii) $f'(x) = (x-3)^2 + x \times 2(x-3)$ (product rule)

$= (x-3)^2 + 2x(x-3)$ — OR

$= (x-3)(x-3+2x)$

$= (x-3)(3x-3)$

$= 3(x-3)(x-1)$ as required — 2

Comment: $f(x) = x(x^2 - 6x + 9)$
$= x^3 - 6x^2 + 9x$
$f'(x) = 3x^2 - 12x + 9$
$= 3(x^2 - 4x + 3)$
$= 3(x-3)(x-1)$

$f''(x) = 3[1(x-1) + 1(x-3)]$ (product rule)

$= 3(x - 1 + x - 3)$

$= 3(2x - 4)$ — OR

$= 3 \times 2(x-2)$

$= 6(x-2)$ as required — 1

Comment: $f'(x) = 3x^2 - 12x + 9$
$f''(x) = 6x - 12$

Note: see page 6 of solutions for (i)

Qn	Solutions	Marks	Comment: Criteria

(iii) stationary points exist when $f'(x) = 0$ — 3

$0 = 3(x-3)(x-1)$

then $x = 3$ or $x = 1$

when $x = 3$, $f''(3) = 6(3-2)$ — OR — Sign Test
$= 6$
> 0 $\therefore$ minimum

$f(3) = 3(3-3)^2$
$= 0$

$\therefore (3, 0)$ is a minimum turning point

when $x = 1$, $f''(1) = 6(1-2)$ — OR — Sign Test
$= -6$
< 0 $\therefore$ maximum

$f(1) = 1(1-3)^2$
$= 4$

$\therefore (1, 4)$ is a maximum turning point

(i) x-intercept when $y = 0$ — 1 — $-\frac{1}{2}$ for a wrong x-value

$0 = x(x-3)^2$

$\therefore x = 0$ or $x = 3$

(iv) point of inflexion exists when $f''(x) = 0$ — 2

$0 = 6(x-2)$

then $x = 2$

when $x = 2$, $y = f(2) = 2(2-3)^2$
$= 2$

concavity test:

x	1.5	2	2.5
$f''(x)$	-3	0	3
		$-$	

Qn	Solutions	Marks	Comment: Criteria

$\therefore (2,2)$ is a point of inflexion. ✓

(v) when $x = -1$, $y = f(-1)$

$= (-1)(-1-3)^2$

$= -16$ ✗

Marks: 2

when $x = 4$, $y = f(4)$

$= 4(4-3)^2$

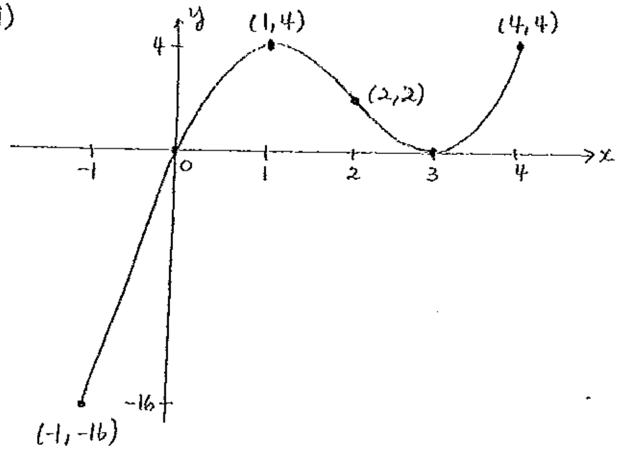
$= 4$ ✗

Comment: $-\frac{1}{2}$ if give min. value is $(-1,-16)$

and since $(3,0)$ is a minimum turning point,

$\therefore$ absolute minimum value of $f(x)$ is $y = -16$. ✓

(vi)



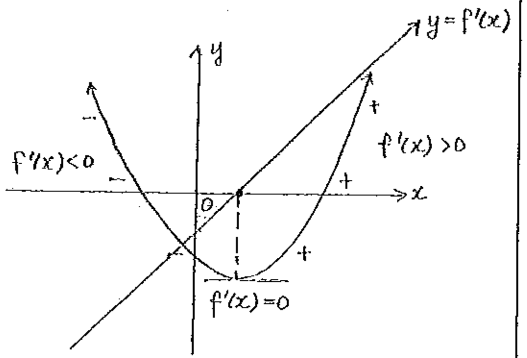
Marks: 2

Comment: 1 for shape

1 for essential features

$-\frac{1}{2}$ for each error

(b) (i) and (ii)



Marks: 1

Qn	Solutions	Marks	Comment: Criteria

6

(a) (i) solve simultaneously:

Marks: 2

$x^2 = \sqrt{x}$

$x^4 = x$

$0 = x^4 - x$

$0 = x(x^3 - 1)$ ✗

then $x = 0$ or $x^3 = 1$

$x = \sqrt[3]{1}$

$x = 1$ ✗

when $x = 0$, $y = 0^2 = 0$ $\therefore (0,0)$ ✗

when $x = 1$, $y = 1^2 = 1$ $\therefore (1,1)$ ✗

$\therefore (0,0)$ and $(1,1)$ are points of intersection.

(ii) Area $= \int_{x_1}^{x_2} y_{top} - y_{bottom}\ dx$

Marks: 3

$= \int_0^1 \sqrt{x} - x^2\ dx$ ✓

$= \int_0^1 x^{\frac{1}{2}} - x^2\ dx$

$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_0^1$ ✓

Comment: 1 for correct integration

$= \frac{1}{3}\left[2x^{\frac{3}{2}} - x^3\right]_0^1$

$= \frac{1}{3}\left\{\left[2(1)^{\frac{3}{2}} - 1^3\right] - \left[2(0)^{\frac{3}{2}} - 0^3\right]\right\}$ ✗

$= \frac{1}{3}\left[(2-1) - 0\right]$

$= \frac{1}{3}$ units^2 ✗

Qn	Solutions	Marks	Comment: Criteria
(b)	$\int_1^k x\sqrt{x}\ dx = \frac{62}{5}$	2	
	$\int_1^k x^1 \times x^{\frac{1}{2}}\ dx = \frac{62}{5}$		
	$\int_1^k x^{\frac{3}{2}}\ dx = \frac{62}{5}$ ✓ $\frac{1}{2}$		
	$\frac{2}{5}\left[x^{\frac{5}{2}}\right]_1^k = \frac{62}{5}$ ✓ $\frac{1}{2}$		
	$\frac{2}{5}\left[k^{\frac{5}{2}} - 1^{\frac{5}{2}}\right] = \frac{62}{5}$		
	$\frac{2}{5}\left(k^{\frac{5}{2}} - 1\right) = \frac{62}{5}$ ✓ $\frac{1}{2}$		
	$k^{\frac{5}{2}} - 1 = 31$		
	$k^{\frac{5}{2}} = 32$		
	$\left(k^{\frac{5}{2}}\right)^{\frac{2}{5}} = 32^{\frac{2}{5}}$		
	$\therefore k = 4$ ✓ $\frac{1}{2}$		
(c)	(i) using Pythagoras' theorem	1	
	$OB^2 = AB^2 - AO^2$		
	$\therefore x^2 = 4^2 - h^2$ ✓		
	(ii) base area = area of square	1	
	$= (2x)^2$		
	$= 4x^2$		
	$= 4(16 - h^2)$ using answer from (i)		
	then $V = \frac{1}{3} \times$ base area $\times$ height		
	$= \frac{1}{3} \times 4(16 - h^2) \times h$		
	$\therefore V = \frac{4h}{3}(16 - h^2)$ as required		

Qn	Solutions	Marks	Comment: Criteria
	(iii) $V = \frac{4h}{3}(16 - h^2)$	3	
	$\frac{dV}{dh} = \frac{4}{3}(16 - h^2) + \frac{4h}{3}(-2h)$ (product rule)		$V = \frac{64h}{3} - \frac{4h^3}{3}$
		OR	$\frac{dV}{dh} = \frac{64}{3} - 4h^2$
	$= \frac{4}{3}(16 - h^2 - 2h^2)$		
	$= \frac{4}{3}(16 - 3h^2)$		
	stationary points when $\frac{dV}{dh} = 0$		
	$0 = \frac{4}{3}(16 - 3h^2)$		
	$0 = 16 - 3h^2$		
	$h^2 = \frac{16}{3}$		
	$h = \pm\frac{4}{\sqrt{3}}$		1 differentiation
	but $h = \frac{4}{\sqrt{3}}$ only since height is positive.		1 stationary points
	$\frac{d^2V}{dh^2} = \frac{4}{3}(-6h)$		$\frac{1}{2}$ test stationary points
	$= -8h$		$\frac{1}{2}$ Maximum volume
	when $h = \frac{4}{\sqrt{3}}$, $\frac{d^2V}{dh^2} = -8 \times \frac{4}{\sqrt{3}}$		
	$= -\frac{32}{\sqrt{3}}$		
	< 0		
	$\therefore$ maximum volume when $h = \frac{4}{\sqrt{3}}$ cm		
	Then greatest volume $= \frac{4}{3} \times \frac{4}{\sqrt{3}}\left[16 - \left(\frac{4}{\sqrt{3}}\right)^2\right]$ ✓		
	$= \frac{512}{9\sqrt{3}}$		
	$= \frac{512\sqrt{3}}{27}$ cm^3		