Student Number

St Catherine's School

Waverley

2012

HIGHER SCHOOL CERTIFICATE

ASSESSMENT TASK 3

HSC Weighting 15%

Extension 1 Mathematics

General Instructions

- Attempt Questions 1-8.
- Working time – 1 hour
- Write using blue or black pen.
- Board- approved calculators may be used.
- All necessary working should be shown.
- Circle the letter of your answer in Q1-4

Total marks - 32

Not all questions are of equal value

1 The number N of animals in a population at time t years is given by $N = 100 + Ae^{kt}$ for constants $A > 0$ and $k > 0$. Which of the following is the correct differential equation?

(A) $\dfrac{dN}{dt} = k(N-100)$ (B) $\dfrac{dN}{dt} = -k(N+100)$

(C) $\dfrac{dN}{dt} = -k(N-100)$ (D) $\dfrac{dN}{dt} = k(N+100)$

2 The velocity of a particle moving along the x axis is given by $v^2 = 24 + 2x - x^2$. Which of the following expressions is the correct equation for the acceleration of the particle in terms of x?

(A) $1-x$ (B) $12x + \dfrac{x^2}{2} - \dfrac{x^3}{6}$ (C) $1-2x$ (D) $24x + x^2 - \dfrac{x^3}{3}$

3 A football is kicked at an angle of α to the horizontal. The position of the ball at time t seconds is given by $x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \dfrac{1}{2}gt^2$ where g m/s^2 is the acceleration due to gravity and v m/s is the initial velocity of projection. What is the maximum height reached by the ball?

(A) $\dfrac{V\sin\alpha}{g}$ (B) $\dfrac{g\sin\alpha}{V}$ (C) $\dfrac{V^2\sin^2\alpha}{2g}$ (D) $\dfrac{g\sin^2\alpha}{2V^2}$

4 A particle moving in a straight line obeys $v^2 = -x^2 + 2x + 8$ where x is its displacement from the origin in metres and v is its velocity in ms^{-1}. The motion is simple harmonic. What is the amplitude?

(A) 2π metres (B) 3 metres (C) 8 metres (D) 9 metres

Question 5: (8 marks)

a) The volume of a cube is increasing at the constant rate of $10cm^3/\sec$. **3**

Find the rate at which the surface area is increasing when the volume is $125cm^3$.

b) The velocity of a particle is given by $\frac{dx}{dt}=5x$. Initially the particle is at $x=1$.

(i) Express x as a function of t **2**

(ii) Find the exact value of the displacement of the particle when $t=1$. **1**

(iii) Find the value of the acceleration when $x=1$. **2**

Question 6: (6 Marks)

If the surrounding air temperature is 20^0C, it takes 15 minutes for a cup of tea at a temperature of 80^0C to cool to a temperature of 40^0C. Given that T is the temperature in degrees Celsius of the tea after t minutes, then Newton's Law of Cooling states that T satisfies the differential equation $\frac{dT}{dt}=k(T-20)$.

a) Show that $T=20+Ae^{kt}$ is a solution of this differential equation. **2**

b) Find the value of A, and show that $k=\frac{-\ln 3}{15}$ **2**

c) Find the temperature of the tea after 30 minutes. **2**

Question 7: (7 Matks)

On the large planet of HEART, a kat sits at a point P on top of a wall which is 3.6 metres high. It sees a nouse on the horizontal ground below. The nouse is exactly 4 metres from the base of the wall. The kat jumps horizontally from the top of the wall with initial velocity 6 metres per second. On HEART the acceleration due to gravity is $20ms^{-2}$. Assuming that the jump can be modelled by a projectile, and air resistance can be ignored;

a) Find expressions for $x(t)$ and $y(t)$, the horizontal and vertical dispplacements of the kat from P after t seconds. **3**

b) Find the time taken for the kat to reach the ground **2**

c) Find the distance by which the kat fails to reach the nouse. **2**

Question 8: (7 Matks)

A particle is moving in a straight line. At time t seconds, its velocity v metres per second and displacement x metres is given by the equation:

$$v^2=48+16x-4x^2$$

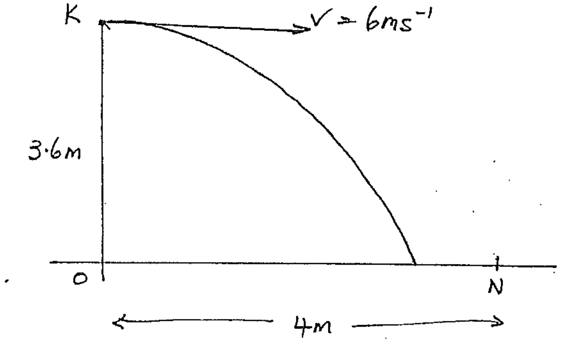
a) Show that motion is Simple Harmonic and state the centre of motion. **2**

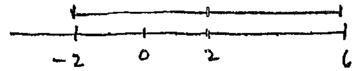
b) Find the amplitude of the motion. **3**

c) Find the value of the maximum velocity. **1**

d) Find the value of the maximum acceleration. **1**

End of Task

Q	Solutions	Marks	Comments
Q7			
	When $t=0$ $x=0$ $V=6$ $y=3\cdot 6$ $\dot{x}=$ $\dot{y}=0$	1	
	a) $\ddot{x}=0$ $\quad \ddot{y}=-g$		
	$\dot{x}=C_1$ $\quad \dot{y}=-gt+C_2$		
	when $t=0$ $\dot{x}=6$ $\quad \dot{y}=0$ $\therefore C_1=6$ $C_2=0$	1	$\frac{1}{2}$ each
	$\therefore \dot{x}=6$ $\quad \dot{y}=-20t$		
	integrating $x=6t+C_3$ $\quad y=-10t^2+C_4$		
	when $t=0$ $x=0$ $\quad y=3\cdot 6 \Rightarrow C_3=0$ $C_4=3\cdot 6$		
	$\therefore \underline{x=6t}$ $\quad \underline{y=-10t^2+3\cdot 6}$	1	$\frac{1}{2}$ each
	b) $3\cdot 6-10t^2=0$	1	
	$10t^2=3\cdot 6$		
	$t^2=\frac{3\cdot 6}{10}$		
	$t=\sqrt{\frac{36}{100}}$		
	$=\frac{3}{5}$	1	
	c) $x=6\times\frac{3}{5}$	1	
	$=3\cdot 6$ m		
	$\therefore$ Kat fails by 40cm.	1	

Q	Solutions	Marks	Comments
Q8	$v^2=48+16x-4x^2$		
	a) $\ddot{x}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)=\frac{d}{dx}\left(24+8x-2x^2\right)$	1	
	$=8-4x$		
	$\therefore \ddot{x}=8-4x$		
	i.e. $\ddot{x}=-4(x-2)$		
	this is in the form $\ddot{x}=-n^2(x-x_0)$		
	$\therefore$ motion is S.H.M.	1	
	b) at extremities $v=0$		
	$\therefore 48+16x-4x^2=0$		
	$12+4x-x^2=0$		
	$(6-x)(2+x)=0$	1	
	$\therefore x=-2,6$		
			
	$\therefore$ amplitude $=4$	1	
	c) Max velocity when $\ddot{x}=0$ ie at C.O.M.	1	
	$v^2=48+16x-4x^2$		
	when $x=2$		
	$v^2=48+32-16$		
	$v^2=64$		
	$\therefore v=\pm 8$		
	$\therefore$ maximum velocity is $8\,ms^{-1}$	1	
	d) Max acceleration where $v=0$ ie $x=6,-2$		
	now $\ddot{x}=-4(x-2)$		
	when $x=6$ $\ddot{x}=-16\,ms^{-2}$		
	$x=-2$ $\ddot{x}=16\,ms^{-2}$		
	$\therefore$ maximum acceleration $16\,ms^{-2}$	1	

Q	Solutions	Marks	Comments
Q1	A	1	
Q2	A	1	
Q3	C	1	
Q4	B	1	
Q5	a) $\frac{dV}{dt} = 10$ $V = x^3$, $\frac{dV}{dx} = 3x^2$ $A = 6x^2$, $\frac{dA}{dx} = 12x$ $V = 125$ $\therefore x^3 = 125$, $x = 5$	1	
	Now $\frac{dA}{dt} = \frac{dV}{dt} \times \frac{dA}{dV}$ $= \frac{dV}{dt} \times \frac{dA}{dx} \times \frac{dx}{dV}$ $= 10 \times 12x \times \frac{1}{3x^2}$	1	
	$= \frac{40}{x}$ when $x=5$ $\frac{dA}{dt} = 8 \text{ cm}^2/\text{s}$	1	
	b) i) $\frac{dx}{dt} = 5x$ $t=0$ $x=1$; $\frac{dt}{dx} = \frac{1}{5x}$; $dt = \frac{1}{5x}dx$	1	
	$\int$ $t = \frac{1}{5}\ln x + c$; when $t=0$ $x=1 \Rightarrow c=0$ NOTE: $\ln 1 = 0$	$\frac{1}{2}$	
	$\therefore t = \frac{1}{5}\ln x$; $5t = \log_e x$; $\therefore x = e^{5t}$	$\frac{1}{2}$	
	(ii) when $t=1$ $x = e^5$	1	
	(iii) $\ddot{x} = \frac{d}{dx}\frac{1}{2}v^2$ $= \frac{d}{dx}\frac{25x^2}{2}$	1	
	$= 25x$ when $x=1$ $\ddot{x} = 25 \text{ ms}^{-2}$	1	

Q	Solutions	Marks	Comments
Q6	$\frac{dT}{dt} = k(T-20)$		
	a) $T = 20 + Ae^{kt} \Longrightarrow Ae^{kt} = T - 20$	1	
	now $\frac{dT}{dt} = kAe^{kt}$ $= k(T-20)$	1	
	b) $T = 20 + Ae^{kt}$; when $t=0$ $T=80$; $80 = 20 + A$; $\therefore A = 60$	$\frac{1}{2}$	
	$\therefore T = 20 + 60e^{kt}$; $40 = 20 + 60e^{kt}$; $20 = 60e^{kt}$; $\frac{1}{3} = e^{kt}$	$\frac{1}{2}$	
	taking log of both sides $\ln\frac{1}{3} = kt$; When $t = 15$ $k = \frac{\ln\frac{1}{3}}{15}$ Note: $\ln\frac{1}{3} = \ln 1 - \ln 3 = -\ln 3$; $\therefore k = -\frac{\ln 3}{15}$	1	
	c) $\therefore T = 20 + 60e^{-\frac{\ln 3}{15}t}$; When $t=30$ $T = 20 + 60e^{-2\ln 3}$	1	
	$= 20 + 60e^{-\ln 9}$ $= 20 + 60e^{\ln\frac{1}{9}}$ Note: $-\ln 9 = \ln 9^{-1} = \ln\frac{1}{9}$; $= 20 + 60 \cdot \frac{1}{9}$; $= \frac{80}{3}$°C	1	