Student Number ____________

St Catherine's School
Waverley

Year 12 Mathematics Extension 1
Task 3
May2015

Time allowed: **90 minutes plus 5 minutes reading time**

Total marks: **50 marks**

Weighting: **20%**

There are 9 questions including 6 multiple choice questions.

- Attempt all questions.
- Attempt Questions 1 – 7 in one booklet. Write the answers to the multiple choice questions in this booklet
- Attempt Questions 8 and 9 in separate booklets.
- Show all necessary working for questions 7 to 9.
- Marks may be deducted for careless or badly arranged work.
- A table of Standard integrals is provided in the last page of this paper.
- Board-approved calculators may be used.

Questions 1 to 7 /19	
Question 8	
Question 9	
Total	

STANDARD INTEGRALS

$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$

$\int \frac{1}{x}\,dx = \ln x,\ x > 0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$

$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE: $\ln x = \log_e x,\ x > 0$

1 Which of the following is an expression for $\int \cos^2 2x dx$?

(A) $x - \frac{1}{4}\sin 4x + c$

(B) $x + \frac{1}{4}\sin 4x + c$

(C) $\frac{x}{2} - \frac{1}{8}\sin 4x + c$

(D) $\frac{x}{2} + \frac{1}{8}\sin 4x + c$

2 Which of the following is the solution to $\int \cos^2 x \sin x dx$?

Use the substitution $u = \cos x$.

(A) $2\cos x \sin x + c$

(B) $\cos^3 x + c$

(C) $\frac{1}{3}\cos^3 x + c$

(D) $-\frac{1}{3}\cos^3 x + c$

3 A particle is moving along the x-axis. Its velocity v at position x is given by $v = \sqrt{8x - x^2}$

What is the acceleration when $x = 3$?

(A) 1

(B) 2

(C) 3

(D) 4

4 The acceleration of a particle is given by $a = 3x^2 - 3$ where x is the displacement. The particle is initially at the origin and has a velocity of 2 cm/s. What is the velocity when the particle is 2 cm from the origin?

(A) $v = 2\sqrt{2}$ cm/sec

(B) $v = -2\sqrt{2}$ cm/sec

(C) $v = 12\ cm/sec$

(D) $v = -12\ cm/sec$

5 The speed v (cm/s) of a particle moving in a straight line is given by $v^2 = 24x - 4x^2 - 32$, where the magnitude of its displacement from a fixed point O is x cm. The motion is simple harmonic. What is the centre of the motion?

(A) $x = -3$

(B) $x = 2$

(C) $x = 3$

(D) $x = -4$

6 The velocity of a particle moving in a straight line is given by $v = 2x$

Initially the particle is at $x = 4$. The displacement x is given by

(A) $x = 4e^t$

(B) $x = 4\ log_e t$

(C) $x = \frac{1}{2}\ log_4 t$

(D) $x = 4e^{2t}$

Question 7 (13 marks)

a) Integrate $\int x\sqrt{x^2+1}\,dx$ Use the substitution $u = x^2 + 1$ 2

b) Show using the substitution $u = sin^2x$, that $\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1+sin^2x}dx = \ln 2$ 3

c) Evaluate the integral $\int_0^7 \frac{x^2}{\sqrt[3]{x+1}}\,dx$ use the substitution $x + 1 = u^3$ 4

d) (i) Use the substitution $x = 2\sin\theta$, 4

to evaluate $\int_0^2 \frac{x^2}{\sqrt{4-x^2}}dx$

Question 8 17 marks Start a new booklet

a) A skydiver is falling at a speed of V metres per second. He opens his parachute when falling at 20 metres per second and thereafter the acceleration is given by the equation $\frac{dV}{dt} = -k\,(V-5)$, where k is a constant.

(i) Show that $V = 5 + Ae^{-kt}$ satisfies $\frac{dV}{dt} = -k\,(V-5)$, 1

(ii) Explain why $A = 15$ 1

(iii) One second after the parachute opens, the skydiver's speed has fallen to 12 metres per second.
Determine the value of k correct to 4 decimal places. 2

(iv) Find the skydiver's speed after 3 seconds. 1

(v) What is the limiting value of the velocity V? Explain your answer. 1

(vi) Sketch a graph of the Velocity vs Time 1

b) The velocity v *metres per second* of a particle moving in simple Harmonic Motion is given by $v^2 = 8 + 2x - x^2$

(i) Between which points is the particle oscillating? 2

(ii) Show that the acceleration is $\ddot{x} = -(x-1)$ 1

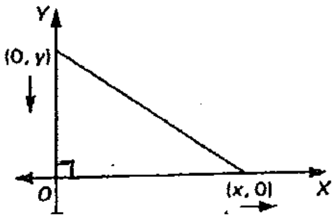
(iv) Find the period of motion 1

(v) Find the maximum speed of the particle 1

(vi) Find the maximum value of the acceleration 1

Question 8 continues next page

c) A ladder 5 metres long has its upper end against a vertical wall and its lower end on the horizontal floor. The lower end moves away from the wall at a constant speed of 1 metre per second. Find the speed at which the upper end moves down the wall 4 seconds after the lower end starts slipping. **4**



Question 9 (14 marks) Start a new booklet

b) The acceleration of a particle is given by

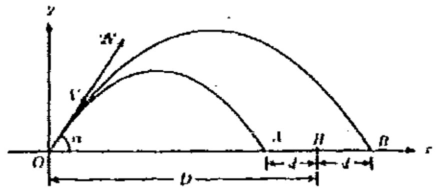
$\ddot{x} = 4(x+1)^3$, where x is the displacement from the origin after t seconds. Initially the particle is at $x = 1$, with a velocity of $-4\sqrt{2}$ metres per second.

(i) Show that the velocity at any position x is given by $v = -\sqrt{2}\,(x+1)^2$ **3**

(ii) Find the time taken by the particle to reach the origin. **3**

Question 9 continues next page

b) Jane hits a golf ball from the origin O, with a velocity of V m/s and at an angle of α to the horizontal towards a hole H. The hole is D metres away from O. The ball lands at a point A, which is d metres short of the hole H.



(i) Use the axes as shown, and assume that there is no air resistance. **2**
Show that $x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$, where g is the acceleration due to gravity.

(ii) Show that $D = \frac{V^2 \sin 2\alpha}{g} + d$ **3**

Later she hits a second ball from O with a velocity of 2V m/s with the same angle of projection. This lands at a point B, which is d metres past the hole H.

See the diagram above.

(iii) By using the trajectory of the second ball, write down another expression for D. **1**

(iv) If the first ball is hit with a velocity $\sqrt{gd}\ m/s$, find α the angle of projection **2**

END OF PAPER

Solutions	Marks	Comments: Criteria
Extension 1 Mathematics Solutions. Task 3 2015.		
$\int \cos^2 2x\, dx = \frac{1}{2}\int (1+\cos 4x)\, dx$ $= \frac{1}{2}\left[x + \frac{\sin 4x}{4}\right] + C$	$\boxed{D}$	
$\int \cos^2 x \sin x\, dx$; $u = \cos x$, $du = -\sin x\, du$ $= \int u^2(-du) = -\frac{u^3}{3} + C$ $= -\frac{\cos^3 x}{3} + C$	$\boxed{D}$	
$u = 8x - x^2$ $v^2 = 8x - x^2$ $a = \frac{d}{dx}\left(4x - \frac{x^2}{2}\right) = 4 - x.$ $x = 3;\ a = 4 - 3 = 1$	$\boxed{A}$	
$a = 3x^2 - 3$ $\frac{v^2}{2} = x^3 - 3x + C$ ($t=0$, $x=0$, $v=2$) $\frac{4}{2} = C$ $\therefore \frac{v^2}{2} = x^3 - 3x + 2$ $v = \pm\sqrt{2x^3 - 6x + 4}$ ($x=0$, $v=2$) $\therefore v = +\sqrt{2x^3 - 6x + 4}$ $x = 2$ $v = \sqrt{16 - 12 + 4}$	$\boxed{A}$	

Qn	Solutions	Marks	Comments: Criteria
5.	$v = 0$; $x = +4$ and $x = +2$ Centre : 3.	$\boxed{C}$	
6	$u = 2x$ $\frac{dt}{dx} = \frac{1}{2x}$ $t = \frac{1}{2}\ln x + C$ ($t=0$, $x=4$) $C = -\frac{\ln 4}{2}$ $t = \frac{1}{2}\ln \frac{x}{4}$ $\ln \frac{x}{4} = 2t$ $x = 4e^{2t}$.	$\boxed{D}$	
7.	$\int x\sqrt{x^2+1}\, dx$; $u = x^2+1$, $du = 2x\, dx$ $\frac{1}{2}$ $= \frac{1}{2}\int \sqrt{u}\, du$ $\frac{1}{2}$ $= \frac{1}{2} \cdot u^{\frac{3}{2}} \cdot \frac{2}{3} + C$ $\frac{1}{2}$ $= \frac{1}{3}(x^2+1)^{3/2} + C$ $\frac{1}{2}$	2	
b)	$u = \sin^2 x$ $du = 2\sin x \cos x\, dx$ $\frac{1}{2}$ $x = 0;\ u = 0$ $x = \frac{\pi}{2};\ u = 1$ $\frac{1}{2}$ $\int_0^{\pi/2} \frac{\sin 2x\, dx}{1+\sin^2 x} = \int_0^1 \frac{du}{1+u}$ $= \left(\ln(1+u)\right)_0^1$ $= \ln 2 - \ln 1$ $= \ln 2$, as required.	3.	

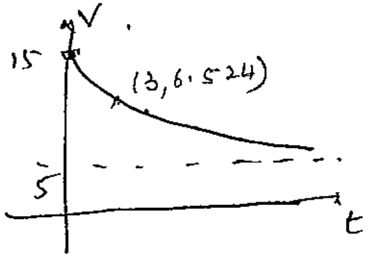
Qn	Solutions	Marks	Comments: Criteria
	$\int_0^7 \frac{x^2}{\sqrt[3]{x+1}}\,dx$ $x+1=u^3$; $dx=3u^2du$ $\frac{1}{2}$; $x=0;\ u=1$; $x=7;\ u=2$ $\frac{1}{2}$.		
	$=\int_1^2 \frac{(u^3-1)^2\cdot 3u^2du}{u}$ $\frac{1}{2}$		
	$=3\int_1^2 u(u^6-2u^3+1)\,du$ $\frac{1}{2}$		
	$=3\int_1^2 (u^7-2u^4+u)\,du$ $\frac{1}{2}$		
	$=3\left(\frac{u^8}{8}-\frac{2u^5}{5}+\frac{u^2}{2}\right)_1^2$ $\frac{1}{2}$		
	$=3\left[\left(32-\frac{2}{5}32+2\right)-\left(\frac{1}{8}-\frac{2}{5}+\frac{1}{2}\right)\right]$ $\frac{1}{2}$	4	
	$=\frac{2517}{40}$ $\frac{1}{2}$.		
	$x=2\sin\theta$; $dx=2\cos\theta\,d\theta$ $\frac{1}{2}$; $x=0;\ \theta=0$; $x=2;\ \theta=\frac{\pi}{2}$ $\frac{1}{2}$; $4-x^2=4-4\sin^2\theta=4\cos^2\theta$		
	$\int_0^2 \frac{x^2\,dx}{\sqrt{4-x^2}}=\frac{1}{2}\int_0^{\pi/2}\frac{4\sin^2\theta\cdot 2\cos\theta\,d\theta}{2\cos\theta}$ $\frac{1}{2}$		
	$=\frac{4}{2}\int_0^{\frac{\pi}{2}}(1-\cos 2\theta)$ $\frac{1}{2}$	4.	
	$=2\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{\pi/2}$ $\frac{1}{2}$		
	$=2\left[\left(\frac{\pi}{2}-\frac{\sin\pi}{2}\right)-(0-0)\right]$ $\frac{1}{2}$ ③		

Qn	Solutions	Marks	Comments: Criteria
	$=2\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{\pi/2}$.		
	$=2\left[\frac{\pi}{2}\right]=\pi$.		
Q.8 (i)	$V=5+Ae^{-kt}$ — ①		
	$\frac{dV}{dt}=-k(Ae^{-kt})$		
	$=-k(V-5)$ from ①.	1	
(ii)	$t=0;\ V=20$		
	$\therefore 20=5+Ae^{0}$ $\therefore A=15$	1	
(iii)	$t=1;\ V=12$		
	$12=5+15e^{-k}$		
	$\frac{7}{15}=e^{-k}$ $\therefore -k=\ln\frac{7}{15}$	1	
	$k=0.7621$ (4 sig fig)	1	
(iv)	$t=3$ $V=5+15e^{-3\times 0.7621}$		
	$=6.524$ (3d.p.)	1	

Solutions	Marks	Comments: Criteria

$t \to \infty$; $e^{-kt} \to 0$ — 1

$\therefore v \to 5$



1.

At the end points $v = 0$.

$8+2x-x^2=0$

$-(x-4)(x+2)=0$

$x=-2$ and $x=4$ — 2

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(4+x-\frac{x^2}{2}\right)$$

$$= 1-x \quad = -(x-1)$$ — 1

Period: $\frac{2\pi}{1} = 2\pi$ — 1

max. speed is at the centre $x=1$

$v^2 = 8+2-1^2$

$= 9$. — 1

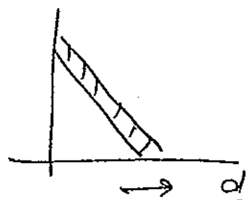
$|v| = 3$ m/s.

max. value of accel is at $x=4$, ~~the c~~

or $x=-2$, at the end points

$|a| = |-(4-1)| = 3$ m/s^2 — 1.

Qn	Solutions	Marks	Comments: Criteria



$\longrightarrow \frac{dx}{dt} = 1$

$x^2 + y^2 = 5^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ — 1

$x = 4$; $y = \sqrt{5^2-4^2} = 3$ — 1

$\therefore 4\times 1 + 3\times\frac{dy}{dt} = 0$ — 1

$\frac{dy}{dt} = -\frac{4}{3}$ — 1

OR

$x^2+y^2 = 5^2$

$y = \sqrt{25-x^2}$ $(y>0)$

$\frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}\sqrt{25-x^2}}$ (1M)

$\left(\frac{dy}{dx}\right)_{x=4} = \frac{-4}{\sqrt{25-16}} = \frac{-4}{\sqrt{9}}$ (1M)

$\left(\frac{dy}{dt}\right)_{x=4} = \left(\frac{dy}{dx}\right)_{x=4} \times \frac{dx}{dt}$ (1M)

$= -\frac{4}{3}\times 1$

$= -\frac{4}{3}$. (1M)

Solutions	Marks	Comments: Criteria
$\ddot{x} = 4(x+1)^3$		
$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4(x+1)^3$		
$\frac{v^2}{2} = 4\frac{(x+1)^4}{4} + c$		
$v^2 = 2(x+1)^4 + 2c$	(1M)	
when $x = 1$; $v = -4\sqrt{2}$		
$2\times 16 = 2\times 16 + c \quad \therefore c = 0$		
$v^2 = 2(x+1)^4$	(2M)	
$v = \pm\sqrt{2}(x+1)^4$	(3M)	
$v = -\sqrt{2}(x+1)^4$ for when $x=1$ $v=-4\sqrt{2}$.		
$v = -\sqrt{2}(x+1)^2$		
$\frac{dx}{dt} = -\sqrt{2}(x+1)^2$		
$\frac{dt}{dx} = -\frac{1}{\sqrt{2}(x+1)^2}$		
$t = -\frac{1}{\sqrt{2}}\int\frac{dx}{(x+1)^2}$	1	
$t = +\frac{1}{\sqrt{2}(x+1)} + c$		
$x=1$, $t=0$: $0 = \frac{1}{2\sqrt{2}} + c \quad \therefore c = -\frac{1}{2\sqrt{2}}$	1	
$t = \frac{1}{\sqrt{2}(x+1)} - \frac{1}{2\sqrt{2}}$	1	
$x=0$; $t = \frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{1}{2\sqrt{2}}$ secs		

Qn	Solutions	Marks	Comments: Criteria
	The only accel. is gravity		Suitable explanations for values / constants. ($\frac{1}{2}$ each)
	$\ddot{x} = 0$; $\ddot{y} = -g$		
	$\dot{x} = \text{Const}$; $\dot{y} = -gt + c$		
	at $t=0$; $\dot{x} = V\cos\alpha$; $t=0$; $\dot{y} = V\sin\alpha$		
	$\therefore \dot{x} = V\cos\alpha$; $\therefore \dot{y} = -gt + V\sin\alpha$		
	$x = V\cos\alpha\, t + c$; $y = -\frac{gt^2}{2} + V\sin\alpha\, t + c$		
	$t=0$; $x=0 \quad \therefore c=0$; $t=0$; $y=0 \quad \therefore c=0$		
	$x = V\cos\alpha\, t$; $y = -\frac{gt^2}{2} + V\sin\alpha\, t$		
	where it lands $y=0$.		
	$-\frac{gt^2}{2} + V\sin\alpha\, t = 0$		
	$t\left(-\frac{gt}{2} + V\sin\alpha\right) = 0$; $t \neq 0$	(1M)	
	$\therefore t = \frac{2V\sin\alpha}{g}$		
	Range is $x = V\cos\alpha \times \frac{2V\sin\alpha}{g}$		
	$= \frac{V^2\sin 2\alpha}{g}$	(1M)	
	In the diagram;	1M.	
	$D = d + \text{Range}$	1	
	$= d + \frac{V^2\sin 2\alpha}{g}$		
	For the second ball, the range is $\frac{(2V)^2\sin 2\alpha}{g}$		

In the diagram.

$$D + d = \frac{4v^2 \sin 2\alpha}{g}$$

$$D = \frac{4v^2 \sin 2\alpha}{g} - d \quad \text{—} (2) \qquad 1M.$$

Using (1) and (2)

$$\frac{v^2 \sin 2\alpha}{g} + d = \frac{4v^2 \sin 2\alpha}{g} - d$$

$$v = \sqrt{gd} \qquad \underline{1M}$$

$$\frac{3v^2 \sin 2\alpha}{g} = 2d$$

$$\frac{3}{\cancel{g}} \cdot \cancel{g} d \cdot \sin 2\alpha = 2d$$

$$\sin 2\alpha = \frac{2}{3}$$

$$2\alpha = \sin^{-1}\frac{2}{3}$$

$$\alpha = \frac{1}{2}\sin^{-1}\frac{2}{3} \qquad \underline{1M.}$$

$$= 21^\circ \text{ (to n. deg.)}$$