Student Number: ____________

St. Catherine's School
Waverley

Mathematics Extension 2

2015

ASSESSMENT TASK 3
25 May

Assessment weight 20%

Reading time - 5minutes Working time – 90 minutes.
Total marks 50

There are 8 questions including 5 multiple choice questions.

- Attempt all questions.
- Attempt Questions 1 – 6 in one booklet. Write the answers to the multiple choice questions in this booklet
- Attempt Questions 7 and 8 in a separate booklet.
- Show all necessary working for questions 6 to 8.
- Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- A table of Standard Integrals is provided on the last page of the paper.
- **Hand in one bundle.**
- **Leave the question paper inside the answer booklets.**

Multiple Choice	/5
Question 6	/15
Question 7	/15
Question 8	/15
Total	/50

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0,\ \text{if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x>0$

Write the answers in the booklet.

1) Which of the following is equivalent to $\int \frac{1}{\sqrt{7-6x-x^2}}dx$

a) $\sin^{-1}\left(\frac{x-3}{2}\right)+c$ b) $\sin^{-1}\left(\frac{x+3}{2}\right)+c$

c) $\sin^{-1}\left(\frac{x-3}{4}\right)+c$ d) $\sin^{-1}\left(\frac{x+3}{4}\right)+c$

2) Which of the following answers is false

a) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^7\theta\, d\theta = 0$ b) $\int_0^1 x(1-x)^{49}dx = \int_0^1(1-x)x^{49}dx$

c) $\int_{-1}^{1} e^{-x^2} = 0$ d) $\int_{-2}^{2}\frac{1}{9-x^2}dx = 2\int_0^2\frac{1}{9-x^2}dx$

3) If $\int_1^5 f(x)dx = 7$, what is the value of $\int_1^5 f(6-x)dx$? (Hint: Use $u = 6-x$)

a) 7 b) −7

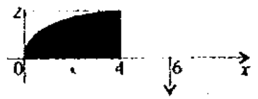
c) −1 d) 3.5

4) Using a suitable substitution, which is the correct expression for $\int_0^{\frac{\pi}{6}}\cos^3 x\sin^4 x\, dx$ in terms of u?

a) $\int_0^{\frac{\sqrt{3}}{2}} u^4 - u^6\, du$ b) $\int_0^{\frac{1}{2}} u^6 - u^4\, du$

c) $\int_0^{\frac{1}{2}} u^4 - u^6\, du$ d) $\int_1^{\frac{\sqrt{3}}{2}} u^6 - u^4\, du$

5)



The curve$y = \sqrt{x}$, bounded by the x $axis$ between $y = 0$ and $x = 4$ is rotated about the line $x = 6$. Which of the following integrals gives the volume of this solid of revolution? (Hint: Use $a^2 - b^2 = (a-b)(a+b)$ in your simplification.)

a) $\int_0^2(4-y^2)(6-y^2)dy$ b) $\int_0^2(2-y^2)(6-y^2)dy$

c) $\int_0^2 2\pi(6-y^2)dy$ d) $\pi\int_0^2(4-y^2)(8-y^2)dy$

Question 6   15 marks   Start a new page

a) By completing the squares, find $\int \frac{dx}{x^2+4x+13}$. 2

b) Use integration by parts to find $\int_0^{\pi} e^x \sin x \, dx$ 3

c) i) Find real numbers a, b and c such that

$$\frac{4x+3}{(x^2+1)(x+2)} = \frac{ax+b}{x^2+1} + \frac{c}{x+2}$$ 2

ii) Hence find

$$\int \frac{4x+3}{(x^2+1)(x+2)} dx$$ 3

d) If $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$

i) Show that

$$I_n = \frac{n-1}{n} I_{n-2}$$ 3

ii) Hence evaluate

$$\int_0^{\frac{\pi}{2}} \cos^5 x \, dx$$ 2

Question 7   15 marks   Start a new booklet.

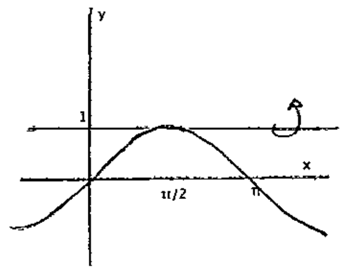
a) Find $\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta}$ by using $t = \tan\frac{\theta}{2}$ 4

b) Use the substitution $u = a - x$, to show that

i) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ 1

ii) Hence or otherwise evaluate $\int_0^1 x(1-x)^7 \, dx$ 2

c)



The area enclosed by $y = \sin x$, $0 \le x \le \pi$ and the x *axis* is rotated about the line $y = 1$.

(i) Copy the diagram and shade the region being rotated

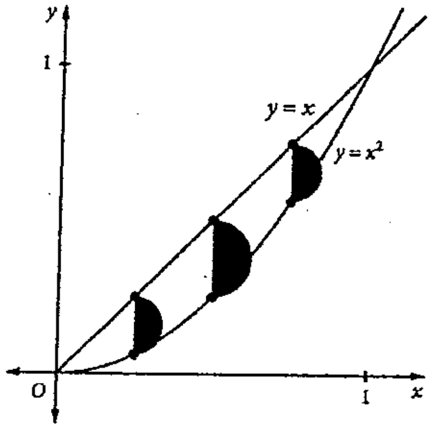
(ii) Show that the volume of the solid generated by taking slices perpendicular to the axis of rotation is given by $V = \int_0^{\pi} 2sinx - sin^2 x \ dx$ 2

(iii) Hence find the volume 2

Question 7 continued.......

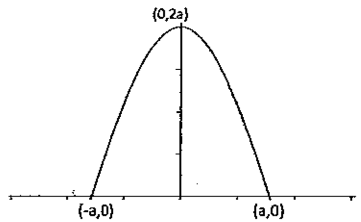
c) A solid S has a base that is the region bounded by $y = x$ and $y = x^2$. **4**

The solid's cross section perpendicular to the base are all semi circles with their diameters on the base. Show that the volume of the solid is $\frac{\pi}{240}\ cm^3$.



Question 8- 15 marks **Start a new booklet**

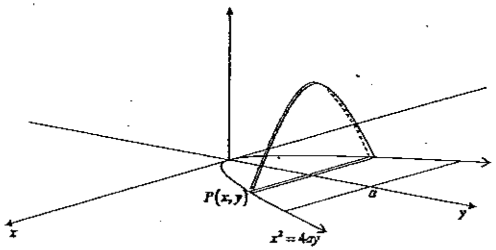
a) The diagram shows a cosine curve between $-a \le x \le a$, with amplitude $2a$.



(i) Show that the equation of this curve is $y = 2a\cos nx$, where $n = \frac{\pi}{2a}$ **1**

(ii) Show that the area enclosed between the x axis and this cosine curve, $y = 2a\cos\frac{\pi}{2a}x$ is $\frac{8a^2}{\pi}$ **2**

(ii) The base of a solid is the region bounded by $x^2 = 4ay$ and its latus rectum $y = a$. **2**



Each slice of thickness Δy taken perpendicular to the axis of symmetry of this parabola is part of a cosine curve as shown, whose amplitude is the length of its base

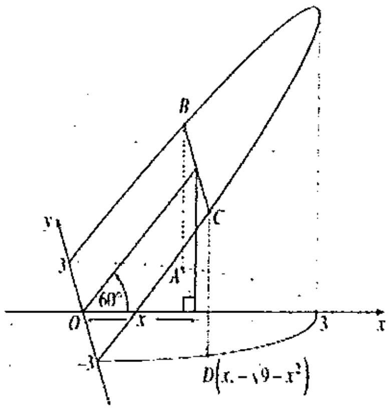
Find the volume of this solid in terms of a.

b) The circle $x^2 + y^2 = 4$ is rotated about the line $x = 6$ to form a torus.

(i) Use the method of cylindrical shells to show that the volume is given by $4\pi \int_{-2}^{2} (6 - x)\sqrt{4 - x^2}\, dx$ **2**

ii) Hence find the volume of this solid. **3**

c) The base of a right cylinder is the circle in the xy-plane with centre O and radius 3. A wedge is obtained by cutting this cylinder with the plane through the y-axis inclined at 60^0 to the xy-plane as shown in the diagram.



A rectangular slice ABCD is taken perpendicular to the base of the wedge at a distance x from the y-axis.

(i) Show that the area of this cross section ABCD is given by $2\sqrt{3}\, x \sqrt{9 - x^2}$ **2**

(ii) Find the volume of the wedge. **3**

END OF PAPER

Qn	Solutions	Marks	Comments: Criteria

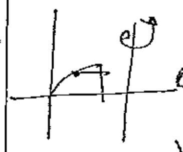
Extension 2 – Assessment Task 3 – 2011

1)
$$7-6x-x^2 = 16-(x+3)^2$$
$$\int \frac{dx}{\sqrt{7-6x-x^2}} = \int \frac{dx}{\sqrt{16-(x+3)^2}}$$
$$= \sin^{-1}\frac{x+3}{4} + C \qquad \boxed{d}$$

2. $$\int_{-1}^{1} e^{x^2} \neq 0. \qquad \boxed{c}$$

3. $$\int_1^5 f(6-x)\,dx \qquad \begin{aligned}6-x&=u\\ -dx&=du.\\ x=1;\ u&=5\\ x=5;\ u&=1\end{aligned}$$
$$= \int_5^1 f(u)\,(-du)$$
$$= \int_1^5 f(u)\,du = 7. \qquad \boxed{a}$$

4. $$\int_0^{\frac{\pi}{6}} \cos^3 x \sin^4 dx = \int_0^{\pi/6} (1-\sin^2 x)\sin^4 x \cos x\,dx$$
$\sin x = u.$
$\cos x\,dx = du$
$x=0;\ u=0$
$x=\frac{\pi}{6};\ u=\frac{1}{2}$
$$= \int_0^{\frac{1}{2}} (1-u^2)u^4.\,du.$$
$$= \int_0^{1/2} (u^4-u^6)\,du \qquad \boxed{c}$$

5. 
dy: suggests – slicing.
$$\Delta V = \pi\left[(6-x)^2 - 2^2\right]\Delta y$$
$$V = \pi\int_0^2 (6-x-2)(6-x+2)\,dy$$
$$= \pi\int_0^2 (4-y^2)(8-y^2)\,dy \qquad \boxed{d}$$

Qn	Solutions	Marks	Comments: Criteria

6. a) $$\int \frac{dx}{x^2+4x+13} = \int \frac{dx}{(x+2)^2+9}$$
$$= \frac{1}{3}\tan^{-1}\frac{(x+2)}{3} + c$$

b) $$\int_0^{\pi} e^x \sin x\,dx.$$
$$= \left(+e^x(-\cos x)\right)_0^{\pi} - \int_0^{\pi} e^x(-\cos x)\,dx$$
$$= (1+e^{\pi}) + \int_0^{\pi} e^x \cos x\,dx$$
$$= (1+e^{\pi}) + \left[\left(e^x \sin x\right)_0^{\pi} - \int_0^{\pi} e^x.\sin x\,dx\right]$$
$$2\int_0^{\pi} e^x \sin x\,dx = (1+e^{\pi}) + 0$$
$$\therefore \int_0^{\pi} e^x \sin x\,dx = \frac{1+e^{\pi}}{2}.$$

c) $$4x+3 \equiv (ax+b)(x+2) + c(x^2+1)$$
$x=-2$ $\quad -5 = 5c \quad \therefore c=-1$
Coeff. of x^2 $\quad a+c=0 \quad \therefore a=1$
Const. $\quad 3 = 2b+c \quad b=2.$

Qn	Solutions	Marks	Comments: Criteria
6 c(ii)	$\therefore \int \frac{4x+3}{(x^2+1)(x+2)}\,dx$		
	$= \int \frac{x+2}{x^2+1} + \frac{-1}{x+2}\,dx$		
	$= \int \frac{x}{x^2+1} + \int \frac{2}{x^2+1} - \int \frac{1}{x+2}\,dx$		
	$= \frac{1}{2}\ln(x^2+1) + 2\tan^{-1}x - \ln\lvert x+2\rvert + C$		
6d.	$I_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$		
	$= \int_0^{\frac{\pi}{2}} \cos^{n-1}x \cdot \cos x\,dx$		
	$= \left(\cos^{n-1}x \cdot \sin x\right)_0^{\frac{\pi}{2}} - \int_0^{\pi/2} (n-1)\cos^{n-2}x \cdot (-\sin x)\sin x\,dx$		
	$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2}x\,(1-\cos^2 x)\,dx$		
	$= (n-1)\left[\,I_{n-2} - I_n\right]$		
	$1+(n-1)\,I_n = (n-1)\,I_{n-2}$		
	$I_n = \frac{n-1}{n}\,I_{n-2}$		

Qn	Solutions	Marks	Comments: Criteria
	$I_5 = \frac{4}{5}\,I_3$		
	$= \frac{4}{5}\cdot\frac{2}{3}\cdot I_1$		
	$= \frac{8}{15}\int_0^{\pi/2} \cos x\,dx$		
	$= \frac{8}{15}\left(\sin x\right)_0^{\pi/2}$		
	$= \frac{8}{15}$		
Q.7 a)	$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta}$		
	$t = \tan\frac{\theta}{2}$		
	$dt = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta$		
	$= \frac{1}{2}\left(1+\tan^2\frac{\theta}{2}\right)d\theta$		
	$= \frac{1}{2}(1+t^2)\,d\theta$		
	$\frac{2\,dt}{1+t^2} = d\theta$		
	$\theta = 0;\ t = 0$		
	$\theta = \frac{\pi}{2};\ t = \tan\frac{\pi}{4} = 1$		
	$2+\cos\theta = 2 + \frac{1-t^2}{1+t^2} = \frac{2+2t^2+1-t^2}{1+t^2} = \frac{3+t^2}{1+t^2}$		

Qn	Solutions	Marks	Comments: Criteria

$$\therefore \int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta}$$

$$= \int_0^1 \frac{2\,dt}{1+t^2} \cdot \frac{1+t^2}{3+t^2}$$

$$= 2\int_0^1 \frac{dt}{3+t^2}$$

$$= \frac{2}{\sqrt{3}}\left(\tan^{-1}\frac{t}{\sqrt{3}}\right)_0^1$$

$$= \frac{2}{\sqrt{3}} \times \frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}.$$

b)

$u = a - x$

$du = -dx.$

$x = 0;\ u = a$

$x = a;\ u = 0$

$$\int_0^a f(a-x)\,dx = \int_a^0 f(u)(-du)$$

$$= \int_0^a f(u)\,du$$

$$= \int_0^a f(x)\,dx.$$

Qn	Solutions	Marks	Comments: Criteria

7b (ii)

$$\int_0^1 x(1-x)^7\,dx$$

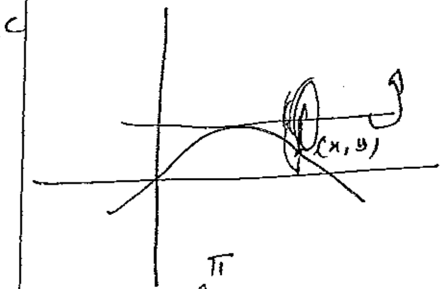
$$= \int_0^1 (1-x)\left(1-(1-x)\right)^7 dx$$

$$= \int_0^1 (1-x)\,x^7\,dx$$

$$= \int_0^1 (x^7 - x^8)\,dx$$

$$= \left(\frac{x^8}{8} - \frac{x^9}{9}\right)_0^1 = \frac{1}{8} - \frac{1}{9} = \frac{1}{72}$$

7c



Slice at (x, y)

Radius: $(1-y)$

$\Delta V = \pi\left[1^2 - (1-y)^2\right]\Delta x$

$$\therefore V = \int_0^{\pi} \pi\left[1 - (1 - 2y + y^2)\right] dx$$

$$= \pi\int_0^{\pi} (2y - y^2)\,dx \qquad y = \sin x$$

$$= \pi\int_0^{\pi} (2\sin x - \sin^2 x)\,dx.$$

Qn	Solutions	Marks	Comments: Criteria

(iii)

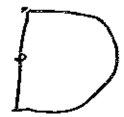
$$\int_0^{\pi} 2\sin x\,dx - \frac{1}{2}\int_0^{\pi}(1-\cos 2x)\,dx$$

$$= \left(-2\cos x\right)_0^{\pi} - \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)_0^{\pi}$$

$$= -2(-1-1) - \frac{1}{2}(\pi)$$

$$= 4 - \frac{\pi}{2}.$$

7c) $x-x^2$



Diam : $x-x^2$

Radius : $\frac{x-x^2}{2}$,

[Take a cross section at height y]

$$\Delta V = \frac{\pi}{2}\left(\frac{x-x^2}{2}\right)^2 \Delta y.$$

$$V = \frac{\pi}{8}\int_0^1 (x^2+x^4-2x^3)\,dx$$

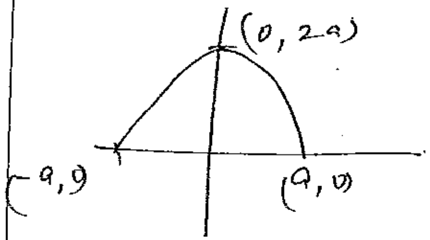
$$= \frac{\pi}{8}\left(\frac{x^3}{3}+\frac{x^5}{5}-\frac{2x^4}{4}\right)_0^1$$

$$= \frac{\pi}{8}\left(\frac{1}{3}+\frac{1}{5}-\frac{1}{2}\right)$$

$$= \frac{\pi}{8}\times\frac{1}{30} = \frac{\pi}{240} \text{ units}^3$$

Qn	Solutions	Marks	Comments: Criteria

8 a)



$y = 2a\cos nx$, amplitude is $2a$

Period : $4a$.

$$\therefore \frac{2\pi}{n} = 4a$$

$$n = \frac{2\pi}{4a} = \frac{\pi}{2a}$$

$$\therefore\ y = 2a\cos\frac{\pi}{2a}x.$$

(ii)

$$A = \int_{-a}^{a} 2a\cos\frac{\pi}{2a}x\,dx$$

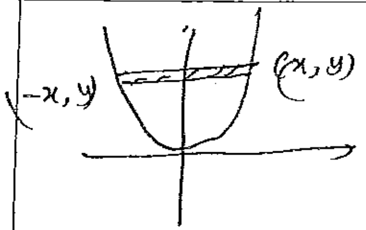
$$= 4a\int_0^a \cos\frac{\pi}{2a}x\,dx \quad [\cos \text{ is an even fn}]$$

$$= 4a\cdot\frac{2a}{\pi}\left(\sin\frac{\pi}{2a}x\right)_0^a$$

$$= \frac{8a^2}{\pi}\left(\sin\frac{\pi}{2}\right)$$

$$= \frac{8a^2}{\pi}.$$

Qn	Solutions	Marks	Comments: Criteria



Amplitude = length of base = $2x$.

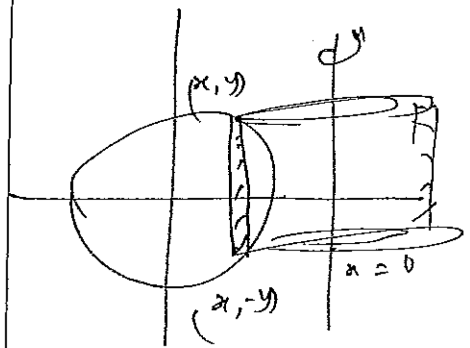
Area of a slice at (x, y) is $\frac{8x^2}{\pi}$ units^2

$$\Delta V = \frac{8x^2}{\pi}\Delta y.$$

$$\therefore V = \frac{8}{\pi}\int_0^a x^2\,dy \qquad x^2 = 4ay$$

$$= \frac{8}{\pi}\cdot 4a\cdot\int_0^a y\,dy$$

$$= \frac{32a}{\pi}\left(\frac{y^2}{2}\right)_0^a = \frac{16a^3}{\pi}.$$



$$\Delta V = 2\pi.(6-x).2y.\Delta x$$
$$= 4\pi(6-x)\cdot y\cdot\Delta x$$

$$x^2+y^2 = 4$$
$$y^2 = 4-x^2$$
$$y = \pm\sqrt{4-x^2}.$$
$$2y = 2\sqrt{4-x^2}$$

$$\therefore \Delta V = 4\pi(6-x)(4-x^2)\,\Delta x$$
$$V = 4\pi\int_{-2}^{2}(6-x)(4-x^2)\,dx$$

Qn	Solutions	Marks	Comments: Criteria

$$V = 4\pi\left[\int_{-2}^{2} 6\sqrt{4-x^2}\,dx - \int_{-2}^{2} x\sqrt{4-x^2}\,dx\right]$$

$$\int_{-2}^{2} x\sqrt{4-x^2}\,dx = 0$$ for $x\sqrt{4-x^2}$ is an odd fn.

$$\therefore V = 24\pi\int_{-2}^{2}\sqrt{4-x^2}\,dx$$

$$= 48\pi\int_0^2\sqrt{4-x^2}\,dx.$$ ($\sqrt{4-x^2}$ is even

Let $x = 2\sin\theta$
$dx = 2\cos\theta\,d\theta$
$x = 0$; $\theta = 0$
$x = 2$; $\theta = \frac{\pi}{2}$
$4-x^2 = 4-4\sin^2\theta$
$= 4\cos^2\theta$

$$\therefore V = 48\pi\int_0^{\pi/2} 2\cos\theta\cdot 2\cos\theta\,d\theta$$

$$= 96\pi\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$$

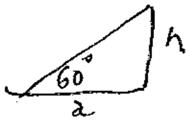
$$= 96\pi\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= 96\pi\times\frac{\pi}{2} = 48\pi^2.$$

Qn	Solutions	Marks	Comments: Criteria

Length $AD = 2y$.

$CD =$ height h, where



$$\frac{h}{x} = \tan 60°$$

$$h = \sqrt{3}\,x.$$

$\therefore$ Area of $ABCD = 2y \times \sqrt{3}\,x$.

$$= 2\sqrt{3}\,x \cdot \sqrt{9-x^2} \qquad (x^2+y^2=9)$$

$$V = 2\sqrt{3}\int_0^3 x\sqrt{9-x^2}\,dx$$

$$= \frac{2\sqrt{3}}{-2}\int_0^3 \sqrt{9-x^2}\,(-2x)\,dx.$$

$$= -\sqrt{3}\cdot\left[(9-x^2)^{3/2}\right]_0^3 \cdot \frac{2}{3}.$$

$$= -\frac{2\sqrt{3}}{3}(0-27)$$

$$= \frac{54\sqrt{3}}{3} \quad \text{or} \quad 18\sqrt{3} \text{ units}^3.$$