Student Number

St Catherine's School

Waverley

HIGHER SCHOOL CERTIFICATE

ASSESSMENT TASK 3 – 15%

Mathematics

General Instructions

- Working time – 55 minutes
- Write using black or blue pen
- Board approved calculators may be used
- All necessary working should be shown
- Start each question in a new booklet

Total marks - 46

The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\ dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\ dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\ dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax \tan ax\ dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\ dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\ dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

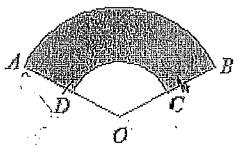
$$\int \frac{1}{\sqrt{x^2-a^2}}\ dx = \ln\left(x+\sqrt{x^2-a^2}\right)\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

QUESTION 1 START A NEW BOOKLET

(a) Choose the correct answer from A,B,C and D and write it in your writing booklet.

(i) A car windscreen wiper traces out the area $ABCD$ where AB and CD are arcs of circles with a centre O and radii 40 cm and 20 cm respectively. Angle AOB measures 120°.

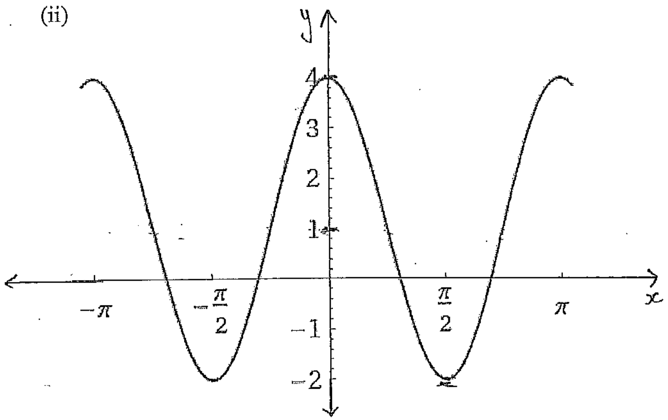


Not to scale

What is the area of $ABCD$?

(A) 419 cm^2 (B) 1257 cm^2 (C) 1676 cm^2 (D) 2095 cm^2

(ii)



The equation of the graph above is:

(A) $y = 4\sin(2x+1)$ (B) $y = 4\cos(2x+1)$

(C) $y = 3\sin(2x)+1$ (D) $y = 3\cos(2x)+1$

(iii) What is the solution to the equation $\log_e(x+2) - \log_e x = \log_e 4$?

(A) $\frac{2}{5}$ (B) $\frac{2}{3}$ (C) $\frac{3}{2}$ (D) $\frac{5}{2}$

(b) Simplify:

(i) $\sin\frac{4\pi}{3}$ /1

(ii) $\log_2 36 - 2\log_2 3$ /2

(c) Find:

(i) $f'(x)$ if $f(x) = e^{\cos x}$ /1

(ii) $f'(\frac{\pi}{3})$ if $f(x) = \tan x$ /2

(iii) $\frac{dy}{dx}$ if $y = e^x \sin x$ /2

(d) Find:

(i) $\int \cos 5x\,dx$ /1

(ii) $\int_{\frac{\pi}{3}}^{\pi} \sin\frac{x}{2}\,dx$ /3

(iii) $\int \frac{x^3}{x^4-1}\,dx$ /2

(iv) $\int \frac{e^{4x}+e^{2x}}{e^{3x}}\,dx$ /2

QUESTION 2 START A NEW BOOKLET

(a) The area of a sector of a circle with radius 6cm is $\frac{9\pi}{2}cm^2$. Find:

(i) the angle at the centre (answer in radians) /2

(ii) the perimeter of the sector /2

(b) (i) Write down the period and amplitude of $y = 3\sin 2x$ /2

(ii) Sketch the curve $y = 3\sin 2x$ for $0 \le x \le 2\pi$ /2

(iii) State the range of $y = 3\sin 2x$ /1

(iv) Without solving, state the number of solutions to $3\sin 2x = 2$, for the domain $0 \le x \le 2\pi$. /1

(c) Find the volume generated when the curve $y = \sqrt{\cos \pi x}$ is rotated about the x-axis for $0 \le x \le \frac{1}{2}$ /3

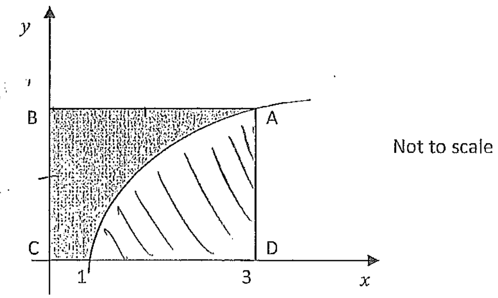
(d) (i) Show that $\frac{d}{dx}\sec x = \sec x \tan x$ /3

(ii) Hence, or otherwise, find the equation of the tangent to $y = \sec x$ at the point where $x = \frac{\pi}{4}$ /2

(e) (i) Find the derivative of $y = \log_e(2 - \cos 3x)$ /1

(ii) Hence, or otherwise find $\int \frac{6\sin 3x}{2 - \cos 3x}\,dx$ /2

(f) The diagram shows the graph of $y = \log_e x$. ABCD is a rectangle.



(i) Find the coordinates of the points A and B in exact form. /2

(ii) Find the shaded area /3

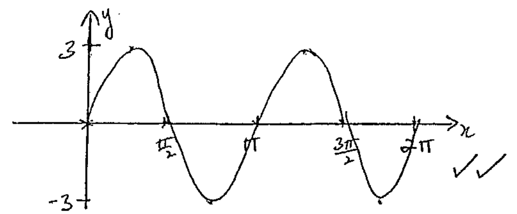
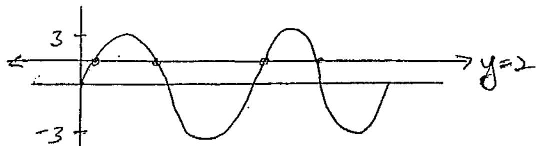
(iii) Evaluate $\int_1^3 \log_e x\,dx$ /1

END OF TEST

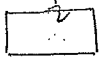
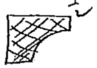
2UNIT TASK 3 2012

Qn	Solutions	Marks	Comments: Criteria
1(i)	$A = \frac{120}{360}\times\pi\times 40^2 - \frac{120}{360}\times\pi\times 20^2$ $= 1257\ cm^2$ (B)	1	
(ii)	(D)	1	
(iii)	$\log_e(x+2) - \log_e x = \log_e 4$ $\log_e\left(\frac{x+2}{x}\right) = \log_e 4$ $\therefore \frac{x+2}{x} = 4$ $x+2 = 4x$ $x = \frac{2}{3}$ (B)	1	
(b) (i)	$\sin\frac{4\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right)$ $= -\sin\frac{\pi}{3}$ $= -\frac{\sqrt{3}}{2}$	1	
(ii)	$\log_2 36 - 2\log_2 3 = \log_2(36) - \log_2(3^2)$ $\frac{1}{2}$ $= \log_2\left(\frac{36}{9}\right)$ $\frac{1}{2}$ $= \log_2 4$ $\frac{1}{2}$ $= 2$ $\frac{1}{2}$	2	
(c) (i)	$f(x) = e^{\cos x}$ $f'(x) = -\sin x\, e^{\cos x}$	1	
(ii)	$f(x) = \tan x$ $f'(x) = \sec^2 x$ ✓ $f'\left(\frac{\pi}{3}\right) = \sec^2\frac{\pi}{3}$ $= (2)^2$ $= 4$ ✓	2	

Qn	Solutions	Marks	Comments: Criteria
(c) (iii)	$y = e^x \sin x$ $u = e^x$ $v = \sin x$ $\frac{du}{dx} = e^x$ $\frac{dv}{dx} = \cos x$ $\frac{dy}{dx} = e^x \sin x + e^x \cos x$ $= e^x(\sin x + \cos x)$	2	
(d) (i)	$\int \cos 5x\, dx = \frac{1}{5}\sin 5x + c$	1	
(ii)	$\int_{\frac{\pi}{3}}^{\pi} \sin\left(\frac{x}{2}\right) dx = \left[-2\cos\left(\frac{x}{2}\right)\right]_{\frac{\pi}{3}}^{\pi}$ ✓ $= -2\cos\left(\frac{\pi}{2}\right) - -2\cos\left(\frac{\pi}{6}\right)$ ✓ $= -2\times 0 + 2\times\frac{\sqrt{3}}{2}$ $= \sqrt{3}$ ✓	3	1 for integral 1 for substitution
(iii)	$\int \frac{x^3}{x^4-1}\, dx = \frac{1}{4}\int \frac{4x^3}{x^4-1}\, dx$ $= \frac{1}{4}\log_e(x^4-1) + c$ ✓	2	
(iv)	$\int \frac{e^{4x}+e^{2x}}{e^{3x}}\, dx = \int \frac{e^{4x}}{e^{3x}} + \frac{e^{2x}}{e^{3x}}\, dx$ $= \int e^x + e^{-x}\, dx$ $= e^x - e^{-x} + c$	2	

Qn	Solutions	Marks	Comments: Criteria
2 (a)	(i) $A=\frac{1}{2}r^2\theta$ $\frac{9\pi}{2}=\frac{1}{2}\times6^2\times\theta$ ✓ $9\pi=36\theta$ $\frac{\pi}{4}=\theta$ ✓	2	Radian $-\frac{1}{2}$
	(ii) $l=r\theta$ $l=6\times\frac{\pi}{4}$ $=\frac{3\pi}{2}$ ✓ $P=6+6+\frac{3\pi}{2}$ $=12+\frac{3\pi}{2}$ units ✓ (1)	2	
(b)	(i) amplitude $=3$ ✓ period $=\frac{2\pi}{2}=\pi$ ✓	2	
	(ii)  ✓✓		amplitude $\frac{1}{2}$ shape $\frac{1}{2}$ [illegible] $\frac{1}{2}$
	(iii) range $=-3\le y\le3$ ✓	1	$\frac{1}{2}$ for $-3\le x\le3$.
	(iv)  $3\sin2x=2$ intersects 4 ✓ times.	1	

Qn	Solutions	Marks	Comments: Criteria
(c)	$y=\sqrt{\cos\pi x}$ $V=\pi\int_a^b y^2\,dx$ $=\pi\int_0^{\frac{1}{2}}(\sqrt{\cos\pi x})^2\,dx$ $=\pi\int_0^{\frac{1}{2}}\cos\pi x\,dx$ ✓ $=\pi\left[\frac{1}{\pi}\sin\pi x\right]_0^{\frac{1}{2}}$ ✓ $=\sin\frac{\pi}{2}-\sin0$ $=1$ ✓	3	
(d)	(i) $\frac{d}{dx}(\sec x)=\frac{d}{dx}\left(\frac{1}{\cos x}\right)$ $\frac{1}{2}$ $=\frac{\cos x\times0--\sin x}{(\cos x)^2}$ ✓ $\frac{1}{2}$ $u=1$ $v=\cos x$ $\frac{du}{dx}=0$ $\frac{dv}{dx}=-\sin x$ $=\frac{\sin x}{(\cos x)^2}$ ✓ $=\frac{\sin x}{\cos x\times\cos x}$ $=\tan x\sec x$ ✓	3	
	(ii) when $x=\frac{\pi}{4}$ $\frac{d}{dx}(\sec x)=\tan\frac{\pi}{4}\sec\frac{\pi}{4}$ $=1\times\sqrt{2}$ $=\sqrt{2}$ ✓ $\frac{1}{2}$ when $x=\frac{\pi}{4}$ $y=\sec\frac{\pi}{4}$ $=\sqrt{2}$ $\frac{1}{2}$ tangent $y-\sqrt{2}=\sqrt{2}\left(x-\frac{\pi}{4}\right)$ $\frac{1}{2}$ $\sqrt{2}x-y-\frac{\sqrt{2}\pi}{4}+\sqrt{2}=0$ ✓ $\frac{1}{2}$	2	

Qn	Solutions	Marks	Comments: Criteria
(e)(i)	$y = \log_e(2-\cos 3x)$ $\dfrac{dy}{dx} = \dfrac{3\sin(3x)}{2-\cos 3x}$ ✓	1	
(ii)	$\displaystyle\int \frac{6\sin(3x)}{2-\cos(3x)}\,dx = 2\int \frac{3\sin(3x)}{2-\cos(3x)}\,dx$ ✓ $= 2\log_e(2-\cos 3x) + c$ ✓	2	incorrect ln $\frac{1}{2}$ ($\ln(2-3x)$ etc) give $\frac{1}{2}$
(f)(i)	$y = \log_e x$ $\underline{\text{at } x=3}$ $\quad y = \log_e 3 \quad \therefore A \text{ is } (3, \log_e 3)$ $\frac{1}{2}$ ✓ $\text{and } B \text{ is } (0, \log_e 3)$ $\frac{1}{2}$ ✓ $\frac{1}{2}$ ✓	2	
(ii)	$A = \displaystyle\int_0^{\log_e 3} x\,dy \quad \text{where } x = e^y$ $= \displaystyle\int_0^{\log_e 3} e^y\,dy$ ✓ $= \left. e^y \right]_0^{\log_e 3}$ $= e^{\log_e 3} - e^0$ ✓ $= 3 - 1$ $= 2 \text{ units}^2$ ✓	3	Simpson's $\Rightarrow \frac{2}{3}$
(iii)	$\displaystyle\int_1^3 \log_e x\,dx =$  $-$  $\frac{1}{2}$ $= (3\times\log_e 3) - 2$ ✓ $=$	1	