DEUS MEUS ET OMNIA

St Clare's College
Waverley

Year 12
Mathematics Extension I
Examination
Term 1
2009

Time Allowed: 1.5 hours (plus 5 mins reading time)

Instructions: Attempt all questions.
Part marks are indicated.
Approved calculators may be used.
Write in blue or black pen only.
Show all necessary working.
Marks may be deducted for careless or poorly arranged work.

Question 1 **(Begin a new page)**

a) (i) Find the exact values of the gradients of the tangents to the curve $y = e^x$ at the points where $x = 0$ and $x = 1$. **2**

(ii) Find the acute angle between these tangents correct to the nearest degree. **2**

b) The interval AB has end points $A(-3,5)$ and $B(3,2)$. Find the coordinates of the point P which divides the interval AB externally in the ratio 2:5. **2**

c) Find the value of $\lim\limits_{x \to 0} \dfrac{\sin 2x}{5x}$. **2**

d) Find the exact value of $\cos 15°$. Express your answer as a single fraction with rational denominator. **2**

e) Prove the identity $\dfrac{1 - \tan^2 A}{1 + \tan^2 A} = \cos 2A$. **2**

Question 2 **(Begin a new page)**

a) (i) Sketch the graph of $y = x^3 - 4x$. 2

(ii) Hence or otherwise, solve $x^3 - 4x > 0$. 1

b) Find the value of x if $\frac{d}{dx}\left(\frac{x+2}{\sqrt{x-1}}\right) = 0$. 3

c) Solve $\frac{x}{x^2 - 4} \le 0$. 4

d) Evaluate $\int_0^{\ln 7} e^{-x}\,dx$ 2

Question 3 **(Begin a new page)**

a) Find $\frac{d^2}{dx^2} e^{x^2}$ 2

b) Evaluate $\log_2 7$ correct to two decimal places. 2

c) Consider the function $f(x) = 1 + \ln x$.

(i) Show that the function $f(x)$ is increasing and the curve $y = f(x)$ is concave down for all values of x in the domain of the function. 2

(ii) Sketch the curve. 1

(iii) Find the equation of the tangent to the curve $y = f(x)$ at the point on the curve where $x = 1$. 2

d) (i) Show that $\frac{1}{1-\tan x} - \frac{1}{1+\tan x} = \tan 2x$ 2

(ii) Evaluate $\frac{1}{1-\tan\frac{\pi}{6}} - \frac{1}{1+\tan\frac{\pi}{6}}$ 1

Question 4 **(Begin a new page)**

a) Use Mathematical Induction to show that for all positive integers $n \geq 1$
$4^n + 14$ is divisible by 6. 4

b) A solid is formed by rotating the curve $y = 1 + \sqrt{2}\cos x$ between $x = 0$ and $x = \frac{\pi}{4}$ about the x- axis.

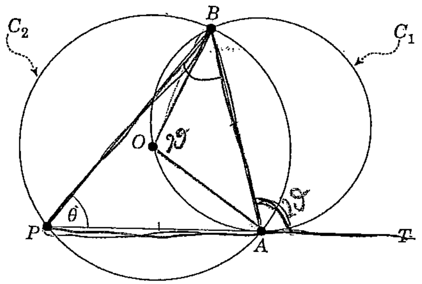
(i) Show that $2\cos^2 x = 1 + \cos 2x$. 1

(ii) Show the volume of the solid can be expressed as

$$V = \pi \int_0^{\frac{\pi}{4}} \left(2 + 2\sqrt{2}\cos x + \cos 2x\right) dx$$ 1

(iii) Hence find the exact volume of the solid. 2

c) Two circles, C_1 and C_2, intersect at points A and B. Circle C_1 passes through the centre O of circle C_2. The point P lies on circle C_2 so that the line PAT is tangent to circle C_1 at the point A.



Let $\angle APB = \theta$.

Copy or trace the diagram.

(i) Find $\angle AOB$ in terms of θ. Give a reason for your answer. 1

(ii) Explain why $\angle TAB = 2\theta$. 1

(iii) Deduce that $PA = BA$. 2

Question 5 **(Begin a new page)**

a) Two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.
The equation of the tangent to the parabola at P is $y = px - ap^2$.
The tangent at P and the line through Q parallel to the y axis intersect at T.

(i) Show this on a sketch. 1

(ii) Find the coordinates of T. 2

(iii) M is the midpoint of PT. Show that the coordinates of M are $(a(p+q), apq)$. 2

(iv) If PQ is a focal chord show that $pq = -1$ 1

(v) Find the locus of M. 1

b) For the polynomial $9x^4 - 25x^2 + 10kx - k^2$

(i) Determine the value of the constant k for which the polynomial is divisible by both $(x-1)$ and $(x+2)$. 3

(ii) With this value of k, find the zeros of the polynomial. 2

End of Exam

Year 12 Extension 1 Term 1 2009

Question 1

1) (i) $y = e^x$

$\frac{dy}{dx} = e^x$

at $x = 0$

$\frac{dy}{dx} = e^0$

$= 1$

$\therefore$ Gradient of tangent at $x = 0$, $m_1 = 1$ (1)

at $x = 1$

$\frac{dy}{dx} = e$

$\therefore$ Gradient of tangent at $x = 1$, $m_2 = e$ (1)

(ii) $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\tan\theta = \left|\frac{1 - e}{1 + 1.e}\right|$

$\tan\theta = \left|\frac{1-e}{1+e}\right|$ (1)

$\theta = \tan^{-1}\left|\frac{1-e}{1+e}\right|$

$\theta \doteqdot 25^\circ$ (1)

b) $A(-3,5)$ $B(3,2)$

$-2:5$

$x = \frac{-2(3)+5(-3)}{-2+5}$ $y = \frac{-2(2)+5(5)}{-2+5}$ (1)

$x = \frac{-6-15}{3}$ $y = \frac{-4+25}{3}$

$x = -7$ $y = 7$

$\therefore P(-7,7)$ (1)

c) $\lim_{x\to 0} \frac{\sin 2x}{5x} = \lim_{x \to 0} \frac{\sin 2x}{2x} \times \frac{2}{5}$ (1)

$= 1 \times \frac{2}{5}$

$= \frac{2}{5}$ (1)

d) $\cos 15^\circ = \cos(45^\circ - 30^\circ)$

$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$ (1)

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$ (1)

e) $LHS = \frac{1-\tan^2 A}{1+\tan^2 A}$

$= \frac{1-\tan^2 A}{\sec^2 A}$ (1)

$= \cos^2 A\,(1 - \tan^2 A)$

$= \cos^2 A - \cos^2 A \times \frac{\sin^2 A}{\cos^2 A}$

$= \cos^2 A - \sin^2 A$

$= \cos 2A$

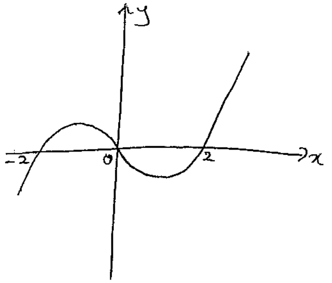
$= RHS$ (1)

Question 2

a) (i) $y = x^3 - 4x$

$y = x(x^2-4)$

$y = x\ (x+2)(x-2)$



(2)

(ii) $x^3 - 4x > 0$

$-2 < x < 0$ and $x > 2$ (1)

b) $$\frac{d}{dx}\left(\frac{x+2}{(x-1)^{\frac{1}{2}}}\right) = \frac{(x-1)^{\frac{1}{2}}.1 - (x+2).\frac{1}{2}(x-1)^{-\frac{1}{2}}}{(x-1)}$$ (1)

$$0 = \frac{\sqrt{x-1} - \frac{x+2}{2\sqrt{x-1}}}{x-1}$$ $(x \neq 1)$

$$\therefore \sqrt{x-1} - \frac{x+2}{2\sqrt{x-1}} = 0$$

$$\frac{2(x-1) - (x+2)}{2\sqrt{x-1}} = 0$$ (1)

$2x - 2 - x - 2 = 0$

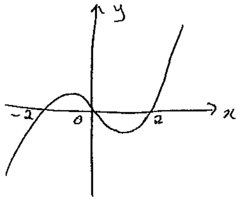
$x = 4$ (1)

c) $\frac{x}{x^2-4} \leq 0$ $x \neq \pm 2$

Multiply both sides by $(x^2-4)^2$ (1)

$x(x^2-4) \leq 0$

$x(x+2)(x-2) \leq 0$ (1)



$x < -2$ and $0 \leq x < 2$ (since $x \neq \pm 2$) (2)

d) $$\int_0^{\ln 7} e^{-x}\,dx = \left[-e^{-x}\right]_0^{\ln 7}$$

$= -e^{-\ln 7} - (-e^0)$

$= -e^{\ln 7^{-1}} + e^0$ (1)

$= -e^{\ln\frac{1}{7}} + 1$

$= -\frac{1}{7} + 1$

$= \frac{6}{7}$ (1)

Question 3

a) $\frac{d}{dx} e^{x^2} = 2x e^{x^2}$ (1)

$\frac{d^2}{dx^2} e^{x^2} = e^{x^2}.2 + 2x.2xe^{x^2}$

$= 2e^{x^2}(1 + 2x^2)$ (1)

b) $\log_2 7 = \frac{\log_e 7}{\log_e 2}$ (1)

$\doteqdot 2 \cdot 81$ (1)

c) $f(x) = 1 + \ln x$ domain $x > 0$

For function to be increasing, $f'(x) > 0$

$f'(x) = \frac{1}{x}$

$\therefore$ for $x > 0$, $\frac{1}{x} > 0$

$\therefore$ function is increasing (1)

For function to be concave down, $f''(x) < 0$

$f''(x) = -\frac{1}{x^2}$

$\therefore$ for $x > 0$, $-\frac{1}{x^2} < 0$

$\therefore$ $f''(x) < 0$

$\therefore$ function is concave down. (1)

"increasing" → gradient → $\frac{dy}{dx}$ → pos → $\frac{dy}{dx} > 0$

"decreasing" → gradient → $\frac{dy}{dx}$ → neg → $\frac{dy}{dx} < 0$

---

concave up → concavity → $\frac{d^2y}{dx^2}$ → pos → $\frac{d^2y}{dx^2} > 0$

---

concave down → concavity → $\frac{d^2y}{dx^2}$ → neg → $\frac{d^2y}{dx^2} < 0$.

) asymptote at y-axis

x-intercept: $1 + \ln x = 0$

$\ln x = -1$

$x = e^{-1}$

$x = \frac{1}{e}$

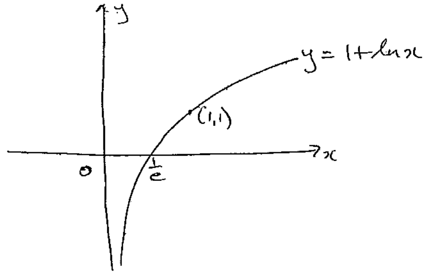
when $x = 1$

$f(1) = 1 + \ln 1$

$= 1 + 0$

$= 1$

$\therefore$ curve passes through $(1,1)$



(1)

(iii) $f(x) = 1 + \ln x$

$f'(x) = \frac{1}{x}$

when $x = 1$

$f'(1) = 1$

$\therefore$ m of tangent $= 1$ (1)

Eqn of tangent:

$y - 1 = 1(x - 1)$

$y - 1 = x - 1$

$y = x$ (1)

d) (i) LHS $= \dfrac{1}{1-\tan x} - \dfrac{1}{1+\tan x}$

$= \dfrac{1+\tan x - (1-\tan x)}{(1-\tan x)(1+\tan x)}$

$= \dfrac{1+\tan x - 1 + \tan x}{1-\tan^2 x}$ (1)

$= \dfrac{2\tan x}{1-\tan^2 x}$

$= \tan 2x$ (1)

$=$ RHS

(ii) $\dfrac{1}{1-\tan\frac{\pi}{6}} - \dfrac{1}{1+\tan\frac{\pi}{6}}$

$= \tan 2\left(\frac{\pi}{6}\right)$

$= \tan\frac{\pi}{3}$

$= \sqrt{3}$ (1)

Question 4

a) Show $4^n + 14$ is divisible by 6.

when $n = 1$

$4^1 + 14 = 4 + 14$
$= 18$

which is divisible by 6. (1)

Assume true for $n = k$

ie $\dfrac{4^k + 14}{6} = M$ (where M is an integer)

$4^k + 14 = 6M$

$4^k = 6M - 14$ (1)

Now prove true for $n = k+1$

ie Prove $4^{k+1} + 14$ is divisible by 6

$4^{k+1} + 14 = 4.4^k + 14$

$= 4(6M - 14) + 14$

$= 24M - 56 + 14$

$= 24M - 42$

$= 6(4M - 7)$ (1)

which is divisible by 6

∴ True for $n = 1$ and if true for $n = k$, then true for $n = k+1$

∴ True for $n = 1+1 = 2$, $n = 2+1 = 3$ etc

∴ $4^n + 14$ is divisible by 6 for all $n \geq 1$ (1)

1) (i) $\cos 2x = 2\cos^2 x - 1$

$\therefore 2\cos^2 x = 1 + \cos 2x$ (1)

(ii) $y = (1 + \sqrt{2}\cos x)$

$y^2 = (1 + \sqrt{2}\cos x)^2$

$y^2 = 1 + 2\sqrt{2}\cos x + 2\cos^2 x$

$y^2 = 1 + 2\sqrt{2}\cos x + 1 + \cos 2x$

$y^2 = 2 + 2\sqrt{2}\cos x + \cos 2x$

$\therefore V = \pi\int_0^{\frac{\pi}{4}}(2 + 2\sqrt{2}\cos x + \cos 2x)\,dx$ (1)

(iii) $V = \pi\left[2x + 2\sqrt{2}\sin x + \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$ (1)

$V = \pi\left\{\left[2\left(\frac{\pi}{4}\right) + 2\sqrt{2}\sin\frac{\pi}{4} + \frac{1}{2}\sin 2\left(\frac{\pi}{4}\right)\right] - \left[2(0) + 2\sqrt{2}\sin 0 + \frac{1}{2}\sin 0\right]\right.$

$V = \pi\left\{\left[\frac{\pi}{2} + 2\sqrt{2}.\frac{1}{\sqrt{2}} + \frac{1}{2}\right] - [0]\right\}$

$V = \pi\left\{\frac{\pi}{2} + 2 + \frac{1}{2}\right\}$

$V = \pi\left(\frac{\pi}{2} + \frac{4}{2} + \frac{1}{2}\right)$

$V = \frac{\pi}{2}(\pi + 5)$ units^3

2) (i) In C_2 $\angle AOB = 2\theta$ (Angle at centre is double angle at circumference subtended by the same arc). (1)

(ii) In C_1 $\angle TAB = 2\theta$ (Angle between tangent and chord is equal to the angle in the alternate segment). (1)

(iii) In $\triangle PBA$

$\angle BAT = \angle BPA + \angle PBA$ (exterior angle sum $\triangle$)

$2\theta = \theta + \angle PBA$ (1)

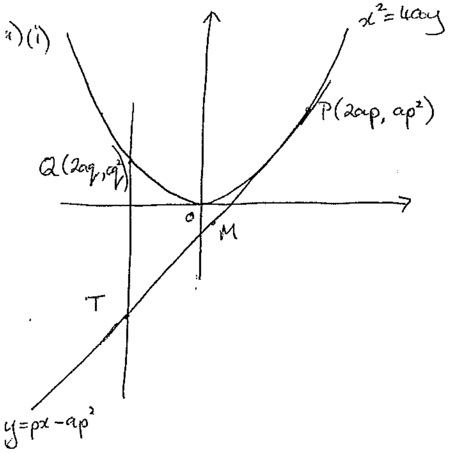
$\therefore \angle PBA = \theta$

$\therefore \triangle PBA$ is isosceles (base angles equal)

$\therefore PA = BA$ (1)

Question 5

a)(i)



(1)

ii) sub $x=2aq$ into

$y=px-ap^2$

$y=p(2aq)-ap^2$ (1)

$y=2apq-ap^2$

$\therefore T(2aq,\ 2apq-ap^2)$ (1)

iii) $M=\left(\dfrac{2ap+2aq}{2},\ \dfrac{ap^2+2apq-ap^2}{2}\right)$ (1)

$M=\left(\dfrac{2a(p+q)}{2},\ \dfrac{2apq}{2}\right)$

$M=\left(a(p+q),\ apq\right)$ (1)

(iv) m of PQ $=\dfrac{ap^2-aq^2}{2ap-2aq}$

$=\dfrac{a(p+q)(p-q)}{2a(p-q)}$

$=\frac{1}{2}(p+q)$

Eqn PQ

$y-ap^2=\frac{1}{2}(p+q)(x-2ap)$

$y-ap^2=\frac{1}{2}(p+q)x-ap(p+q)$

when $x=0$ $y=a$ (since it is a focal chord)

$a-ap^2=-ap^2-apq$

$a=-apq$

$pq=-1$ (1)

) $x=a(p+q)$

$y=apq$

but $pq=-1$

$\therefore y=-a$ (1)

This is independent of the parameters $\therefore$ $y=-a$ is the locus.

c) $P(x)=9x^4-25x^2+10kx-k^2$

$P(1)=9-25+10k-k^2=0$

$-16+10k-k^2=0$ ① (1)

$P(-2)=144-100-20k-k^2=0$

$44-20k-k^2=0$ ② (1)

$k^2=-16+10k$ ①

$k^2=44-20k$ ②

Sub ① in ②

$-16+10k=44-20k$

$30k=60$

$k=2$ (1)

) $(x-1)(x+2) = x^2 + x - 2$

$$\begin{array}{rl} & 9x^2 - 9x + 2 \\ \cline{2-2} x^2+x-2 \big) & 9x^4 - 25x^2 + 20x - 4 \end{array}$$

(1)

$\therefore\ P(x) = (x^2+x-2)(9x^2-9x+2)$

$= (x^2+x-2)(3x-2)(3x-1)$

$\therefore\ x = 1, -2, \frac{2}{3}, \frac{1}{3}$ (1)