SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12 ASSESSMENT TASK 3

JUNE 2002

MATHEMATICS

EXTENSION 1

Time Allowed: 70 minutes

Instructions:

* Attempt all questions
* Answers to be written on the paper provided.
* Start each question on a new page.
* All necessary working should be shown.
* Marks may not be awarded for careless or badly arranged working.
* This question paper must be stapled on top of your answers.
* Marks shown are for guidance and may be changed slightly if needed.
* Standard integrals are attached and may be removed for your convenience.

Name:____________________ Teacher:____________________

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Total
9	8	8	5	5	5	40

Question 1

a) Differentiate $y = \cos^{-1} 2x$ **1**

b) Find $\dfrac{d}{dx}(2^x)$ **1**

c) Find as an exact value $\sin^{-1}\dfrac{1}{2} + \cos^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$ **2**

d) Solve the equation $\ln(x+7) = 2\ln(x+1)$ **3**

e) (i) Sketch, without the use of calculus, the polynomial $P(x) = (x-1)^2(x+1)^3$ showing the x and y intercepts. **3**

(ii) Hence solve the inequation $P(x) \geq 0$

Question 2

a) Consider the function $f(x) = e^{x+2}$ **4**

(i) Find the inverse function $f^{-1}(x)$

(ii) Sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same number plane. Clearly label each graph and show the intercepts.

b) The polynomial $P(x) = x^3 + 2x^2 + ax + b$ has a factor of $(x+2)$ and when divided by $(x-2)$ there is a remainder of 12. Find the values of a and b. **4**

Question 3

a) Find $\dfrac{d}{dx}\log_e(\sin^{-1} x)$ **2**

b) Find $\displaystyle\int_0^{2/5} \frac{dx}{\sqrt{16-25x^2}}$ as an exact value **3**

c) Use the substitution $u = e^x$ to find the exact value of **4**

$$\int_0^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}}\,dx$$

Question 4

Consider the function $y = \log_e\left(\frac{2x}{2+x}\right)$ where $x < -2, x > 0$

a) Find the value of x for which $y = 0$. **1**

b) Show that $\frac{dy}{dx} = \frac{2}{x(2+x)}$ and hence state why the function is increasing for all x in the given domain. **2**

c) Are there any points of inflexion? Justify your answer. **2**

(You may use $\frac{d^2y}{dx^2} = \frac{1}{(2+x)^2} - \frac{1}{x^2}$)

d) Determine the equation of the horizontal asymptote. **1**

e) Sketch the graph of the function showing the features from (a) to (d) above. **2**

Question 5

a) (i) Find $\frac{d}{dx}(x\tan^{-1}x)$ **4**

(ii) Hence find the exact value of $\int_0^1 \tan^{-1}x \, dx$

b) Two of the zeros of the cubic polynomial $P(x) = 3x^3 - bx^2 - 27x + 9$ are reciprocals of each other, and two of the zeros of $P(x)$ are opposite in sign. **4**

(i) Find the value of b.

(ii) Factorise $P(x)$ completely.

Question 6

Consider the function $f(x) = \sin^{-1}(x - 1)$

(i) Evaluate $f(0)$ 1

(ii) State the domain and range of $y = f(x)$ 2

(iii) Draw the graph of $y = f(x)$ 1

(iv) The area bounded by the curve $y = f(x)$, the y axis and the line $y = \frac{\pi}{2}$ is rotated about the y axis. Find the volume of the solid formed. 4

Question 1 (10 marks)

a) $\dfrac{dy}{dx} = \dfrac{-2}{\sqrt{1-4x^2}}$ (1)

b) $\ln 2 . 2^x$ (1)

c) $\sin^{-1}\frac{1}{2} + \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$= \frac{\pi}{6} + \pi - \cos^{-1}\frac{\sqrt{3}}{2}$

$= \frac{\pi}{6} + \pi - \frac{\pi}{6}$

$= \pi$ (2)

d) $\ln(x+7) = 2\ln(x+1)$

$\ln(x+7) = \ln(x+1)^2$

$x+7 = x^2+2x+1$ (1)

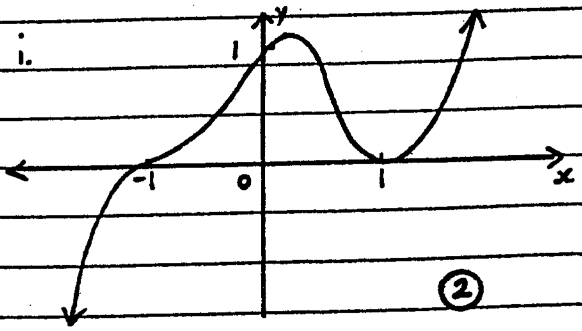
$0 = x^2+x-6$

$0 = (x+3)(x-2)$

$x = 2 \text{ or } -3$ (1)

$\therefore$ $x = 2$ only ($x=-3$ gives $\ln(-2)$ which is undefined) (1)

e) i.



(2)

ii. $x \geq -1$ (1)

Question 2 (10 marks)

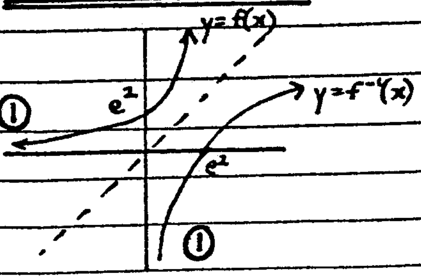
a) i. $y = e^{x+2}$

$x = e^{y+2}$ (1)

$\log_e x = y+2$

$y = \log_e x - 2$ (1)

ii.



(1) (1)

b) $P(x) = x^3 + 2x^2 + ax + b$

$(x+2)$ is a factor so $P(-2) = 0$ (1)

$-8 + 8 - 2a + b = 0$

$-2a + b = 0$ — (1)

also $P(2) = 12$ (1)

$8 + 8 + 2a + b = 12$

$2a + b = -4$ — (2)

solve simultaneously — (1) + (2)

$2b = -4$

$b = -2$ (1)

$2a - 2 = -4$

$a = -1$ (1)

$\therefore$ $a = -1$ and $b = -2$

Question 3 (9 marks)

a) $\dfrac{\frac{1}{\sqrt{1-x^2}}}{\sin^{-1}x} = \dfrac{1}{\sqrt{1-x^2}\,\sin^{-1}x}$ (1) (1)

b) $\displaystyle\int_0^{2/5} \frac{dx}{\sqrt{16-25x^2}}$

$= \displaystyle\int_0^{2/5} \frac{dx}{\sqrt{25\left(\frac{16}{25}-x^2\right)}}$ (1)

$= \displaystyle\frac{1}{5}\int_0^{2/5} \frac{dx}{\sqrt{\frac{16}{25}-x^2}}$

$= \frac{1}{5}\left[\sin^{-1}\frac{x}{4/5}\right]_0^{2/5}$

$= \frac{1}{5}\left[\sin^{-1}\frac{5x}{4}\right]_0^{2/5}$ (1)

$= \frac{1}{5}\left(\sin^{-1}\frac{1}{2} - \sin^{-1}0\right)$

$= \frac{1}{5}\left(\frac{\pi}{6} - 0\right)$

$= \frac{\pi}{30}$ (1)

c) $\displaystyle\int_0^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}}\,dx$

$u = e^x$

$\frac{du}{dx} = e^x$

$= \displaystyle\int_1^{\sqrt{3}} \frac{du}{1+u^2}$ (1)

$du = e^x\,dx$

$x=0,\ u=1$

$x=\ln\sqrt{3},\ u=\sqrt{3}$ (1)

$= \left[\tan^{-1} u\right]_1$

$= \tan^{-1}\sqrt{3} - \tan^{-1} 1$

$= \frac{\pi}{3} - \frac{\pi}{4}$

$= \frac{\pi}{12}$ ①

Question 4 (8 marks)

$y = \log_e\left(\frac{2x}{x+2}\right) \quad x < -2,\ x > 0$

a) $0 = \log_e\left(\frac{2x}{2+x}\right)$

$1 = \frac{2x}{2+x}$

$2 + x = 2x$

$\therefore x = 2$ ①

b) $y = \log_e 2x - \log_e(2+x)$

$\frac{dy}{dx} = \frac{2}{2x} - \frac{1}{2+x}$

$= \frac{1}{x} - \frac{1}{2+x}$ ①

$= \frac{2+x-x}{x(2+x)}$

$\therefore \frac{dy}{dx} = \frac{2}{x(2+x)}$

$\frac{2}{x(2+x)} > 0$ for all x in the given domain

$\therefore$ function is increasing for all x in the given domain as $\frac{dy}{dx} > 0$ ①

c) Inflexions : $\frac{d^2y}{dx^2} = 0$ & concavity changes

$0 = \frac{1}{(2+x)^2} - \frac{1}{x^2}$

$\frac{1}{x^2} = \frac{1}{(2+x)^2}$

$x^2 = x^2 + 4x + 4$

$x = -1$ (out of domain) ①

$\therefore$ no points of inflexion ①

d) $\lim_{x\to\infty}\left[\log_e \frac{2x}{2+x}\right]$

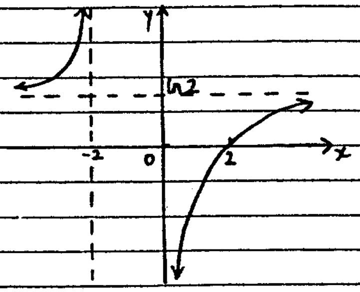
$= \lim_{x\to\infty}\left[\log_e \frac{\frac{2x}{x}}{\frac{2}{x} + \frac{x}{x}}\right]$

$= \lim_{x\to\infty}\left[\log_e \frac{2}{\frac{2}{x}+1}\right]$

$= \log_e 2$

$\therefore y = \log_e 2$ is horizontal asymptote ①

e)



②

Question 5 (8 marks)

a) i. $u = x \quad v = \tan^{-1} x$

$u' = 1 \quad v' = \frac{1}{1+x^2}$

$\frac{d}{dx}(x\tan^{-1}x) = \tan^{-1}x + \frac{x}{1+x^2}$ ①

ii. $\tan^{-1}x = \frac{d}{dx}(x\tan^{-1}x) - \frac{x}{1+x^2}$

$\int_0^1 \tan^{-1}x\,dx = \int_0^1 \frac{d}{dx}(x\tan^{-1}x)\,dx - \int_0^1 \frac{x}{1+x^2}\,dx$ ①

$= \left[x\tan^{-1}x - \frac{1}{2}\ln(1+x^2)\right]_0^1$ ①

$= \left(\tan^{-1}1 - \frac{1}{2}\ln 2\right) - 0$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$ ①

b) $P(x) = 3x^3 - bx^2 - 27x + 9$

let the roots be $\alpha, \frac{1}{\alpha}, -\alpha$

i. sum of roots :

$\alpha + \frac{1}{\alpha} - \alpha = \frac{b}{3}$

$\frac{1}{\alpha} = \frac{b}{3}$

$b = \frac{3}{\alpha}$ ①

product of roots :

$\alpha \cdot \frac{1}{\alpha} \cdot -\alpha = -\frac{9}{3}$

$-\alpha = -3$

$\alpha = 3$

$\therefore b = 1$ ①

ii. the roots are $3, \frac{1}{3}, -3$ ①

$\therefore P(x) = (x+3)(x-3)(3x-1)$ ①

Question 6 (8 marks)

$f(x) = \sin^{-1}(x-1)$

i. $f(0) = \sin^{-1}(-1)$

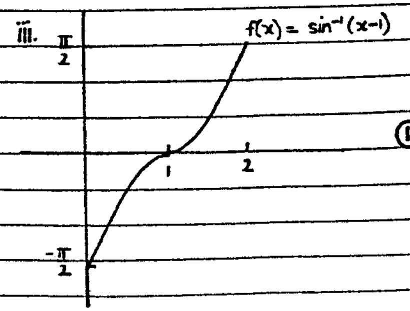
$= -\frac{\pi}{2}$ ①

ii. $-1 \le x-1 \le 1$

$D: 0 \le x \le 2$ ①

$R: -\frac{\pi}{2} \le y \le \frac{\pi}{2}$ ①

iii.



①

iv. $y = \sin^{-1}(x-1)$

$\sin y = x - 1$

$x = \sin y + 1$

$V = \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin y + 1)^2\,dy$ ①

$= \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 y + 2\sin y + 1\,dy$ *

* using $\cos 2A = 1 - 2\sin^2 A$

$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

$= \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} - \frac{1}{2}\cos 2y + 2\sin y + 1\,dy$ ①

$= \pi\left[\frac{y}{2} - \frac{1}{4}\sin 2y - 2\cos y + y\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$ ①

$= \pi\left[\left(\frac{\pi}{4} - \frac{1}{4}\sin\pi - 2\cos\frac{\pi}{2} + \frac{\pi}{2}\right) - \left(-\frac{\pi}{4} - \frac{1}{4}\sin(-\pi) - 2\cos\left(-\frac{\pi}{2}\right) - \frac{\pi}{2}\right)\right]$

$= \pi\left[\frac{3\pi}{2}\right]$

$= \frac{3\pi^2}{2}$ units^3 ①