St George Girls High School

Trial Higher School Certificate Examination

2009

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Begin each question on a new booklet
- Write your student number on each page
- All necessary working must be shown.
- Diagrams are not to scale.
- Board-approved calculators may be used.
- The mark allocated for each question is listed at the side of the question.
- Standard Integrals sheet may be detached.

Total Marks – 84

- Attempt ALL questions.
- All questions are of equal value.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1 – (12 marks) – Start a New Booklet

Marks

a) Differentiate $\frac{1}{\sqrt{x^2+x}}$ 2

b) Solve $\frac{1}{x-1} \leq 3$ 3

c) Differentiate $\ln(e^x \cos x)$ 2

d) Find $\frac{d}{dx}\left(3\sin^{-1}\frac{x}{2}\right)$ 2

e) Use the substitution $u = 1 + 2x$ to evaluate $\int_0^1 \frac{x}{1+2x}\,dx$ 3

Question 2 – (12 marks) – Start a New Booklet

Marks

a) A is the point $(-4, 2)$ and B is the point $(3, -1)$. Find the coordinates of the point P which divides the interval AB externally in the ratio $2:1$ 2

b) (i) Find the gradient of the tangent to the curve $y = x^2 + 3$ at the point $(1, 4)$. 1

(ii) Find the acute angle between the line $y = 3x + 1$ and the curve $y = x^2 + 3$ at the point of intersection $(1, 4)$. Give your answer to the nearest degree. 2

c) Consider the function $f(x) = 4\cos^{-1}(\sqrt{3}x)$

(i) Evaluate $f\left(-\frac{1}{2}\right)$ 1

(ii) State the domain and range of $f(x)$ 2

(iii) Draw a neat labelled sketch of $f(x)$ 2

d) Prove $\frac{2}{\tan A + \cot A} = \sin 2A$ 2

Question 3 – (12 marks) – Start a New Booklet

Marks

a) If α, β and γ are the roots of $2x^3 + 4x^2 - 3x + 1 = 0$, find the values of:

(i) $\alpha + \beta + \gamma$ 1

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma$ 1

(iii) $\alpha^2 + \beta^2 + \gamma^2$ 2

b) (i) Write down the expansion of $\tan(A + B)$ 1

(ii) Find the exact value of $\tan\left(\frac{7\pi}{12}\right)$ in simplest form. 3

c) Prove by induction that 4

$$\frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \cdots - \frac{1}{x^n} = \frac{1}{x^n(x-1)}$$

for all positive integers n, $x \neq 0,\ 1$

Question 4 – (12 marks) – Start a New Booklet

Marks

a) For the curve $y = (x-2)^2 - 1$

(i) Find the largest positive domain such that the graph defines a function $f(x)$ which has an inverse. 1

(ii) Write down the inverse function $f^{-1}(x)$ and state its domain. 2

(iii) State a domain for which $f(x)$ does not have an inverse that is a function. Give a brief reason for your answer. 2

b) Calculate the volume of the solid formed when the area bounded by $y = \frac{3}{\sqrt{x+5}}$, the x-axis, $x = -4$ and $x = 2$ is rotated about the x-axis. 3

c) Chris borrows \$50 000 to pay for a new car. She plans to repay the loan by making 60 equal monthly instalments. Interest is charged at the rate of 0.6% per month on the balance owing.

(i) Show that immediately after making two monthly instalments of $\$M$ the balance owing is given by $\$(50\ 601.80 - 2.006M)$. 2

(ii) Calculate the value of each monthly instalment. 2

Question 5 – (12 marks) – Start a New Booklet

Marks

a) (i) Show that $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d^2x}{dt^2}$ 2

(ii) Show that $\frac{d}{dx}(x\log_e x) = 1 + \log_e x$ 1

(iii) The acceleration of a particle moving in a straight line and starting from rest, 1 unit from the origin is given by 3

$$\frac{d^2x}{dt^2} = 1 + \log_e x$$

Calculate the velocity v when displacement is $x = e^2$.

b) The equation $\sin x = 1 - \frac{x}{2}$ has a root near $x = 0.6$. Use one application of Newton's method to find another approximation correct to 2 decimal places. 3

c) A spherical mothball evaporates such that its volume $V\text{cm}^3$ and radius r cm after t weeks are related by the equation

$$\frac{dV}{dt} = -4k\pi r^2, \text{ where } k \text{ is a positive constant.}$$

(i) Show that $\frac{dr}{dt} = -k$ 1

(ii) If the initial radius of the mothball is 1m and the radius is $\frac{1}{2}$ cm after 10 weeks, express r in terms of t. 2

Question 6 – (12 marks) – Start a New Booklet

Marks

a) A particle's displacement x centimetres from the Origin O at time t seconds is given by:

$$x = 5\sin\left(3t + \frac{\pi}{6}\right)$$

(i) Show that the motion of this particle is Simple Harmonic i.e. satisfies the condition $\ddot{x} = -n^2x$. 2

(ii) Write down the period of the motion. 1

(iii) Find the maximum speed of the particle and the time taken to first reach this speed. 3

b) The rate of change of the population P of Kogarah is proportional to $(P - A)$

$$\text{i.e.} \quad \frac{dP}{dt} = k(P - A)$$

(i) Show that $P = A + Ce^{kt}$ satisfies the condition, where A, C and k are constants. 1

(ii) Initially the population is 10 000. 3

A is the <u>excess</u> of the initial population over 8000.

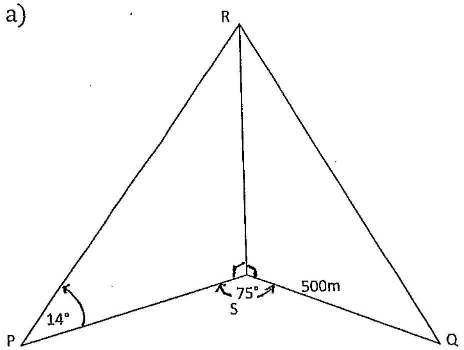
After 10 years the population is 20 000. Find the population 5 years later.

(iii) Find when the population reaches 50 000. 2

Question 7 – (12 marks) – Start a New Booklet

Marks

a)



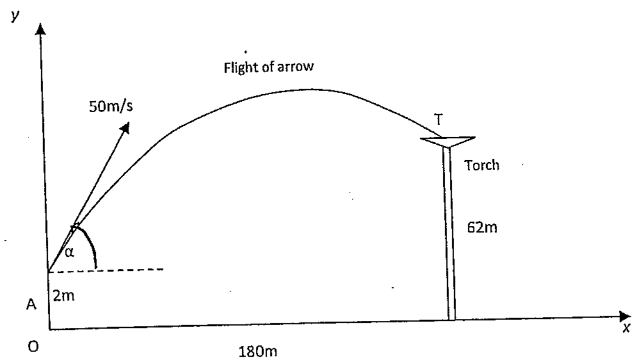
Two houses P and Q lie in the same plane as S, the foot of a hill RS. 3

The height of the hill is known to be 200m and from P the angle of elevation of the top of the hill is 14°.

If PQ subtends an angle of 75° at the foot of the hill S and if Q is 500m from S, how far apart are the two houses?

b) Find the constant term in the expansion of $\left(2x^2 - \frac{1}{x}\right)^9$ 3

c)



An archer fires an arrow at 50m/s at an angle α to the horizontal from a height of 2 metres above the ground. The archer is 180m from a 62m high Olympic torch. He aims to land the arrow in the top of the torch as shown in the diagram. The acceleration due to gravity can be assumed to be 10m/s/s.

(i) Given that $\frac{d^2x}{dt^2} = 0$ and $\frac{d^2y}{dt^2} = -10$, show that $x = 50t\cos\alpha$ and $y = 2 + 50t\sin\alpha - 5t^2$ are the horizontal and vertical displacements of the arrow in metres from A at time t seconds after firing. 2

(ii) Assuming $\alpha > 45°$ and the arrow reaches its maximum height 4 seconds after its release, find the angle of projection, α, and the time for the arrow to reach the top, T, of the torch. 4

End of Paper

Extension I Mathematics Year 12 Trial

Q1

a) $$\frac{d}{dx}\left(\frac{1}{\sqrt{x^2+x}}\right) = \frac{d}{dx}(x^2+x)^{-1/2}$$
$$= -\tfrac{1}{2}(x^2+x)^{-3/2}.(2x+1)$$
$$= \frac{-(2x+1)}{2(x^2+x)\sqrt{x^2+x}}$$

b) $$(x-1)^2 \times \frac{1}{x-1} \le 3 \times (x-1)^2$$
$$x-1 \le 3x^2-6x+3$$
$$0 \le 3x^2-7x+4$$
$$0 \le (3x-4)(x-1)$$

OR

$$x-1 \le 3(x-1)^2$$
$$0 \le 3(x-1)^2-(x-1)$$
$$0 \le (x-1)[3x-3-1]$$
$$0 \le (x-1)(3x-4)$$

$\therefore$ $x \le 1$ OR $x \geq 1\frac{1}{3}$, But $x \neq 1$

$\therefore$ Solution is $x < 1$ OR $x \geq 1\frac{1}{3}$

c) $$\frac{d}{dx}\ln(e^x\cos x) = \frac{d}{dx}(\ln e^x) + \frac{d}{dx}\ln(\cos x)$$
$$= \frac{d}{dx}x + \frac{1}{\cos x}.-\sin x$$
$$= 1 - \tan x$$

d) $$\frac{d}{dx}\left(3\sin^{-1}\frac{x}{2}\right) = \frac{3}{\sqrt{1-\left(\frac{x}{2}\right)^2}}.\frac{1}{2} \quad \text{OR} \quad 3.\frac{1}{\sqrt{4-x^2}}$$
(from standard Integrals sheet)
$$= \frac{3}{2\sqrt{1-\frac{x^2}{4}}}$$
$$= \frac{3}{2\times\frac{1}{2}\sqrt{4-x^2}}$$
$$= \frac{3}{\sqrt{4-x^2}}$$

e) $$\int_0^1 \frac{x}{1+2x}\,dx = \frac{1}{2}\int_1^3 \frac{(u-1)}{u}.\tfrac{1}{2}\,du$$

Let $u = 1+2x \Rightarrow x = \frac{u-1}{2}$

$du = 2\,dx$

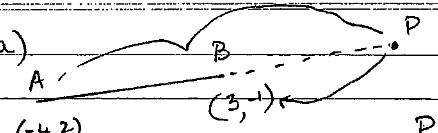
$\frac{1}{2}du = dx$

when $x=0$, $u=1$

when $x=1$, $u=3$

$$= \frac{1}{4}\int_1^3\left(1-\frac{1}{u}\right)du$$
$$= \frac{1}{4}\Big[u-\ln u\Big]_1^3$$
$$= \frac{1}{4}\Big[(3-\ln 3)-(1-\ln 1)\Big]$$
$$= \frac{1}{4}\Big[2-\ln 3\Big] \quad , \ln 1 = 0$$

Q2. a)



$m : n = 2 : -1$

$$P\left(\frac{-1\times{}^{-}4 + 2\times 3}{2+-1}, \frac{-1\times 2 + 2\times -1}{2+-1}\right)$$

$$\text{ie } P\left(\frac{4+6}{1}, \frac{-2-2}{1}\right)$$

$$P(10, -4)$$

b) (i) $y = x^2 + 3$

$Dy = 2x \quad (= 2 \text{ when } x = 1)$

ie gradient of tangent at $(1,4)$ is 2.

(ii) $m_1 = 3$ (= gradient of line $y = 3x+1$)

$m_2 = 2$ (= gradient tangent at $(1,4)$ on $y = x^2+3$)

Let θ be angle between curves.

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{3-2}{1+3\times 2}\right|$$

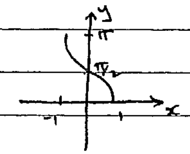
$$= \left|\frac{1}{1+6}\right|$$

$$= \frac{1}{7} \qquad \therefore \theta = 8\cdot130\ldots^\circ$$

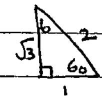
ie $\theta \doteqdot 8^\circ$ (nearest degree)

c) $f(x) = 4\cos^{-1}(\sqrt{3}\,x)$

(i)



$$f\left(-\tfrac{1}{2}\right) = 4\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$



$$= 4 \times \left(\pi - \frac{\pi}{6}\right)$$

$$= \frac{10\pi}{3}$$

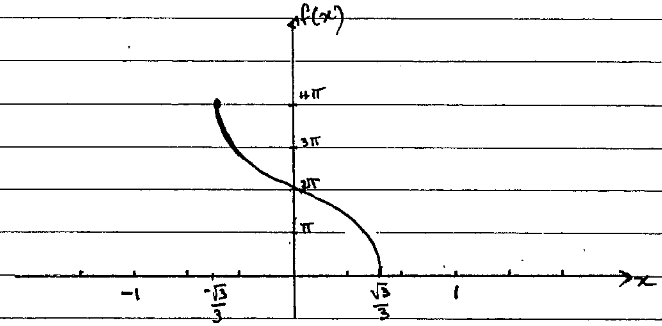
(ii) Domain $= \{x : -1 \le \sqrt{3}x \le 1\}$

$$= \left\{x : -\frac{1}{\sqrt{3}} \le x \le \frac{1}{\sqrt{3}}\right\}$$

Range $= \left\{y : 0 \le \frac{y}{4} \le \pi\right\}$

$$= \{y : 0 \le y \le 4\pi\}$$

(iii)



d) LHS $= \dfrac{2}{\tan A + \cot A}$

$$= \frac{2}{\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}} \times \frac{\sin A\cos A}{\sin A\cos A}$$

$$= \frac{2\sin A\cos A}{\sin^2 A + \cos^2 A}$$

$$= \frac{2\sin A\cos A}{1}, \qquad \sin^2 A + \cos^2 A = 1$$

$$= \sin 2A$$

$= \text{RHS}$ $\qquad \therefore \dfrac{2}{\tan A + \cot A} = \sin 2A$

Q3. a) $P(x) = 2x^3 + 4x^2 - 3x + 1 = 0$

(i) $\alpha + \beta + \gamma = \frac{-b}{a} = -\frac{4}{2}$

$= -2$

(ii) $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

$= \frac{-3}{2}$

(iii) $(\overline{\alpha + \beta} + \gamma)^2 = (\alpha+\beta)^2 + 2(\alpha+\beta)\gamma + \gamma^2$

$= \alpha^2 + \beta^2 + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma + \gamma^2$

$= \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$\therefore (-2)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2\left(-\frac{3}{2}\right)$

$4 = \alpha^2 + \beta^2 + \gamma^2 - 3$

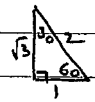
$\therefore 7 = \alpha^2 + \beta^2 + \gamma^2$

b) (i) $\operatorname{Tan}(A+B) = \frac{\operatorname{Tan} A + \operatorname{Tan} B}{1 - \operatorname{Tan} A \operatorname{Tan} B}$

(ii) Let $A = \frac{3\pi}{12} = \frac{\pi}{4}$, $B = \frac{4\pi}{12} = \frac{\pi}{3}$, Thus $A+B = \frac{7\pi}{12}$

Thus, $\operatorname{Tan}\frac{7\pi}{12} = \operatorname{Tan}\left(\frac{\pi}{4} + \frac{\pi}{3}\right) = \frac{\operatorname{Tan}\frac{\pi}{4} + \operatorname{Tan}\frac{\pi}{3}}{1 - \operatorname{Tan}\frac{\pi}{4} . \operatorname{Tan}\frac{\pi}{3}}$

$= \frac{1 + \sqrt{3}}{1 - 1 \times \sqrt{3}}$



full marks to here. $\rightarrow$ $= \frac{1+\sqrt{3}}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}}$

$= \frac{1 + 2\sqrt{3} + 3}{1 - 3}$

$= \frac{4 + 2\sqrt{3}}{-2}$

$= -2 - \sqrt{3}$

c) To prove: $\frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \dots - \frac{1}{x^n} = \frac{1}{x^n(x-1)}$, n +ve integer and $x \neq 0, 1$

Test Proposition,

For $n=1$, LHS $= \frac{1}{x-1} - \frac{1}{x}$ $\qquad$ RHS $= \frac{1}{x^1(x-1)}$

$= \frac{x - (x-1)}{x(x-1)}$ $\qquad$ $= \frac{1}{x(x-1)}$

$= \frac{1}{x(x-1)}$ $\qquad$ $\therefore$ LHS = RHS.

ie Proposition is true for $n=1$

Assume proposition is true for $n=k$

ie $\frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \dots - \frac{1}{x^k} = \frac{1}{x^k(x-1)}$

Thus we need to prove that the proposition is true for $n = k+1$,

(ie $\frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \dots - \frac{1}{x^k} - \frac{1}{x^{k+1}} = \frac{1}{x^{k+1}(x-1)}$)

Now, $\frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \dots - \frac{1}{x^k} - \frac{1}{x^{k+1}} = \frac{1}{x^k(x-1)} - \frac{1}{x^{k+1}}$

$= \frac{x^{k+1} - x^k(x-1)}{x^k(x-1)(x^{k+1})}$

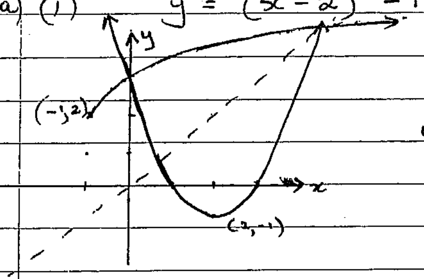
$= \frac{x^{k+1} - x^{k+1} + x^k}{x^k(x-1)(x^{k+1})}$

$= \frac{x^k}{x^k(x-1)x^{k+1}}$

$= \frac{1}{x^{k+1}(x-1)}$

Since proposition is true for $n=k$ and is proven true for $n=k+1$ and since it is true for $n=1$ then the proposition is true for all n, positive integers

Q4 a) (i) $y=(x-2)^2-1$



Largest domain positive for which $f(x)$ has an inverse is

$\{x: x \geq 2\}$ 1 /1

(ii) $y+1=(x-2)^2 \Rightarrow x+1=(y-2)^2$ $\quad x=(y-2)^2-1$

$\pm\sqrt{x+1} = y-2$ $\quad \pm\sqrt{x+1}=y-2$

$2\pm\sqrt{x+1}=y$ $\quad y=2\pm\sqrt{x+1}$ 1 /2

ie $f^{-1}(x)=2+\sqrt{x+1}$, Domain $=\{x: x\geq -1\}$ 1

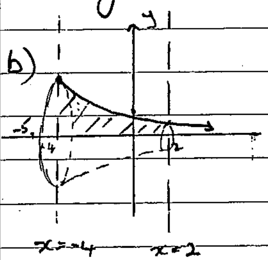
(iii)

A domain for which $f(x)$ does not have an inverse could be:

$\{x: 1 \leq x \leq 3\}$ 1

For this domain the function $y=(x-2)^2-1$ is decreasing and increasing within the domain thus on reflection about $y=x$ it will not be single valued for y for every x and thus can not be a function. 1 /2

b)



$V=\pi\int_{-4}^{2} \frac{9}{(\sqrt{x+5})^2}\,dx$ 1

$=9\pi\int_{-4}^{2}\frac{1}{x+5}\,dx$

$=9\pi\Big[\ln(x+5)\Big]_{-4}^{2}$ 1 /3

$=9\pi\Big[\ln 7-\ln 1\Big]$

Volume $=9\pi\ln 7$ units^3 or 55.0 units^3 (1 d.p.) 1

c) Let A_n be amount owed at end of each month

(i) $A_0 = \$50\,000$

$A_1 = 50\,000\left(1+\frac{0.6}{100}\right) - M$

$=50\,000(1.006)-M$ 1

$A_2=\Big[50000(1.006)-M\Big]1.006-M$

$=50000\times1.006^2 - 1.006M - M$ /2

$=50000\times1.006^2 - M(1.006+1)$

ie Balance owing after two monthly instalments is $\$(50\,601.80 - 2.006M)$ 1

(ii) Now $A_3=\Big[50000\times1.006^2-1.006M-M\Big]1.006-M$

$=50000\times1.006^3-1.006^2M-1.006M-M$

$=50000\times1.006^3-M(1+1.006+1.006^2$

. . . .

ie $A_n=50000\times1.006^n-M(1+1.006+1.006^2+\dots 1.006^{n-1})$

This is an G.P. since 1

Thus $A_n=50000\times1.006^n-M\left[\frac{1(1-1.006^n)}{1-1.006}\right]$ $\quad \left.\begin{array}{l}\frac{T_2}{T_1}=\frac{1.006}{1}=1.006\\ \frac{T_3}{T_2}=\frac{1.006^2}{1.006}=1.006\end{array}\right\}$ ie common ratio of $r=1.006$, $a=1$

When $A_n=0$, loan is repaid and this is when $n=60$

Thus, $50000\times1.006^{60}-M\left(\frac{1-1.006^{60}}{1-1.006}\right)=0$ /2

$\frac{(1-1.006)\,50000\times1.006^{60}}{(1-1.006^{60})}=M$ 1

$994.78474\ldots=M$

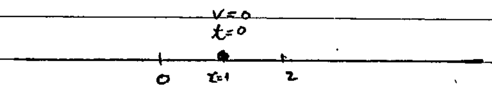
ie monthly repayment is \$994.78

Q5 a) (i)

$$\frac{d^2x}{dt^2} = \frac{d\left(\frac{dx}{dt}\right)}{dt}$$
$$= \frac{dv}{dt}$$
$$= \frac{dv}{dx} \cdot \frac{dx}{dt}$$
$$= \frac{dx}{dt} \cdot \frac{dv}{dx}$$
$$= v \cdot \frac{dv}{dx}$$
$$= \frac{d(\frac{1}{2}v^2)}{dv} \cdot \frac{dv}{dx}$$
$$= \frac{d(\frac{1}{2}v^2)}{dx} \quad \text{ie} \quad \frac{d^2x}{dt^2} = \frac{d(\frac{1}{2}v^2)}{dx}$$

(ii) $$\frac{d(x \cdot \log_e x)}{dx} = \log_e x \cdot 1 + x \cdot \frac{1}{x}$$
$$= \log_e x + 1$$

(iii)



$$\frac{d^2x}{dt^2} = 1 + \log_e x$$
$$\text{ie} \quad \frac{d(\frac{1}{2}v^2)}{dx} = 1 + \log_e x$$
$$\frac{1}{2}v^2 = \int (1 + \log_e x)\, dx$$
$$\frac{1}{2}v^2 = x \log_e x + C$$

when $x = 1$, $v = 0$,

$$0 = 1 \log_e 1 + C \Rightarrow C = 0$$
$$\text{ie} \quad \frac{1}{2}v^2 = x \log_e x$$
$$v^2 = 2x \log_e x$$

Thus when $x = e^2$, $v^2 = 2e^2 \log_e e^2 = 4e^2 \log_e e$

$\therefore$ velocity $= \pm 2e$ units/sec.

b) $$\sin x = 1 - \frac{x}{2}$$

$$\text{ie} \quad \sin x - 1 + \frac{x}{2} = 0 \Rightarrow \begin{cases} y = 0 \\ y = \sin x - 1 + \frac{x}{2} \end{cases}$$

$$\begin{cases} f(x) = \sin x - 1 + \frac{x}{2} \\ f'(x) = \cos x + \frac{1}{2} \end{cases}$$

$$\begin{cases} f(0.6) = \sin 0.6 - 1 + 0.3 \\ = -0.135\ldots \end{cases}$$

$$f'(0.6) = \cos 0.6 + \frac{1}{2}$$
$$= 1.32533\ldots$$

$\therefore$ New (better approximation) $= 0.6 - \dfrac{f(0.6)}{f'(0.6)}$

$$= 0.70213\ldots$$
$$\doteqdot 0.702 \text{ (3 d.pl.)}$$

c) $$\frac{dV}{dt} = -4k\pi r^2, \quad k > 0$$

(i) $$\frac{dr}{dt} = \frac{dr}{dV} \cdot \frac{dV}{dt} \quad \text{where} \quad V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dr} = 4\pi r^2$$
$$= \frac{1}{4\pi r^2} \times -4k\pi r^2$$
$$= -k$$

(ii)

$$\therefore r = -k \int 1\, dt$$
$$r = -kt + A$$

when $t = 0$, $r = 100$ $\therefore 100 = A$

ie $r = -kt + 100$

when $t = 10$, $r = \frac{1}{2}$, $\therefore \frac{1}{2} = -10k + 100$

$-99.5 = -10k \Rightarrow k = 9.95$

Thus, $r = 100 - 9.95t$

Q6 (i) SHM is defined by acceleration is proportional to the displacement at any time and directed towards the centre of the motion

ie $\ddot{x} \propto -x$

ie $\ddot{x} = -n^2 x$

Now for $x = 5\sin(3t + \frac{\pi}{6})$

$v = \frac{dx}{dt} = 5\cos(3t + \frac{\pi}{6}) . 3$

$= 15\cos(3t + \frac{\pi}{6})$

$\ddot{x} = \frac{d^2x}{dt^2} = 15\left[-\sin(3t + \frac{\pi}{6})\right].3$

$= -45\sin(3t + \frac{\pi}{6})$

$= -9\left[5\sin(3t + \frac{\pi}{6})\right]$

$\ddot{x} = -9x$

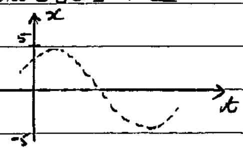
ie in the form of SHM.

(ii) Period $= \frac{2\pi}{n}$

$= \frac{2\pi}{3}$ seconds

(iii) Max speed occurs as particle passes the centre point of the motion.

ie when $x = 0$ in this case



ie $5\sin(3t + \frac{\pi}{6}) = 0$

$\sin(3t + \frac{\pi}{6}) = 0$

$3t + \frac{\pi}{6} = 0$ or π or 2π or 3π ....

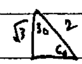
$3t = -\frac{\pi}{6}$ or $\frac{5\pi}{6}$ or ....

$t = -\frac{\pi}{18}$ or $\frac{5\pi}{18}$ or ...

ie time taken to first reach max. velocity is $\frac{5\pi}{18}$ secs

at $t = \frac{5\pi}{18}$, vel $= 15\cos(3 \times \frac{5\pi}{18} + \frac{\pi}{6})$

$= 15\cos\pi = -15$ cm/s



ie Max speed 15 cm/sec

OR ALTERNATIVELY: Maximum velocity occurs when $\ddot{x} = 0$

ie $-45\sin(3t + \frac{\pi}{6}) = 0$ (solution follows)

b) $\frac{dP}{dt} \propto P - A$ ie $\frac{dP}{dt} = k(P - A)$

(i) If $P = A + Ce^{kt}$

$\frac{dP}{dt} = Ce^{kt} . k$

$= k(P - A)$

ie $P = A + Ce^{kt}$ is a solution of $\frac{dP}{dt} = k(P - A)$

(ii) $A = 10000 - 8000$

$= 2000$

$\therefore P = 2000 + Ce^{kt}$

when $t = 0$, $10000 = 2000 + Ce^{0}$

$\therefore C = 8000$

$\therefore P = 2000 + 8000e^{kt}$

when $t = 10$, $P = 20000$

$20000 = 2000 + 8000e^{10k}$

$18000 = 8000e^{10k}$

$\frac{18000}{8000} = e^{10k}$

$\frac{9}{4} = e^{10k}$

$\ln\frac{9}{4} = 10k$

$\therefore k = \frac{1}{10}\ln 2.25$

when $t = 15$,

$P = 2000 + 8000e^{\frac{t}{10}\ln 2.25}$

$P = 2000 + 8000e^{\frac{15}{10}\ln 2.25}$

$P = 29000$

ie after 15 years the population is 29000

(iii) When $P = 50000$, $50000 = 2000 + 8000e^{\frac{t}{10}\ln 2.25}$

$\frac{48000}{8000} = e^{\frac{t}{10}\ln 2.25}$

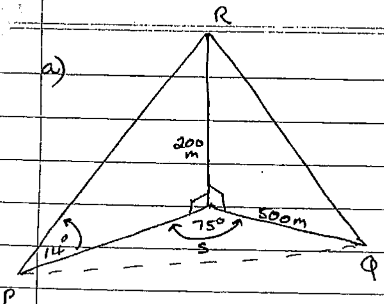
$6 = e^{\frac{t\ln 2.25}{10}}$

$10\ln 6 = t\ln 2.25$

$\frac{10\ln 6}{\ln 2.25} = t$

$22.095 = t$ ie in the 23rd year

11. a)



In $\triangle RPS$,

$$\tan 14^\circ = \frac{200}{PS}$$

$$PS = \frac{200}{\tan 14^\circ} \qquad 1$$

$\therefore$ In $\triangle PSQ$,

$$PQ^2 = \left(\frac{200}{\tan 14^\circ}\right)^2 + 500^2 - 2\left(\frac{200}{\tan 14^\circ}\right).500 \cos 75^\circ \qquad 1$$

$$PQ^2 = 685841.25$$

$$PQ = 828. \qquad 1$$

b) $\left(2x^2 - \frac{1}{x}\right)^9$ has a general term

$$T_{k+1} = {}^9C_k (2x^2)^{9-k}(-1)^k\left(\frac{1}{x}\right)^k$$

$$= {}^9C_k\, 2^{9-k}\, x^{18-2k}\, (-1)^k \frac{1}{x^k}$$

$$= {}^9C_k\, 2^{9-k}.(-1)^k\, x^{18-3k}$$

This term is constant when $18-3k=0$

ie $k=6$

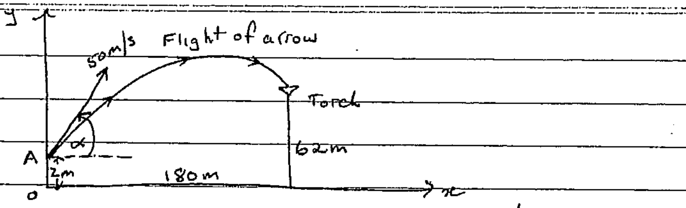
$$\therefore T_7 = {}^9C_6\, 2^3.(-1)^6$$

$$= \frac{9\times8\times7}{1\times2\times3} \times 2^3$$

$$= 672$$

1

c)



Horizontal

(i) $\ddot{x} = 0$

$\dot{x} = V\cos\alpha$

$= 50\cos\alpha$

$x = 50t\cos\alpha + B$

when $x=0$, $t=0 \therefore B=0$

$x = 50t\cos\alpha$

1

Vertical

$\ddot{y} = -g = -10$

$\dot{y} = -10t + A$

when $t=0$, $\dot{y} = V\sin\alpha = 50\sin\alpha$

ie $50\sin\alpha = A$

$\therefore \dot{y} = -10t + 50\sin\alpha$

$y = -5t^2 + 50t\sin\alpha + C$

when $t=0$, $y=2 \therefore C=2$

ie $y = -5t^2 + 50t\sin\alpha + 2$

1

(ii) If arrow reaches its maximum height when $t=4$ secs and this occurs when $\frac{dy}{dt} = 0$ 1

ie $-10\times4 + 50\sin\alpha = 0$

$50\sin\alpha = 40$ 1

$\sin\alpha = \frac{4}{5}$

$\therefore \alpha = 53^\circ 8'$

Arrow reaches target, ie top of Torch, T, when $x = 180$, $y = 62$

ie $(50\cos\alpha)t = 180$

$50\times0.6\,t = 180$ 1

$30t = 180$

$t = 6$

where $\sin\alpha = 0.8$

$\cos\alpha = \sqrt{1-\sin^2\alpha}$

$= \sqrt{1-0.64}$

$= \sqrt{0.36}$

$= 0.6$

time taken is 6 secs. 1.