St George Girls High School

Year 11 – Higher School Certificate Course

Assessment Task 1

2007

Mathematics Extension 2

Time Allowed: 90 Minutes
(plus 5 minutes reading time)

Instructions to Candidates

1. Attempt all questions.
2. All necessary working must be shown.
3. Start each question on a new page.
4. All diagrams are to be at least $\frac{1}{3}$ page each.

Question 1 – (25 marks) – Start a New Page Marks

a) Given that $z_1 = 3 - 2i$ and $z_2 = 4 + 3i$, express each of the following in the form $a + bi$, where a and b are real.

(i) $z_2 - z_1$ 1

(ii) $z_1 z_2$ 1

(iii) $\dfrac{z_1}{z_2}$ 2

(iv) $(\bar{z}_1)^3$ 2

b) (i) Express $\sqrt{3} - i$ in mod-arg form. 3

(ii) Hence express $\left(\sqrt{3} - i\right)^4$ in the form $a + ib$ (where a, b are real) 2

c) (i) Express $\sqrt{16 - 30i}$ in Cartesian form (ie $a + ib$ with a, b real). 3

(ii) Hence, express the roots of $z^2 - (1 - i)z + 7i - 4 = 0$ in the form $x + iy$ (x, y are real).

d) Draw neat sketches on separate Argand diagrams (of at least ⅓ page in size) of the locus of a point representing a complex number z if:

(i) $\mathrm{Im}(z) < 1$

(ii) $|z - 1 + 2i| \geq 1$

(iii) $|z + 2| = |z - 3i|$

(iv) $z + \bar{z} > \mathrm{Im}(3z) \cap -\dfrac{\pi}{4} < \arg z < \dfrac{\pi}{4}$

Question 2 – (25 marks) – Start a new page **Marks**

a) (i) Prove that for any complex number z, $z\bar{z} = |z|^2$

(ii) Prove that for any complex numbers z_1 and z_2, $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(iii) Suppose that z_1, z_2 and z_3 are three complex numbers of modulus 1 such that $z_1 + z_2 + z_3 = 0$

Suppose also that z is a complex number of modulus 3.

Using the results in parts (i) and (ii),

(α) show $|z - z_1|^2 = 10 - (z\bar{z}_1 + \bar{z}z_1)$

(β) show $|z - z_1|^2 + |z - z_2|^2 + |z - z_3|^2 = 30$

b) (i) Use de Moivre's theorem to solve the equation $z^5 = 1$

(ii) Show that the points representing the five roots of this equation form the vertices of a regular pentagon when they are plotted on the Argand diagram.

(iii) Find the area of this pentagon.

c) Given that $z = \cos\theta + i\sin\theta$,

(i) show that $z^n + z^{-n} = 2\cos n\theta$ using de Moivre's theorem.

(ii) Hence, solve the equation $2z^4 - z^3 + 3z^2 - z + 2 = 0$

(hint: divide throughout by z^2 and use the result $\cos 2\theta = 2\cos^2\theta - 1$)

Question 3 – (25 marks) – Start a new page **Marks**

a) Sketch the locus of the complex number z if

(i) $\arg(z - 2i) = \arg(z + 3 - i)$ 3

(ii) $\arg\left(\dfrac{z - 2i}{z + 3 - i}\right) = \dfrac{\pi}{2}$ 3

b) (i) If w is a complex cube root of unity, show that the other complex root is w^2. 2

(ii) Prove that $1 + w + w^2 = 0$ by using two completely different methods. 4

(iii) Evaluate $(1 + 2w + 3w^2)(1 + 2w^2 + 3w^4)$ 2

c) (i) Express the roots of the equation $z^5 + 32 = 0$ in modulus/argument form.

(ii) Hence, show that

$$z^4 - 2z^3 + 4z^2 - 8z + 16 = \left(z^2 - 4\cos\frac{\pi}{5}z + 4\right) \times \left(z^2 - 4\cos\frac{3\pi}{5}z + 4\right)$$

(iii) By equating coefficients in (ii) above, find the values of:

(α) $\cos\dfrac{\pi}{5} + \cos\dfrac{3\pi}{5}$

(β) $\cos\dfrac{\pi}{5} \cdot \cos\dfrac{3\pi}{5}$

(iv) Hence, find the exact values of $\cos\dfrac{\pi}{5}$ and $\cos\dfrac{3\pi}{5}$ in simplest surd form.

End of Paper

EXT 2

Q1 (a) (i) $z_2 - z_1 = (4-3) + (3-(-2))i$
$= 1 + 5i$ — 1

(ii) $z_1 z_2 = 12 + 9i - 8i - 6i^2$
$= 18 + i$ — 1

(iii) $\frac{z_1}{z_2} = \frac{(3-2i)}{(4+3i)} \times \frac{(4-3i)}{(4-3i)}$ — 1
$= \frac{6-17i}{16+9}$
$= \frac{6}{25} - \frac{17}{25}i$ — 1

(iv) $(\bar{z}_1)^3 = (5+12i)(3+2i)$ — 1
$= 15 + 10i + 36i - 24$
$= -9 + 46i$ — 1

or $\bar{z}^3 = [\sqrt{13}(\text{cis}\,\theta)]^3$
$= 13\sqrt{13}\,[\cos 3\theta + i\sin 3\theta]$
$= 13\sqrt{13}\,[\cos + i\sin \ldots]$
$= -9 + 46i$

calc.

(b) (i) $\sqrt{3} - i = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$ 1
$\left.\begin{matrix}\cos\theta = \frac{\sqrt{3}}{2} \\ \sin\theta = -\frac{1}{2}\end{matrix}\right\}\ \theta = -\frac{\pi}{6}$ — 1
$\therefore \sqrt{3} - i = 2\left(\text{cis}\,\frac{-\pi}{6}\right)$ — 1

(ii) $(\sqrt{3} - i)^4 = 2^4\,\text{cis}\left(-\frac{\pi}{6} \times 4\right)$
$= 2^4\,\text{cis}\left(-\frac{2\pi}{3}\right)$ — 1
$= 16\left[\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right]$
$= 16\left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)$ — 1
$= -8 - 8\sqrt{3}i$

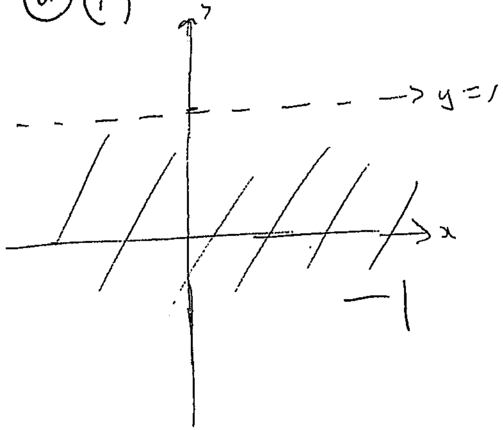
(c) (i) let $\sqrt{16 - 30i} = a + ib$
$16 - 30i = (a^2 - b^2) + 2abi$ — 1
Equate Real Im
$a^2 - b^2 = 16$ --- A
$2ab = -30$ --- B ... $b = \frac{-15}{a}$

So $a^2 - \frac{225}{a^2} = 16$
$a^4 - 16a^2 - 225 = 0$ — 1 a is real.
$(a^2 - 25)(a^2 + 9) = 0$
$a = \pm 5 \quad \therefore b = \mp 3$ — 1
ie $5 - 3i$, $-5 + 3i$

(ii) $z = \frac{(1+i) \pm \sqrt{-2i - 28i + 16}}{2}$ — 1
$= \frac{(1-i) \pm \sqrt{16 - 30i}}{2}$

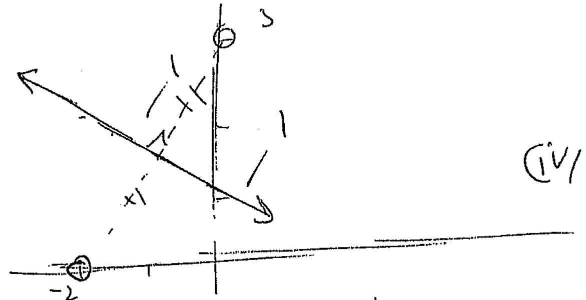
$= 3 - 2i$ — 1 , $-2 + i$ — 1

(d) (i)



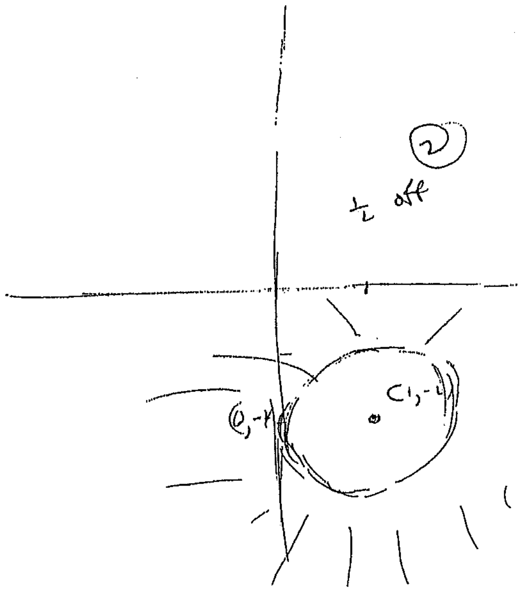
— 1

(ii) $|z - -2| = |z - 3i|$



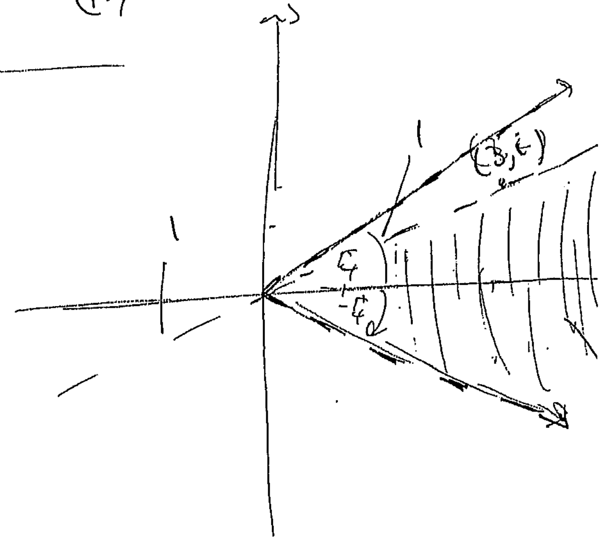
$|x + iy - (-2)| = |x + iy - (3i)|$
$|(x+2) + iy| = |x + (y-3)i|$
$(x+2)^2 + y^2 = x^2 + (y-3)^2$
$x^2 + 4x + 4 + y^2 = x^2 + y^2 - 6y + 9$
$4x + 6y - 5 = 0$
$y = -\frac{2}{3}x + \frac{5}{6}$
$z + \bar{z} = 2x$
$2x > 3y$

(iii) $|z - (1-2i)| \geq 1$



(2)
$\frac{1}{2}$ off

(iv)



Q2

(a) (i) let $z = x+iy$ then $\bar{z} = x - iy$

so $z\bar{z} = (x+iy)(x-iy)$ & $|z| = \sqrt{x^2+y^2}$
$= (x^2+y^2)$ $\therefore |z|^2 = x^2+y^2$

so $z\bar{z} = |z|^2$

(ii) $(x_1+iy_1)-(x_2+iy_2)$
$= (x_1-x_2)+(y_1-y_2)i$
$\therefore \overline{z_1-z_2} = (x_1-x_2)-(y_1-y_2)i$

& $\bar{z}_1 - \bar{z}_2 = (x_1-iy_1)-(x_2-iy_2)$
$= (x_1-x_2)-(y_1-y_2)i$

LHS = RHS.

(iii) (α) $|z-z_1|^2 = (z-z_1)\overline{(z-z_1)}$ from a(i) a(ii)
$= (z-z_1)(\bar{z}-\bar{z}_1)$
$= z\bar{z} - z\bar{z}_1 - z_1\bar{z} + z_1\bar{z}_1$
$= |3|^2 - z\bar{z}_1 - z_1\bar{z} + |1|^2$
$= 10 - (z\bar{z}_1 + \bar{z}z_1)$

(β) $|z-z_1|^2 + |z-z_2|^2 + |z-z_3|^2$

from α $= 10 - (z\bar{z}_1 + \bar{z}z_1) + 10 - (z\bar{z}_2 + \bar{z}z_2) + 10 - (z\bar{z}_3 + \bar{z}z_3)$
$= 30 - [z(\bar{z}_1+\bar{z}_2+\bar{z}_3) + \bar{z}(z_1+z_2+z_3)]$

Given $z_1+z_2+z_3 = 0$
$\bar{z}_1+\bar{z}_2+\bar{z}_3 = 0$

$\therefore (x_1+x_2+x_3) + i(y_1+y_2+y_3) = 0$
$\therefore (x_1+x_2+x_3) - i(y_1+y_2+y_3) = 0$
Equate real / im.

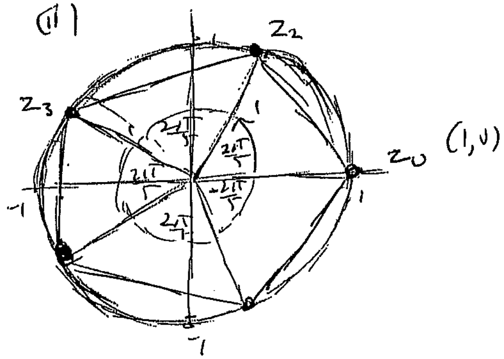
$= 30 - (z \times 0 + \bar{z} \times 0)$
$= 30$

(b) (i) if $\operatorname{cis}\theta = 1 + 0i$
Then $\cos\theta = 1$ & $\sin\theta = 0$
$\therefore \theta = 2k\pi$ ($k = 0, 1, 2, 3, 4$ .. repeat)

so $z^5 = (\cos 2k\pi + i\sin 2k\pi)$
$z = (\cos \frac{2k\pi}{5} + i\sin(\frac{2k\pi}{5}))$ by De Moivre

$k=0$, $z_0 = 1$
$k=1$, $z_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = \operatorname{cis}\frac{2\pi}{5}$
$k=2$, $z_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} = \operatorname{cis}\frac{4\pi}{5}$ $[-\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}]$
$k=3$, $z_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5} = \operatorname{cis}\frac{6\pi}{5} = \operatorname{cis}(-\frac{4\pi}{5}) = \bar{z}_2$
$k=4$, $z_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} = \operatorname{cis}\frac{8\pi}{5} = \operatorname{cis}(-\frac{2\pi}{5}) = \bar{z}_1$

(ii)



five points equally spaced at $\frac{2\pi}{5}$ radians on unit circle.

(iii) $A_1 = \frac{1}{2} \times 1 \times 1 \times \sin(\frac{2\pi}{5})$ $\therefore$ Total area
$= 5[\frac{1}{2}\sin(\frac{2\pi}{5})]$
$= \frac{5}{2}\sin(\frac{2\pi}{5})$ sq units

(c) $z^n = (\cos\theta + i\sin\theta)^n$ by De Moivre
$= \cos n\theta + i\sin n\theta$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$
$= \cos n\theta - i\sin(n\theta)$

Since $\cos(-\theta) = \cos\theta$
$\sin(-\theta) = -\sin\theta$

$\therefore\ z^n + z^{-n} = (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta)$

$= 2\cos n\theta$

(ii) $2z^4 - z^3 + 3z^2 - z + 2 = 0$

$\div\ z^2$

$\therefore\ 2z^2 - z + 3 - z^{-1} + 2z^{-2} = 0$

So $2(z^2 + z^{-2}) - (z + z^{-1}) + 3 = 0$

$4\cos 2\theta - 2\cos\theta + 3 = 0$

$4\,[2\cos^2\theta - 1] - 2\cos\theta + 3 = 0$

$8\cos^2\theta - 2\cos\theta - 1 = 0$

quadratic $\quad \cos\theta = \dfrac{2 \pm \sqrt{4 - 4\times 8\times -1}}{2\times 8}$

$= \dfrac{2 \pm \sqrt{36}}{16}$

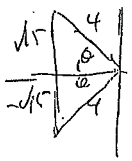
$= \dfrac{-4}{16}$ and $\dfrac{8}{16}$

$= -\dfrac{1}{4}$ $\qquad \dfrac{1}{2}$.

$\cos\theta = -\frac{1}{4}$

2nd and 3rd qd.

$\theta = \pi - \ ,\ \pi +$



$\therefore\ z = -\frac{1}{4} \pm \frac{\sqrt{15}}{4} i$

$z_1 = -\frac{1}{4}(1 - \sqrt{15} i)$

$z_2 = -\frac{1}{4}(1 + \sqrt{15} i)$

$\cos\theta = \frac{1}{2}$

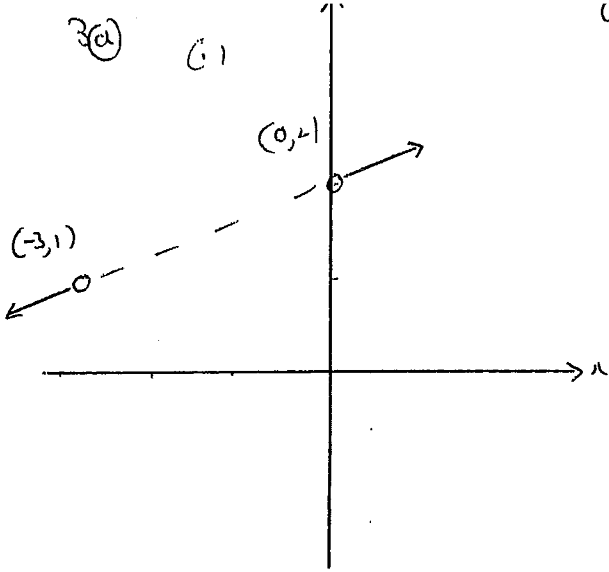
In 1st / 4th qu

$\theta = \frac{\pi}{3}$

$z_3 = \frac{1}{2} + \frac{\sqrt{3}}{2} i$

$z_4 = \frac{1}{2} - \frac{\sqrt{3}}{2} i$

3(a) (i)



$\arg(z - 2i) = \arg(z - (-3+i))$.

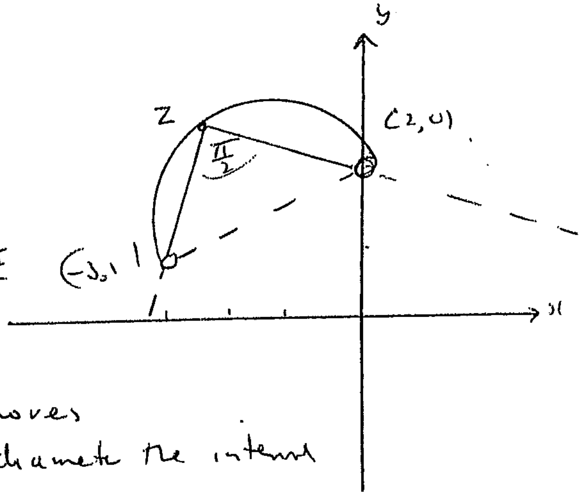
- line through (0,2) and (-3,1), excluding interval between points

(ii) $\arg\left(\dfrac{z-2i}{z-(-3+i)}\right) = \dfrac{\pi}{2}$

then $\arg(z-2i) + \arg(z - (-3+i)) = \frac{\pi}{2}$

so $\arg(z-2i) = \arg(z-(-3+i)) + \frac{\pi}{2}$

- By ext. $\angle$ of Δ, z moves on semi-circle with diameter the interval joining (0,2) to (-3,1)



(b) If $\omega^3 = 1$

then $\omega^3 - 1 = 0$; $(\omega - 1)(\omega^2 + \omega + 1) = 0$

Now $\omega^2 + \omega + 1 = 0$ use quad. formula $\quad \omega = \dfrac{-1 \pm \sqrt{1-4}}{2}$

(use $i^2 = -1$) $\quad = -\dfrac{1}{2} \pm \dfrac{i\sqrt{3}}{2}$

$\omega = 1,\ -\frac{1}{2} - \frac{i\sqrt{3}}{2},\ -\frac{1}{2} + \frac{i\sqrt{3}}{2}$

then $\omega^2 = \left(\frac{1}{4} - \frac{3}{4}\right) - 2i\frac{\sqrt{3}}{4}$

[complex] $= -\frac{1}{2} - i\frac{\sqrt{3}}{2} \rightarrow = \bar{\omega}$

(ii) Method ①

$$1+\omega+\omega^2 = 1+\left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)+\left(-\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i\right)$$
$$= 1-1$$
$$= 0 \quad \text{as required}$$

② $1+\omega+\omega^2$ is a Geo P.

$$\therefore S_3 = \frac{1(\omega^3-1)}{\omega-1}$$
$$= \frac{\omega^3-1}{\omega-1} \quad \text{since } \omega \text{ is a cube root then } \omega^3-1=0$$
$$S_3 = 0$$

so $1+\omega+\omega^2 = 0$ as required

* or equate

$$(z-1)(z-\omega)(z-\omega^2) = (z^2-(1+\omega)z+\omega)(z-\omega^2)$$
$$= z^3-(1+\omega)z^2+\omega z - z^2\omega^2 - (1+\omega)\omega^2 z + \omega^3$$
$$= z^3-(1+\omega+\omega^2)z^2 + (\omega^2+\omega^3+\omega)z + \omega^3$$

Now equate coeff on z^2 } $1+\omega+\omega^2=0$

(iii) $(1+2\omega+3\omega^2)(1+2\omega^2+3\omega^4)$ $\boxed{\omega^4=\omega}$

$$= (1+2\omega+3\omega^2)(1+2\omega^2+3\omega)$$
$$= [2(1+\omega+\omega^2)-1+\omega^2][2(1+\omega+\omega^2)-1+\omega]$$
$$= (\omega^2-1)(\omega-1)$$
$$= \omega^3-\omega^2-\omega+1$$
$$= 2-(\omega+\omega^2)$$
$$= 3-[1+\omega+\omega^2]$$
$$= 3.$$

(e) (i) If $r\operatorname{cis}\theta$ is a root then $r\operatorname{cis}\theta = -32+0i$

$$32\cos\theta = -32 \quad \& \quad 32\sin\theta = 0 \qquad k=0,1,2,3,4$$
$$r=2^5, \quad \cos\theta=-1$$
$$\theta=(2k+1)\pi$$
$$\therefore z^5 = +2^5\left(\cos(2k+1)\pi + i\sin(2k+1)\pi\right)$$
$$z = +2\left(\cos\frac{(2k+1)\pi}{5} + i\sin\frac{(2k+1)\pi}{5}\right)$$

$k=0$, $z_0 = 2\operatorname{cis}\frac{\pi}{5} = 2[\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}]$

$k=1$, $z_1 = 2\operatorname{cis}\frac{3\pi}{5} = 2[\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}]$

$k=2$, $z_2 = 2\operatorname{cis}\pi = -2$

$k=3$, $z_3 = 2\operatorname{cis}\frac{7\pi}{5} = 2[\cos\frac{3\pi}{5}-i\sin\frac{3\pi}{5}] = \bar{z}_1 = 2\operatorname{cis}(-\frac{3\pi}{5})$

$k=4$, $z_4 = 2\operatorname{cis}\frac{9\pi}{5} = 2[\cos\frac{\pi}{5}-i\sin\frac{\pi}{5}] = \bar{z}_0 = 2\operatorname{cis}(-\frac{\pi}{5})$

(ii)
$$z^5+2^5 = (z-z_0)(z-z_1)(z-z_2)(z-z_3)(z-z_4)$$
$$= (z-z_0)(z-\bar{z}_0)(z-z_1)(z-\bar{z}_1)(z-2)$$
$$= [z^2-(z_0+\bar{z}_0)z+z_0\bar{z}_0][z^2-(z_1+\bar{z}_1)z+z_1\bar{z}_1](z$$
$$= [z^2-4\cos\tfrac{\pi}{5}z+4][z^2-4\cos\tfrac{3\pi}{5}z+4][z+$$

Note. $(z+\bar{z}) = 2\operatorname{Re}(z)$ & $z\bar{z} = |z|^2 = 4$

& $(z+2)(z^4-2z^3+4z^2-8z+16)$
$$= z^5 - \cancel{2z^4} + \cancel{4z^3} - \cancel{8z^2} + \cancel{16z} + \cancel{2z^4} - \cancel{4z^3} + \cancel{8z^2} - \cancel{16z} + 32$$
$$= z^5+32$$

Let $\cancel{(z+2)}(z^4-2z^3+4z^2-8z+16) = \cancel{(z+2)}[z^2-4\cos\frac{\pi}{5}z+4][z^2-4\cos\frac{3\pi}{5}$

$$\therefore z^4-2z^3+4z^2-8z+16 = [z^2-4\cos\tfrac{\pi}{5}z+4][z^2-4\cos\tfrac{3\pi}{5}$$

(iii) Equate coeff

(x) $z^4 - 4z^3\cos\frac{3\pi}{5} + 4z^2 - 4z^3\cos\frac{\pi}{5} + 16z^2\cos\frac{\pi}{5}\cos\frac{3\pi}{5}$

$\quad - 16z\cos\frac{\pi}{5} + 4z^2 - 16z\cos\frac{3\pi}{5} + 16$

$= z^4 - z^3\left(4\cos\frac{3\pi}{5} + 4\cos\frac{\pi}{5}\right) + z^2\left(16\cos\frac{\pi}{5}\cos\frac{3\pi}{5} + 8\right)$

$\quad - 16z\left(\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}\right) + 16$.

(α) on z^3 $\quad -\left(4\cos\frac{\pi}{5} + 4\cos\frac{3\pi}{5}\right) = -2$

$\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$

β) on z^2

$16\cos\frac{\pi}{5}\cos\frac{3\pi}{5} + 8 = 4$

$\cos\frac{\pi}{5}\cos\frac{3\pi}{5} = -\frac{1}{4}$

(iv) (α) $\cos\frac{3\pi}{5} = \frac{1}{2} - \cos\frac{\pi}{5} \implies \cos\frac{\pi}{5}\left(\frac{1}{2} - \cos\frac{\pi}{5}\right) = -\frac{1}{4}$

$\frac{1}{2}\cos\frac{\pi}{5} - \cos^2\frac{\pi}{5} = -\frac{1}{4}$

$4\cos^2\left(\frac{\pi}{5}\right) - 2\cos\frac{\pi}{5} - 1 = 0$

$\therefore \cos\frac{\pi}{5} = \frac{2 \pm \sqrt{4+16}}{8}$

$= \frac{2 \pm 2\sqrt{5}}{8}$

$= \frac{1+\sqrt{5}}{4}$ & $\frac{1-\sqrt{5}}{4}$

also

$\cos\frac{3\pi}{5}$ gives same results

$\therefore \cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4} \quad > 0$

$\cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{4} \quad < 0$