St George Girls' High School

Trial Higher School Certificate Examination

2003

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- All questions may be attempted.
- Begin each question on a new page
- All necessary working must be shown.
- Marks may be deducted for careless or poorly presented work.
- Board-approved calculators may be used.
- A list of standard integrals is included at the end of this paper.
- The mark allocated for each question is listed at the side of the question.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Question 1 – (15 marks) – Start a new page **Marks**

a) Show that $y = xe^{-x}$ has a stationary point at the point $\left(1, \frac{1}{e}\right)$. 2

b) On separate diagrams sketch the following curves. Indicate clearly any turning points, asymptotes and intercepts with the coordinate axes. 9

(i) $y = xe^{-x}$

(ii) $y = x^2e^{-2x}$

(iii) $y = \dfrac{1}{x^2e^{-2x}}$

(iv) $y = \log_e(xe^{-x})$

(v) $y = e^{xe^{-x}}$

c) Solve for x:

$$\frac{3-x}{\sqrt{x-1}} \geq \frac{\sqrt{x-1}}{3-x}$$ 4

Question 2 – (15 marks) – Start a new page **Marks**

a) Evaluate the following definite integrals:

(i) $\int_0^1 \frac{2x}{1+2x}\,dx$ 3

(ii) $\int_2^3 \frac{x+1}{\sqrt{x^2+2x+5}}\,dx$ 3

(iii) $\int_2^4 \frac{dx}{x\sqrt{x-1}}$ 3

b) The cubic equation $x^3-2x+4=0$ has roots α, β, γ.

(i) Prove that the cubic equation with roots $\alpha^2, \beta^2, \gamma^2$ is $x^3-4x^2+4x-16=0$. 3

(ii) Factorise $x^3-4x^2+4x-16$ into linear factors and hence prove that only one of α, β, γ is real. 3

Question 3 – (15 marks) – Start a new page **Marks**

a) Let $P(z)=z^7-1$.

(i) Find all the complex roots of $P(z)=0$. (Call them $z_0, z_1, \ldots, z_6$ leaving all answers in terms of π). 3

(ii) Plot the points representing $z_0, z_1, \ldots, z_6$ on the Argand Diagram.

(iii) Factorise $P(z)$ over the complex numbers. 1

(iv) Factorise $P(z)$ over the real numbers (leave answers in terms of π). 2

(v) Show that $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}=-\frac{1}{2}$ 2

b) The area bounded by the curve $y=x(2-x)$ and the x axis is rotated about the y axis. 5

(i) By considering cylindrical shells with generators parallel to the y axis, show that the volume V units^3 of the solid so generated is given by $V=\int_0^2 2\pi xy\,dx$.

(ii) Hence, determine the volume of this solid.

Question 4 – (15 marks) – Start a new page **Marks**

a) (i) Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ 2

(ii) With reference to an appropriate diagram, show geometrically why the above result is true. 2

(iii) Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ 3

b) If $I_n = \int_0^1 x^n e^{-x}\,dx \quad (n \geq 0)$

(i) Prove that $I_n = n\,I_{n-1} - \frac{1}{e} \quad (n \geq 1)$ 2

(ii) Hence, evaluate $\int_0^1 x^3 e^{-x}\,dx$ 2

c) Using the substitution $t = \tan\frac{\theta}{2}$, or otherwise, calculate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cot\theta \tan\frac{\theta}{2}\,d\theta$ 4

Question 5 – (15 marks) – Start a new page **Marks**

a) Find all pairs of real numbers x and y that satisfy $(x+iy)^2 = -12+16i$ 3

b) Consider the polynomial equation

$$z^3 + (6+i)z^2 + (17+8i)z + 33i + 30 = 0$$

(i) This equation has a root $\alpha = pi$ where p is real. Find the value of p. 3

(ii) If the other roots are β and γ, use the relationships for the sum and product of the roots (or otherwise) to find β and γ.
[Hint: Use your answer from part (a)]. 3

(iii) Show that the points representing α, β, γ on the Argand diagram are the vertices of a right-angled triangle. 1

c) The complex number z and its conjugate $\bar{z}$ satisfy

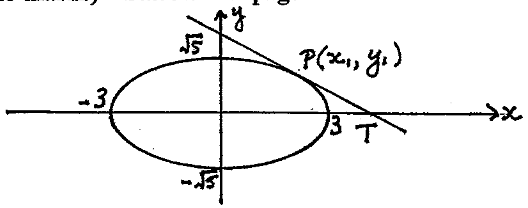
$$(z-\bar{z})^2 + 8(z+\bar{z}) = 16$$

(i) Prove that the point which represents z on the Argand diagram lies on a parabola.

(ii) Sketch the locus of z and hence show that $-\frac{\pi}{4} \leq \arg z \leq \frac{\pi}{4}$

Question 6 – (15 marks) – Start a new page **Marks**

a)



The point $P(x_1, y_1)$ lies on the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1 \quad (x_1 \neq 0)$

The tangent to the ellipse at P cuts the x axis at T.

(i) Find the eccentricity, e, of the ellipse. 1

(ii) Find the coordinates of the foci, S and S^1 1

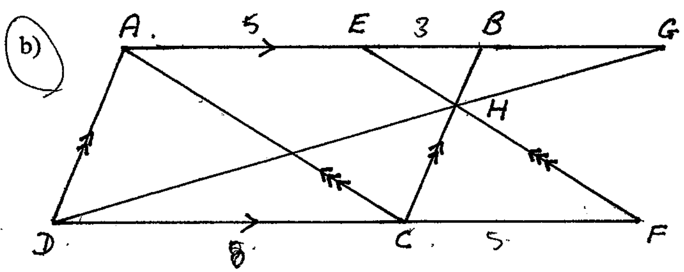
(iii) Find the equations of the directrices. 1

(iv) Show that the equation of the tangent at P is $\frac{x x_1}{9}+\frac{y y_1}{5}=1$ 2

(v) Hence write down the coordinates of T. 1

(vi) Using the focus-directrix definition of the ellipse, or otherwise, show that $\frac{PS}{PS^1}=\frac{TS}{TS^1}$ 4

b)



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$ABCD$ and $AEFC$ are parallelograms, $AE = 5$cm and $EB = 3$cm.

(i) Find, giving brief reasons, the length of AG. 13.

(ii) Find the ratio of the area of ΔCDH to the area of the parallelogram $ABCD$.

Question 7 – (15 marks) – Start a new page **Marks**

a) The acceleration of a particle which is moving along the x axis is given by

$$\frac{d^2x}{dt^2}=2x^3-10x$$

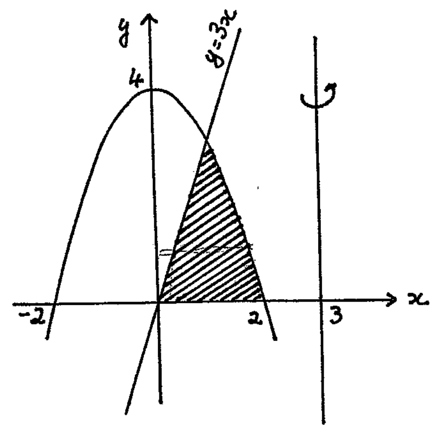
(i) If the particle starts at the origin with velocity u show that its velocity v is given by $v^2-u^2=x^4-10x^2$ 2

(ii) (α) If $u=3$ show that the particle oscillates within the interval $-1\le x\le 1$. 4

(β) Is this an example of simple harmonic motion? Give a clear reason for your answer. 1

(iii) If $u=6$ carefully describe the motion of the particle. 3

b)



The area enclosed by the curve $y=4-x^2$, the line $y=3x$ and the x axis (for $x\ge 0$) is rotated about the line $x=3$.

Calculate the volume of the solid generated. 5

Question 8 – (15 marks) – Start a new page **Marks**

a) (i) x, y and z are positive integers. Show that if x is a factor of both y and z then x is a factor of $z-y$. 1

(ii) Show that $2^{2^{k+1}} = \left(2^{2^k}\right)^2$ 1

(iii) $F_n = 2^{2^n} + 1$ defines a set of positive integers called Fermat numbers for $n = 0, 1, 2, \ldots$ i.e. $F_0 = 2^{2^0} + 1 = 2^1 + 1 = 3$ 5

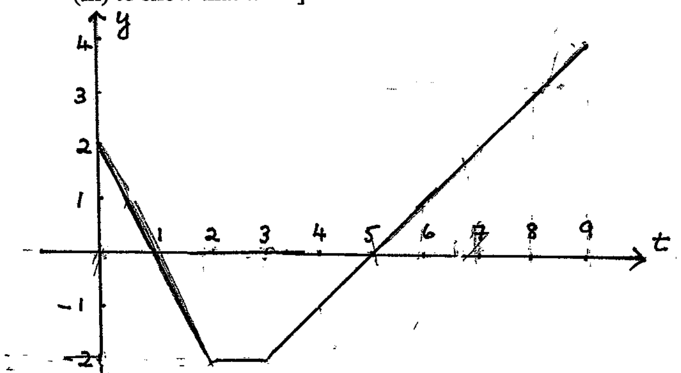
Use mathematical induction to show that

$F_n = F_0 F_1 F_2 \ldots F_{n-1} + 2$ for $n \geq 1$

(iv) Hence, or otherwise, show that the highest common factor of any two Fermat numbers is 1. 3

[Hint: Let k be a common factor of F_m and F_n, where $m < n$, and use (i) and (iii) to show that $k = 1$]

b) 5



The graph of $y = f(t)$ $0 \leq t \leq 9$ is shown. If $F(x) = \int_0^x f(t)\,dt$ find:

(i) the values of x for which $F(x) = 0$

(ii) the coordinates of any stationary points.

TRIAL HSC EXTENSION 2 2003 SOLUTIONS

Question 1

(a) $y = xe^{-x}$

$\frac{dy}{dx} = 1.e^{-x} + x.(-e^{-x})$

$= e^{-x}(1-x)$

Stat pts when $\frac{dy}{dx} = 0$

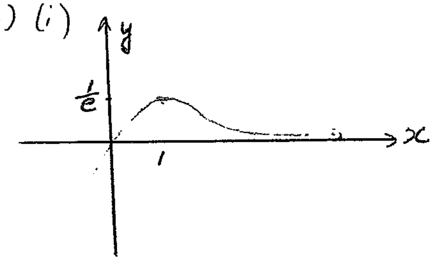
$e^{-x}(1-x) = 0$

$x = 1$

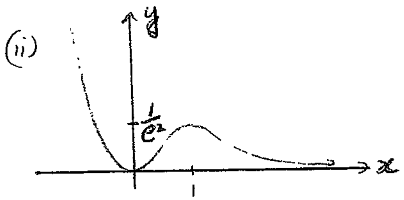
$y = 1.e^{-1} = \frac{1}{e}$

$\therefore (1, \frac{1}{e})$ is a stat pt

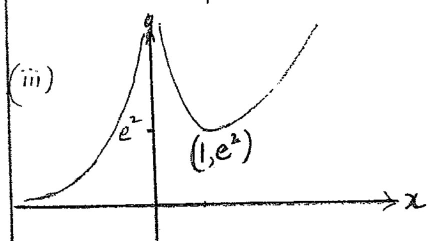
(b) (i)



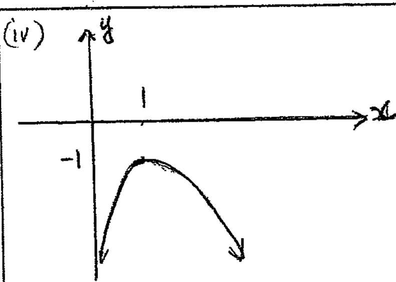
(ii)



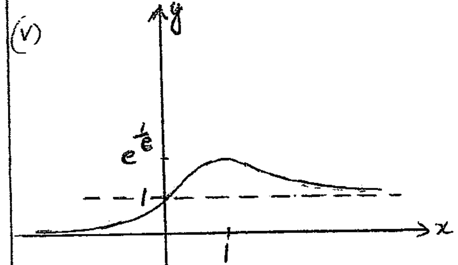
(iii)



(iv)



(v)



(c) $\frac{3-x}{\sqrt{x-1}} \geq \frac{\sqrt{x-1}}{3-x}$ $x > 1$, $x \neq 3$

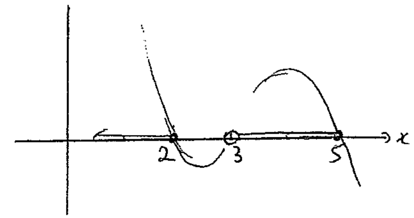
$\frac{3-x}{1} \geq \frac{(\sqrt{x-1})^2}{3-x}$

$\frac{3-x}{1} - \frac{x-1}{3-x} \geq 0$

$\frac{9-6x+x^2-x+1}{3-x} \geq 0$ $\left(\times \frac{(3-x)^2}{(3-x)^2}\right)$

$(x^2-7x+10)(3-x) \geq 0$

$(x-2)(x-5)(3-x) \geq 0$



$1 < x \leq 2$ or $3 < x \leq 5$

Question 2

(a) (i) $\int_0^1 \frac{2x}{1+2x}\,dx$

$= \int_0^1 \frac{1+2x-1}{1+2x}\,dx$

$= \int_0^1 1 - \frac{1}{1+2x}\,dx$

$= \left[x - \frac{1}{2}\log_e|1+2x|\right]_0^1$

$= \left(1-\frac{1}{2}\log_e 3\right) - \left(0 - \frac{1}{2}\log_e 1\right)$

$= 1 - \frac{1}{2}\log_e 3$

(ii) $\int_2^3 \frac{x+1}{\sqrt{x^2+2x+5}}\,dx$

$= \frac{1}{2}\int_2^3 (2x+2)(x^2+2x+5)^{-1/2}\,dx$

$= \frac{1}{2}\left[\frac{(x^2+2x+5)^{1/2}}{1/2}\right]_2^3$

$= \left[\sqrt{x^2+2x+5}\right]_2^3$

$= \sqrt{20} - \sqrt{13}$ $\quad (= 2\sqrt{5} - \sqrt{13})$

(iii) $\int_2^4 \frac{dx}{x\sqrt{x-1}}$ $\qquad u = \sqrt{x-1}$, $x = u^2+1$, $dx = 2u\,du$; $x=2$, $u=1$; $x=4$, $u=\sqrt{3}$

$= \int_1^{\sqrt{3}} \frac{2u\,du}{(u^2+1)\,u}$

$= 2\int_1^{\sqrt{3}} \frac{1}{u^2+1}\,du$

$= 2\left[\tan^{-1} u\right]_1^{\sqrt{3}}$

$= 2(\tan^{-1}\sqrt{3} - \tan^{-1} 1)$

$= 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$= \frac{\pi}{6}$

OR $\quad$ Let $x = \sec^2\theta$

ie $\theta = \cos^{-1}\left(\frac{1}{\sqrt{x}}\right)$

$x = 2 \quad \theta = \frac{\pi}{4}$

$x = 4 \quad \theta = \frac{\pi}{3}$

$dx = 2\sec\theta \cdot \sec\theta\tan\theta\,d\theta$

$\int_2^4 \frac{dx}{x\sqrt{x-1}}$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2\sec^2\theta\tan\theta\,d\theta}{\sec^2\theta\sqrt{\sec^2\theta - 1}}$ $\qquad \left(\sqrt{\sec^2\theta-1} = \sqrt{\tan^2\theta} = \tan\theta\right)$

$\int_{\frac{\pi}{4}}^{\pi/3} 2\,d\theta = \left[2\theta\right]_{\pi/4}^{\pi/3}$

$= 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{6}$

(b)(i) $x^3 - 2x + 4 = 0$ has roots α, β, γ

Let $P(x) = x^3 - 2x + 4$

Eq^n with roots $\alpha^2, \beta^2, \gamma^2$ is

$P(\sqrt{x}) = 0$

$(\sqrt{x})^3 - 2\sqrt{x} + 4 = 0$

$x\sqrt{x} - 2\sqrt{x} = -4$

$\sqrt{x}(x-2) = -4$

$x(x-2)^2 = 16$

$x^3 - 4x^2 + 4x = 16$

$x^3 - 4x^2 + 4x - 16 = 0$

(ii) $x^3 - 4x^2 + 4x - 16$

$= x^2(x-4) + 4(x-4)$

$= (x-4)(x^2+4)$

$= (x-4)(x^2 - 4i^2)$

$= (x-4)(x-2i)(x+2i)$

Since $x^3 - 4x^2 + 4x - 16 = 0$ has roots $\alpha^2, \beta^2, \gamma^2$ then

$(x-\alpha^2)(x-\beta^2)(x-\gamma^2)$

$= (x-4)(x-2i)(x+2i)$

Thus, without loss of generality

$\alpha^2 = 4, \quad \beta^2 = 2i, \quad \gamma^2 = -2i$

ie only α^2 and hence α is real

If $\alpha^2 = 4$ then $\alpha = 2$ or -2

[$P(2) = 2^3 - 2\times 2 + 4 = 8 \neq 0$

$P(-2) = (-2)^3 - 2(-2) + 4 = 0$

$\therefore \alpha = -2$]

Thus only one of α, β, γ is real.

Question 3

(a) $\quad P(z) = z^7 - 1$

(i) $z^7 - 1 = 0$

$z^7 = 1 \quad \Rightarrow \quad |z| = 1$

Let $z = \cos\theta + i\sin\theta$

$(\cos\theta + i\sin\theta)^7 = 1$

$\cos 7\theta + i\sin 7\theta = 1$

$\cos 7\theta = 1 \quad \sin 7\theta = 0$

$7\theta = 2k\pi$

$\theta = \frac{2k\pi}{7}$

$k=0 \quad z_0 = \cos 0 + i\sin 0 = 1$

$k=1 \quad z_1 = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

$k=2 \quad z_2 = \cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7}$

$k=3 \quad z_3 = \cos\frac{6\pi}{7} + i\sin\frac{6\pi}{7}$

$k=4 \quad z_4 = \cos\frac{8\pi}{7} + i\sin\frac{8\pi}{7}$

$= \cos\frac{6\pi}{7} - i\sin\frac{6\pi}{7}$

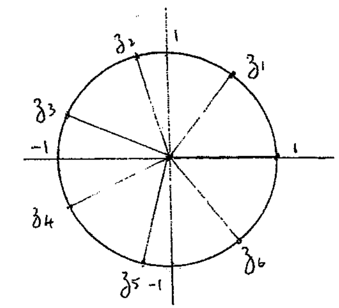
$k=5 \quad z_5 = \cos\frac{10\pi}{7} + i\sin\frac{10\pi}{7}$

$= \cos\frac{4\pi}{7} - i\sin\frac{4\pi}{7}$

$k=6 \quad z_6 = \cos\frac{12\pi}{7} + i\sin\frac{12\pi}{7}$

$= \cos\frac{2\pi}{7} - i\sin\frac{2\pi}{7}$

(ii)



(iii) $P(z) = (z-1)(z-z_1)(z-z_2)(z-z_3)(z-z_4)\times(z-z_5)(z-z_6)$

where $z_1, \ldots, z_6$ are indicated in (i)

(iv) $(z-z_1)(z-z_6) = z^2 - (z_1+z_6)z + z_1 z_6$

$= z^2 - 2\cos\frac{2\pi}{7}z + 1$

Similarly we find $(z-z_2)(z-z_5)$ and $(z-z_3)(z-z_4)$

$\therefore P(z) =$

$(z-1)\left(z^2 - 2\cos\frac{2\pi}{7}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{7} + 1\right)\times\left(z^2 - 2\cos\frac{6\pi}{7} + 1\right)$

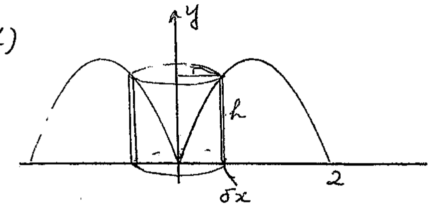
(iv) $z_0 + z_1 + \ldots + z_6 = \frac{-\text{coeff } z^6}{\text{coeff } z^7} = 0$

$\therefore 1 + 2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{6\pi}{7} = 0$

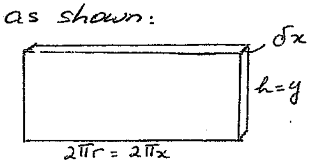
$2\left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right) = -1$

$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$

(b)



When the cylindrical shell of radius $r = x$, height $h = y$ and width δx is cut parallel to y axis and "flattened out" its volume δV is approx that of a rectangular prism as shown:



$\delta V \doteq 2\pi x y\,\delta x$

$V = \lim_{\delta x \to 0} \sum_{x=0}^{2} 2\pi x y\,\delta x = \int_0^2 2\pi x y\,dx$

(ii) $xy = x\,x(2-x) = 2x^2 - x^3$

$V = 2\pi\int_0^2 2x^2 - x^3\,dx$

$= 2\pi\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2$

$= 2\pi\left(\frac{16}{3} - \frac{16}{4} - 0\right)$

Volume $= \frac{8\pi}{3}$ units^3

Question 4

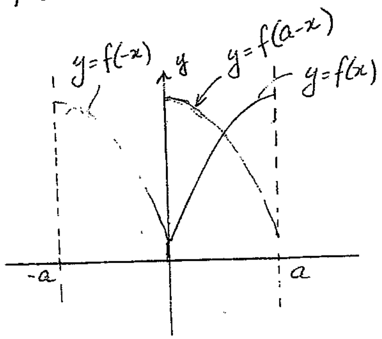
(a) (i) $\int_0^a f(a-x)\,dx$ Let $u=a-x$, $du=-dx$, $dx=-du$; $x=0$, $u=a$; $x=a$, $u=0$

$= \int_a^0 f(u).(-1)\,du$

$= \int_0^a f(u)\,du$

$= \int_0^a f(x)\,dx$

(ii)



If $y=f(x)$ $0 \le x \le a$ is reflected in the y axis we get $y=f(-x)$ $-a \le x \le 0$
If this is then translated 'a' units to the right we get
$y = f(-(x-a))$ $0 \le x \le a$
$= f(a-x)$ $0 \le x \le a$

From the graph it can be seen that the area under $y=f(x)$ from $x=0$ to $x=a$ ($\int_0^a f(x)dx$) equals the area under $y=f(-x)$ from $x=-a$ to $x=0$ and this is in turn equal to the area under $y=f(a-x)$ from $x=0$ to $x=a$ (ie $\int_0^a f(a-x)\,dx$)

(iii) $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

$= \int_0^{\pi/2} \frac{\sqrt{\sin(\frac{\pi}{2}-x)}}{\sqrt{\sin(\frac{\pi}{2}-x)}+\sqrt{\cos(\frac{\pi}{2}-x)}}\,dx$ (by (i))

$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$

Now $\int_0^{\frac{\pi}{2}} \left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}} + \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\right)dx$

$= \int_0^{\pi/2} 1\,dx$

$= [x]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

Since $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$

$= \frac{\pi}{4}$

(b) $I_n = \int_0^1 x^n e^{-x}\,dx$ $n \ge 0$

(i) $= \int_0^1 \frac{d(e^{-x})}{dx}.\,x^n\,dx$

$= \left[-e^{-x}.x^n\right]_0^1 - \int_0^1 -e^{-x}.\,nx^{n-1}\,dx$

$= (-e^{-1} \times 1 - 0) + n\int_0^1 x^{n-1}e^{-x}\,dx$

$= n\,I_{n-1} - \frac{1}{e}$

(ii) $\int_0^1 x^3 e^{-x}\,dx = I_3$
$= 3I_2 - \frac{1}{e}$
$= 3(2I_1 - \frac{1}{e}) - \frac{1}{e}$
$= 6(1.I_0 - \frac{1}{e}) - \frac{4}{e}$
$= 6I_0 - \frac{10}{e}$

$I_0 = \int_0^1 e^{-x}\,dx$
$= \left[-e^{-x}\right]_0^1$
$= -e^{-1} - -e^0$
$= 1 - \frac{1}{e}$

$\int_0^1 x^3 e^{-x}\,dx = 6(1-\frac{1}{e}) - \frac{10}{e}$
$= 6 - \frac{16}{e}$

Question 5

(c) $\int_{\frac{\pi}{3}}^{\pi/2} \cot\theta \tan\frac{\theta}{2}\,d\theta$

$t = \tan\frac{\theta}{2}$ $\theta = 2\tan^{-1}t$
$d\theta = \frac{2}{1+t^2}\,dt$

$\theta = \frac{\pi}{3}$ $t = \tan\frac{\pi}{6} = \frac{1}{\sqrt3}$
$\theta = \frac{\pi}{2}$ $t = \tan\frac{\pi}{4} = 1$

$\cot\theta = \frac{1-t^2}{2t}$

$\int_{\frac{\pi}{3}}^{\pi/2} \cot\theta.\tan\frac{\theta}{2}\,d\theta$

$= \int_{\frac{1}{\sqrt3}}^1 \frac{1-t^2}{2t}.\,t.\,\frac{2}{1+t^2}\,dt$

$= \int_{\frac{1}{\sqrt3}}^1 \frac{1-t^2}{1+t^2}\,dt$

$= \int_{\frac{1}{\sqrt3}}^1 \frac{2-(1+t^2)}{1+t^2}\,dt$

$= \int_{\frac{1}{\sqrt3}}^1 \frac{2}{1+t^2} - 1\,dt$

$= \left[2\tan^{-1}t - t\right]_{1/\sqrt3}^1$

$= (2\tan^{-1}1 - 1) - (2\tan^{-1}\frac{1}{\sqrt3} - \frac{1}{\sqrt3})$

$= (2\times\frac{\pi}{4} - 1) - (2\times\frac{\pi}{6} - \frac{1}{\sqrt3})$

$= \frac{\pi}{6} - 1 + \frac{1}{\sqrt3}$

(a) $(x+iy)^2 = -12+16i$ (x,y real)
$x^2 - y^2 + 2ixy = -12+16i$
$x^2 - y^2 = -12$ — ①
$2xy = 16$
$y = \frac{8}{x}$ — ②
Subst ② in ①
$x^2 - \frac{64}{x^2} = -12$

$x^4 + 12x^2 - 64 = 0$
$(x^2+16)(x^2-4) = 0$
$x = 2, -2$ (x real)
$y = 4, -4$
$x=2$ $y=4$ or $x=-2$, $y=-4$

(b) $z^3 + (6+i)z^2 + (17+8i)z + 33i + 30 = 0$
(i) $\alpha = pi$ is a root
$(pi)^3 + (6+i)(pi)^2 + (17+8i)pi + 33i + 30 = 0$
$-p^3 i - 6p^2 - ip^2 + 17pi - 8p + 33i + 30 = 0$
Equating real and imag. parts
$-p^3 - p^2 + 17p + 33 = 0$ — ①
$-6p^2 - 8p + 30 = 0$ — ②
From ② $3p^2 + 4p - 15 = 0$
$(3p-5)(p+3) = 0$
$p = 5/3, -3$
Test in ①
$p \ne 5/3$ since $5 \not| 33 \times 3 \times 1$
$p=-3$ $LHS = -(-3)^3 - (-3)^2 + 17\times -3 + 33$
$= 27 - 9 - 51 + 33$
$= 0 = RHS$
$\therefore\ \alpha = -3i$

(ii) $\alpha+\beta+\gamma = -(6+i)$
$-3i + \beta + \gamma = -6 - i$
$\beta + \gamma = -6 + 2i$ — ①

$\alpha\beta\gamma = -(33i+30)$
$-3i\beta\gamma = -3i(11-10i)$

From ① $\gamma = -6+2i-\beta$

Subst in ②

$\beta(-6+2i-\beta) = 11-10i$

$(-6+2i)\beta - \beta^2 = 11-10i$

$\beta^2 + (6-2i)\beta + 11-10i = 0$

$\beta^2 + (6-2i)\beta + (3-i)^2 = -11+10i+(3-i)^2$

$(\beta + (3-i))^2 = -11+10i+9-6i-1$

$= -3+4i$

$\beta + (3-i) = \pm\sqrt{-3+4i}$ (*)

$= \pm(1+2i)$

$\beta = -3+i+1+2i, \; -3+i-1-2i$

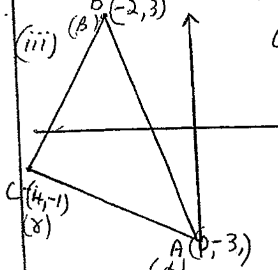
$= -2+3i, \; -4-i$

* $\sqrt{-12+16i} = 2\sqrt{-3+4i}$

$\pm(2+4i) = 2\sqrt{-3+4i}$

$\therefore \sqrt{-3+4i} = \pm(1+2i)$

(iii)



Grad BC $= \frac{3--1}{-2--4} = \frac{4}{2} = 2$

Grad AC $= \frac{-1--3}{-4-0} = \frac{2}{-4} = -\frac{1}{2}$

Grad BC × Grad AC $= 2 \times -\frac{1}{2} = -1$

$\therefore$ A, B, C vertices of right-angled Δ

(c) $(z-\bar{z})^2 + 8(z+\bar{z}) = 16$

Let $z = x+iy$

$\therefore \bar{z} = x-iy$

$z-\bar{z} = 2iy$

$z+\bar{z} = 2x$

$\therefore$ Eqn becomes

$(2iy)^2 + 8.2x = 16$

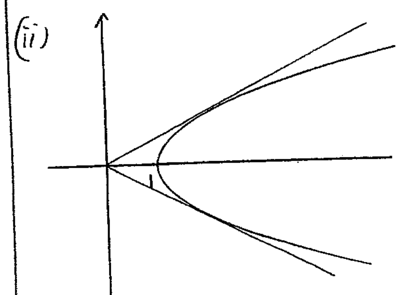
$-4y^2 + 16x = 16$

$16x = 16+4y^2$

$x = 1 + \frac{y^2}{4}$

which is a parabola

(ii)



Maximum and minimum values of arg z occur when $y = mx$ is a tangent to $x = 1+\frac{y^2}{4}$

$y = mx$ —①

$x = 1+\frac{y^2}{4}$ —②

Subst ① in ②

$x = 1+\frac{(mx)^2}{4}$

$4x = 4+m^2x^2$

$m^2x^2 - 4x + 4 = 0$

Line is a tangent when quadratic has equal roots ie $\Delta = 0$

$(-4)^2 - 4\times m^2 \times 4 = 0$

$16 - 16m^2 = 0$

$m^2 = 1$

$m = \pm 1$

$\tan\theta = \pm 1$

$\theta = \pm\frac{\pi}{4}$

$\therefore -\frac{\pi}{4} \le \arg z \le \frac{\pi}{4}$

Question 6

(a) $\frac{x^2}{9} + \frac{y^2}{5} = 1$

$a^2 = 9 \qquad b^2 = 5$

$a = 3 \qquad b = \sqrt{5}$

(i) $b^2 = a^2(1-e^2)$

$5 = 9(1-e^2)$

$e^2 = \frac{4}{9}$

$e = \frac{2}{3} \quad (e>0)$

(ii) $ae = 3\times\frac{2}{3} = 2$

$S(2,0) \quad S'(-2,0)$

(iii) $\frac{a}{e} = \frac{3}{2/3} = \frac{9}{2}$

Directrices. $x = \frac{9}{2}, \; x = -\frac{9}{2}$

(iv) $\frac{x^2}{9} + \frac{y^2}{5} = 1$

$\frac{2x}{9} + \frac{2y}{5}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x}{9}\times\frac{5}{2y}$

$= \frac{-5x}{9y}$

At $P(x_1,y_1)$ $\frac{dy}{dx} = \frac{-5x_1}{9y_1}$

Eqn of tangent at P is

$y - y_1 = \frac{-5x_1}{9y_1}(x-x_1)$

$\frac{yy_1}{5} - \frac{y_1^2}{5} = -\frac{xx_1}{9} + \frac{x_1^2}{9}$

$\frac{xx_1}{9} + \frac{yy_1}{5} = \frac{x_1^2}{9} + \frac{y_1^2}{5}$

$= 1$ (since $P(x_1,y_1)$ lies on ellipse)

$\frac{xx_1}{9} + \frac{yy_1}{5} = 1$

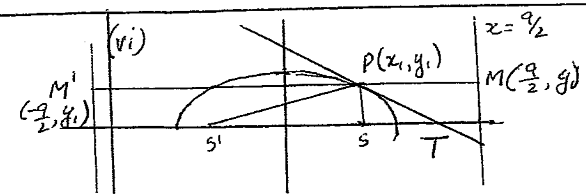
(v) When $y = 0$

$\frac{xx_1}{9} = 1$

$x = \frac{9}{x_1}$

$\therefore$ T has coords $\left(\frac{9}{x_1}, 0\right)$

(vi)



Let $M(\frac{9}{2}, y_1)$ and $M'(-\frac{9}{2}, y_1)$ be the feet of the perpendiculars from P to the directrices

$\frac{PS}{PM} = \frac{PS'}{PM'}$ $(= e)$ (focus-directrix definition)

$\therefore \frac{PS}{PS'} = \frac{PM}{PM'} = \frac{\frac{9}{2}-x_1}{\frac{9}{2}+x_1}$

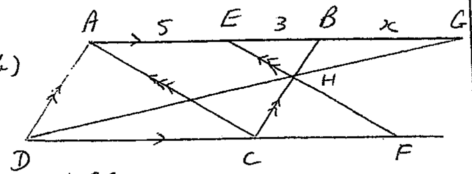
$= \frac{9-2x_1}{9+2x_1}$

Now $\frac{TS}{TS_1} = \frac{\frac{9}{x_1}-2}{\frac{9}{x_1}+2}$

$= \frac{9-2x_1}{9+2x_1}$

$= \frac{PS}{PS'}$

(b)



Let GB = x

In Δ BAC

$\frac{BH}{HC} = \frac{BE}{EA}$ (line parallel to one side of Δ divides other 2 sides in same ratio)

$= \frac{3}{5}$ (OR use similar triangles ABC, EBH)

In Δs BGH, CDH

$B\hat{G}H = C\hat{D}H$ (alt ∠s equal BG ∥ CD)

$G\hat{B}H = D\hat{C}H$

$\therefore$ Δ BGH ||| Δ CDH (equiangular)

$\therefore \frac{BG}{CD} = \frac{BH}{CH}$ (corresp sides in similar Δs in proportion)

$\frac{x}{8} = \frac{3}{5}$ (CD = AB = 8 - opp sides of parallelogram)

$5x = 24$

$x = 4.8$

$\therefore AG = 8+4.8$

$= 12.8$

(ii) Area $\triangle CDH = \frac{1}{2} \times CH \times DC \sin D\hat{C}H$

Area $ABCD = BC \times CD \sin D\hat{C}H$

$$\frac{\text{Area } \triangle CDH}{\text{Area } ABCD} = \frac{\frac{1}{2} \cdot CH . DC \sin D\hat{C}H}{BC \times DC . \sin D\hat{C}H}$$

$$= \frac{\frac{1}{2} CH}{BC}$$

$$= \frac{1}{2} \times \frac{5}{8}$$

$$= \frac{5}{16}$$

Question 7

(a) (i) $\frac{d(\frac{1}{2}v^2)}{dx} = 2x^3 - 10x$

$\frac{1}{2}v^2 = \frac{2x^4}{4} - \frac{10x^2}{2} + c$

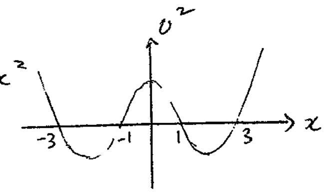
When $x = 0$ $v = u$

$\frac{1}{2}u^2 = c$

$\frac{1}{2}v^2 = \frac{x^4}{2} - \frac{10x^2}{2} + \frac{u^2}{2}$

$v^2 - u^2 = x^4 - 10x^2$

(ii) (α) If $u = 3$ then $v^2 - 9 = x^4 - 10x^2$



$v^2 = x^4 - 10x^2 + 9$

$= (x^2 - 1)(x^2 - 9)$

$= (x-1)(x+1)(x-3)(x+3)$

Since $v^2 \geq 0$ then $(x-1)(x+1)(x-3)(x+3) \geq 0$

$\therefore$ $x \leq -3$ or $-1 \leq x \leq 1$ or $x \geq 3$

Since particle starts at $x = 0$ with $v = 3$ it then moves to the right (positive direction). At $x = 1$ $v = 0$ and $\ddot{x} = -8$ and so particle will then move to the left until it reaches $x = -1$ where $v = 0$ and $\ddot{x} = 8$. This means particle will then move to the right until $v = 0$ again at $x = 1$. Thus particle oscillates between $x = 1$ and $x = -1$ ie within the interval $-1 \leq x \leq 1$

(β) Not SHM since $\ddot{x} \neq -n^2(x - b)$

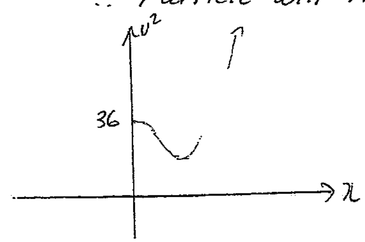
(iii) If $u = 6$ then $v^2 - 36 = x^4 - 10x^2$

$v^2 = x^4 - 10x^2 + 25 + 11$

$= (x^2 - 5)^2 + 11$

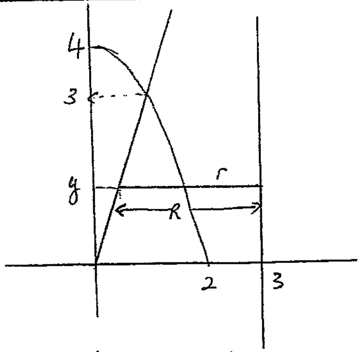
$\therefore$ $v^2 > 0$ for all x in domain

$\therefore$ Particle will never stop



Moves to the right with decreasing velocity until it passes $x = \sqrt{5}$ (with velocity $\sqrt{11}$) and continues to the right with increasing velocity

(b)



$y = 4 - x^2$

$y = 3x$

$x^2 + 3x - 4 = 0$

$(x+4)(x-1) = 0$

$x = -4,\ 1$

$y = -12,\ 3$

Let $A(y)$ be the area of the cross-section at height y

$A(y) = \pi R^2 - \pi r^2$ $R = 3 - \frac{y}{3}$ $y = 3x$

$r = 3 - \sqrt{4-y}$ $y = 4 - x^2$

$R^2 = \left(3 - \frac{y}{3}\right)^2 = 9 - 2y + \frac{y^2}{9}$

$r^2 = \left(3 - \sqrt{4-y}\right)^2 = 9 - 6\sqrt{4-y} + 4 - y$

$A(y) = \pi\left[\left(9 - 2y + \frac{y^2}{9}\right) - \left(9 - 6\sqrt{4-y} + 4 - y\right)\right]$

$= \pi\left(\frac{y^2}{9} - y + 6\sqrt{4-y} - 4\right)$

$V = \int_0^3 \pi\left(\frac{y^2}{9} - y + 6\sqrt{4-y} - 4\right)dy$

$= \pi\left[\frac{y^3}{27} - \frac{y^2}{2} + \frac{6(4-y)^{3/2}}{\frac{3}{2}\times -1} - 4y\right]_0^3$

$= \pi\left\{\left(\frac{27}{27} - \frac{9}{2} - 4\times 1 - 12\right) - \left(-4\times 4^{3/2}\right)\right]$

$= \pi\left\{-\frac{7}{2} - 16 + 32\right\}$

$= \frac{25\pi}{2}$

OR Use cylindrical shells – establish $V = 2\pi\int_0^1 (3-x)3x\,dx + 2\pi\int_1^2 (3-x)(4-x^2)\,dx$ using $\delta V \doteqdot 2\pi r h\,\delta x$ with $r = 3-x$ and $h = 3x$ or $h = 4 - x^2$

Question 8

(a)(i) If x is a factor of both y & z then $y = ax$ and $z = bx$ where a, b are integers

$z - y = bx - ax = (b-a)x$

ie x is a factor of $z - y$

(ii) $2^{2^{k+1}} = 2^{2^k \times 2^1} = \left(2^{2^k}\right)^2$

(iii) $F_n = 2^{2^n} + 1$ $n = 0, 1, 2, \ldots$

Rtp $F_n = F_0F_1\ldots F_{n-1} + 2$ $n \geq 1$

$F_0 = 3$

$n = 1$ $F_1 = 2^{2^1} + 1$

$= 2^2 + 1$

$= 5$

$= 3 + 2$

$= F_0 + 2$

$\therefore$ Proposition is true for $n = 1$

Let k be a value for which proposition is true ($k \geq 1$)

ie $F_k = F_0F_1\ldots F_{k-1} + 2$

Aim to show that proposition is then true for $n = k+1$

ie $F_{k+1} = F_0F_1\ldots F_k + 2$

$F_{k+1} = 2^{2^{k+1}} + 1$

$= \left(2^{2^k}\right)^2 + 1$

$= (F_k - 1)^2 + 1$

$= F_k^2 - 2F_k + 1 + 1$

$= F_k(F_k - 2) + 2$

$= F_k(F_0F_1\ldots F_{k-1}) + 2$ (by inductive hypothesis)

$= F_0F_1\ldots F_{k-1}F_k + 2$

which is of the required form

$\therefore$ If proposition is true for $n = k$ it is also true for $n = k+1$. Since it is true for $n = 1$ it is also true for $n = 1+1 = 2$ and hence by induction it is true for all positive integers

(iv) If $m < n$ then

$F_n = F_0F_1\ldots F_m\ldots F_{n-1} + 2$

Let k be the hcf of F_n and F_m

$\therefore$ k is a factor of F_n and $F_0F_1\ldots F_m\ldots F_{n-1}$

Hence k is a factor of their difference

ie k is a factor of 2

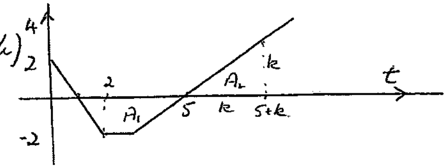
ie $k = 1$ or 2

But $F_n = 2^{2^n} + 1$ is odd

$\therefore$ $k \neq 2$

ie $k = 1$

(b)



(i) $F(x) = 0$ when $x = 0, 2, 5+k$ where k is such that $A_1 = A_2$

$A_1 = \frac{1}{2}(1+3)\times 2 = 4$

$A_2 = \frac{1}{2}k^2$

$\therefore$ $k^2 = 8$

$k = \sqrt{8} = 2\sqrt{2}$

ie $x = 0,\ 2,\ 5 + 2\sqrt{2}$

(ii) Stationary points at $x = 1$ and $x = 5$

$F(1) = \frac{1}{2}\times 1\times 2 = 1$ $F(5) = -4$

$\therefore$ $(1,1)$ and $(5,-4)$ are stationary points