

Trial Higher School Certificate Examination

2001



Mathematics

Extension 2

Time Allowed: 3 hours
(Plus 5 minutes reading time)

Directions to Candidates

- All 8 questions may be attempted.
- Begin each question on a new page.
- All necessary working must be shown.
- All questions are of equal value.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Question 1 (15 Marks) – Start A New Page**Marks**

a) Find $\int \sin^6 x \cos^3 x \, dx$

3

b) Use completion of squares and the table of standard integrals to find

2

$$\int \frac{dx}{\sqrt{x^2 + 8x + 17}}$$

c) Use the substitution $t = \tan \frac{\theta}{2}$ to find the exact value of

5

$$\int_0^{\frac{\pi}{3}} \frac{d\theta}{5 + 3\sin\theta + 4\cos\theta}$$

d) $I_n = \int_0^1 x^n e^{-x} \, dx$

5

(i) Show that $I_n = -\frac{1}{e} + nI_{n-1}$ (ii) Hence, or otherwise, find the exact value of $\int_0^1 x^4 e^{-x} \, dx$

Question 2 (15 Marks) – Start A New Page

Marks

- a) (i) Find $\sqrt{8-6i}$ in the form $a+ib$ where a, b , are real and $a > 0$.
(ii) Hence solve $2z^2 + (1-3i)z - 2 = 0$ giving answers in the form $c+id$ where c, d are real.

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- b) (i) Sketch the arc of a circle of all points z such that

$$\arg\left(\frac{z+1}{z-3}\right) = -\frac{\pi}{3}$$



- (ii) Find the centre and radius of the circle.

5

- c) (i) Prove that $z\bar{z} = |z|^2$
(ii) Suppose that z_1, z_2 and z_3 are three complex numbers of modulus 1 such that $z_1 + z_2 + z_3 = 0$. If z is a complex number of modulus 3 prove that

6

(α) $|z - z_1|^2 = 10 - (z\bar{z}_1 + \bar{z}z_1)$

(β) $|z - z_1|^2 + |z - z_2|^2 + |z - z_3|^2 = 30$

Question 3 (15 Marks) – Start A New Page

Marks

- a) Find a, b if $(x-1)^2$ is a factor of $P(x) = x^5 + 2x^4 + ax^3 + bx^2$.
b) (i) Prove that if the polynomial $Q(x)$ has a zero of multiplicity m then the polynomial $Q'(x)$ has the same zero of multiplicity $m-1$.

3

8

- (ii) Show that the polynomial $H(x) = x^n + px - q$ has a multiple root if

$$\left(\frac{p}{n}\right)^n + \left(\frac{q}{n-1}\right)^{n-1} = 0$$

and find this root.

- c) The cubic equation $x^3 + px^2 + q = 0$ where p, q are real numbers has roots α, β and γ .
The equation $x^3 + ax^2 + bx + c = 0$ has roots α^2, β^2 and γ^2 .
Find a, b, c in terms of p and q .

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Question 4 (15 Marks) – Start A New Page

Marks

- a) Draw a neat sketch of the ellipse

3

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

On your sketch (which should be at least one third of a page) you must clearly show the x and y intercepts, the coordinates of the foci and the equations of the directrices.

- b) Consider the ellipse, E , with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a > b$)

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- (i) Show that the equation of the tangent to the ellipse, E , at the point $P(a \cos \theta, b \sin \theta)$ is $bx \cos \theta + ay \sin \theta = ab$.
- (ii) Find the equation of the normal to E at P .
- (iii) The tangent (in (i)) and normal (in (ii)) cut the y -axis at A and B respectively. Find the coordinates of A and B .
- (iv) Show that a focus, S , lies on the circumference of the circle which has AB as diameter (for each choice of P).

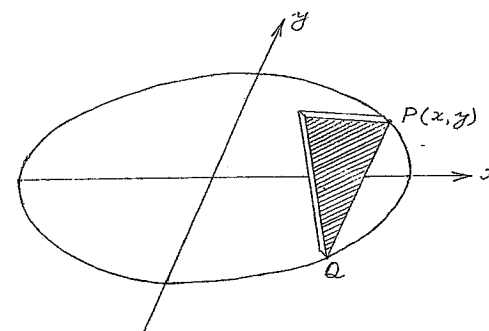
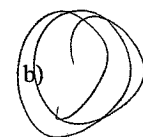
Question 5 (15 Marks) – Start A New Page

Marks

- a) The vertices of $\triangle ABC$ are $A(0, 2)$, $B(1, 1)$ and $C(-1, 1)$. H is the point $(0, y)$ where $1 < y < 2$. The line through H parallel to the x -axis cuts AB and AC at X and Y respectively.

6

- (i) Find the length of XY as a function of y .
- (ii) Hence or otherwise find the volume of the solid formed by rotating $\triangle ABC$ about the x -axis through one complete revolution.



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The base of a particular solid is an ellipse with major and minor axes of 10 cm and 8 cm respectively. Every cross-section perpendicular to the major axis is an equilateral triangle one side of which lies in the base of the solid as shown above.

- (i) Show that the cross-sectional area shaded above is $A(y) = y^2 \sqrt{3}$.
- (ii) Hence find the volume of the slice of thickness Δx (as shown) as a function of x .
- (iii) Find the volume of the solid.
- c) Find the number of different ways of seating n people around 2 circular tables if there are to be k people at one table and the remainder at the other table.

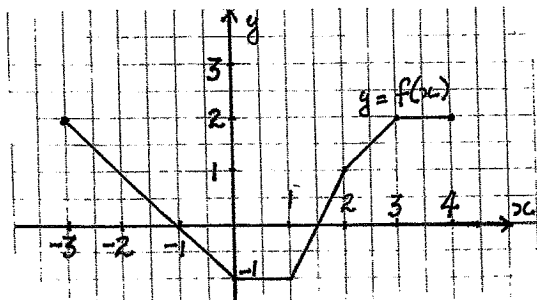
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Question 6 (15 Marks) – Start A New Page

Marks

a)

8



The graph of $y = f(x)$ $-3 \leq x \leq 4$ is shown.

Draw neat sketches of the graphs of the following on separate diagrams.

Each sketch should be approximately one third of a page and should clearly show all important features, including endpoints and discontinuities.

(i) $y = |f(x)|$

(ii) $y^2 = f(x)$

(iii) $y = \frac{1}{f(x)}$

(iv) $y = f^{-1}(x)$

b) For the function in (a) find $\int_{-3}^4 f(x) dx$.

2

(c) Find the equation of the tangent to the curve $x^3 + y^3 - 3xy = 3$ at the point (2, 1).

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(d) (i) Express 42000 as the product of powers of prime numbers.

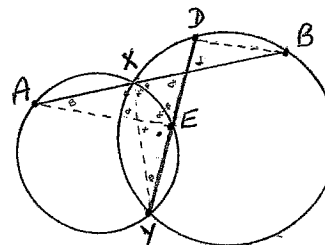
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(ii) Hence or otherwise find the number of factors of 42000.

Question 7 (15 Marks) – Start A New Page

Marks

(a)

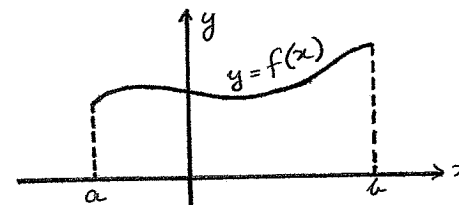


The two circles intersect at X and Y.
AXB and DEY are straight lines.

3

Copy the diagram into your booklet and prove that AE is parallel to DB.

b)



For $y = f(x)$ $a \leq x \leq b$, L, the length of the curve, is given by

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$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Using this formula find the exact value of the length of the curve $y = \frac{1}{2}(e^x + e^{-x})$ from $x = 0$ to $x = 1$.

(c)

A positive integer is said to be blue if no two adjacent digits are the same, so that 7, 30, 242, 695 are examples of blue integers.

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Let $B(n)$ be the number of n -digit blue integers,
 $C(n)$ be the number of even n -digit blue integers and
 $D(n)$ be the number of odd n -digit blue integers.

(i) Explain why $B(n) = 9^n$.

(ii) Explain why $C(k+1) = 4C(k) + 5D(k)$ (k is a positive integer).

(iii) Using induction prove that $C(n) = \frac{9^n + (-1)^n}{2}$

(iv) Hence, or otherwise, find $D(5)$, the number of odd 5-digit blue integers.

Question 8 (15 Marks) - Start A New Page

Marks

a) A particle of mass m kg is dropped from rest in a medium which causes a resistance of mkv .

8

(i) Find the terminal velocity V_T .

(ii) Find the time taken to reach a velocity of $\frac{1}{2}V_T$.

(iii) Find the distance travelled in this time.

b) A mass of 1 kg moves in a straight line from the positive x -axis towards the origin. When its displacement from the origin is x metres it experiences a force of

7

$$-\frac{k}{x^2} N \text{ (ie directed towards the origin).}$$

If the mass starts from rest with displacement p metres, find the time required to reach the origin.

Question 1

$$\begin{aligned} \text{(a)} \int \sin^6 x \cos^3 x \, dx &= \int \sin^6 x \cdot \cos^2 x \cdot \cos x \, dx \\ &= \int \sin^6 x (1 - \sin^2 x) \cos x \, dx \\ &= \int (\sin^6 x - \sin^8 x) \cos x \, dx \\ &= \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + C \end{aligned}$$

$$\begin{aligned} \text{(b)} \int \frac{dx}{\sqrt{x^2 + 8x + 17}} &= \int \frac{dx}{\sqrt{(x+4)^2 + 1}} \\ &= \log_e \left\{ x+4 + \sqrt{(x+4)^2 + 1} \right\} \\ &= \log_e \left\{ x+4 + \sqrt{x^2 + 8x + 17} \right\} \end{aligned}$$

$$\begin{aligned} \text{(c)} \int_0^{\frac{\pi}{3}} \frac{d\theta}{5 + 3\sin\theta + 4\cos\theta} & \quad t = \tan \frac{\theta}{2} \\ & \quad \theta = 2 \tan^{-1} t \\ & \quad d\theta = \frac{2}{1+t^2} dt \\ & \quad \text{When } \theta = 0 \quad t = 0 \\ & \quad \theta = \frac{\pi}{3} \quad t = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} \\ & \quad 5 + 3\sin\theta + 4\cos\theta \\ & \quad = 5 + 3 \times \frac{2t}{1+t^2} + \frac{4(1-t^2)}{1+t^2} \\ & \quad = \frac{5 + 6t + 4 - 4t^2}{1+t^2} \\ & \quad = \frac{t^2 + 6t + 9}{1+t^2} \\ & \quad = \frac{(t+3)^2}{1+t^2} \\ & \quad = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{\frac{(t+3)^2}{1+t^2}} \times \frac{2}{1+t^2} dt \\ & \quad = \int_0^{\frac{1}{\sqrt{3}}} \frac{2}{(t+3)^2} dt \\ & \quad = 2 \left[\frac{(t+3)^{-1}}{-1} \right]_0^{\frac{1}{\sqrt{3}}} \\ & \quad = 2 \left[\frac{-1}{t+3} \right]_0^{\frac{1}{\sqrt{3}}} \\ & \quad = -2 \left(\frac{1}{\frac{1}{\sqrt{3}} + 3} - \frac{1}{3} \right) \\ & \quad = 2 \left(\frac{1}{3} - \frac{\sqrt{3}}{3\sqrt{3} + 1} \times \frac{3\sqrt{3} - 1}{3\sqrt{3} - 1} \right) \\ & \quad = 2 \left(\frac{1}{3} - \frac{9 - \sqrt{3}}{26} \right) \\ & \quad = \frac{2}{3} - \frac{9 - \sqrt{3}}{13} = \frac{26 - 27 + 3\sqrt{3}}{39} = \frac{3\sqrt{3} - 1}{39} \end{aligned}$$

$$(d) I_n = \int_0^1 x^n e^{-x} dx$$

$$(i) I_n = \int_0^1 x^n \cdot \frac{d(e^{-x})}{dx} dx$$

$$= [-e^{-x} \cdot x^n]_0^1 - \int_0^1 n x^{n-1} (-e^{-x}) dx$$

$$= (-e^{-1} \cdot 1 - 0) + n \int_0^1 x^{n-1} e^{-x} dx$$

$$= -\frac{1}{e} + n I_{n-1}$$

$$(ii) \int_0^1 x^4 e^{-x} dx = I_4$$

$$I_4 = -\frac{1}{e} + 4I_3$$

$$= -\frac{1}{e} + 4\left(-\frac{1}{e} + 3I_2\right)$$

$$= -\frac{5}{e} + 12\left(-\frac{1}{e} + 2I_1\right)$$

$$= -\frac{17}{e} + 24\left(-\frac{1}{e} + I_0\right)$$

$$= -\frac{41}{e} + 24I_0$$

$$I_0 = \int_0^1 e^{-x} dx$$

$$= [-e^{-x}]_0^1$$

$$= -e^{-1} + e^0$$

$$= 1 - \frac{1}{e}$$

$$I_4 = -\frac{41}{e} + 24\left(1 - \frac{1}{e}\right)$$

$$= 24 - \frac{65}{e}$$

$$(a) (i) \sqrt{8-6i} = a+ib \quad a, b \in \mathbb{R}, a > 0$$

$$8-6i = (a+ib)^2$$

$$= a^2 - b^2 + 2iab$$

$$\therefore a^2 - b^2 = 8 \quad (1)$$

$$2ab = -6$$

$$ab = -3 \quad (2)$$

$$\text{From (2) } b = -\frac{3}{a}$$

Subst in (1)

$$a^2 - \frac{9}{a^2} = 8$$

$$a^4 - 8a^2 - 9 = 0$$

$$(a^2 - 9)(a^2 + 1) = 0$$

$$(a+3)(a-3)(a^2+1) = 0$$

$$a = -3, 3$$

But $a > 0$

$$\therefore a = 3$$

$$\therefore b = -\frac{3}{3} = -1$$

$$\sqrt{8-6i} = 3-i$$

$$(ii) 2z^2 + (-3i)z - 2 = 0$$

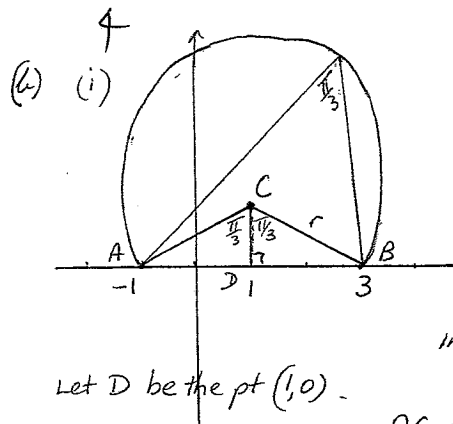
$$z = \frac{-(-3i) \pm \sqrt{(-3i)^2 - 4 \times 2 \times -2}}{2 \times 2}$$

$$= \frac{-1+3i \pm \sqrt{1-6i+9i^2+16}}{4}$$

$$= \frac{-1+3i \pm \sqrt{8-6i}}{4}$$

$$= \frac{-1+3i \pm (3-i)}{4} = \frac{2+2i}{4}, \frac{-4+4i}{4} = \frac{1+i}{2}, -1+i$$

$$= \frac{1}{2} + \frac{1}{2}i, -1+i$$



$$\arg(z+1) - \arg(z-3) = -\frac{\pi}{3}$$

$$\arg(z-3) - \arg(z+1) = \frac{\pi}{3}$$

(ii) Let C be the centre of circle
C lies on perp bisector of
interval joining A(-1,0) and B(3,0)
ie C lies on $x=1$

Let D be the pt (1,0).

$$AC = BC = r$$

$$\angle BCA = \frac{2\pi}{3} \quad (\text{angle at centre = twice angle at circumf standing on same arc})$$

$$\therefore \angle BCD = \angle ACD = \frac{\pi}{3}$$

$$\frac{BD}{CD} = \tan \frac{\pi}{3}$$

$$\frac{BD}{r} = \sin \frac{\pi}{3}$$

$$\frac{2}{CD} = \sqrt{3}$$

$$\frac{2}{r} = \frac{\sqrt{3}}{2}$$

$$CD = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$r = \frac{4}{\sqrt{3}}$$

\therefore Centre is $(1, \frac{2\sqrt{3}}{3})$

$$\text{Radius} = \frac{4\sqrt{3}}{3}$$

(c) (i) Let $z = x+iy$

$$|z| = \sqrt{x^2+y^2}$$

$$z\bar{z} = (x+iy)(x-iy)$$

$$= x^2 - i^2y^2$$

$$= x^2 + y^2$$

$$= |z|^2$$

(ii) $|z_1| = |z_2| = |z_3| = 1$
 $z_1 + z_2 + z_3 = 0$
 $|z_1| = 3$

(2) $|z-z_1|^2 = (z-z_1)(\overline{z-z_1})$
 $= (z-z_1)(\bar{z}-\bar{z}_1)$
 $= z\bar{z} - z\bar{z}_1 - \bar{z}z_1 + z_1\bar{z}_1$
 $= z\bar{z} + z_1\bar{z}_1 - (z\bar{z}_1 + \bar{z}z_1)$
 $= |z|^2 + |z_1|^2 - (z\bar{z}_1 + \bar{z}z_1)$
 $= 9 + 1 - (z\bar{z}_1 + \bar{z}z_1)$
 $= 10 - (z\bar{z}_1 + \bar{z}z_1)$

(3) $|z-z_1|^2 + |z-z_2|^2 + |z-z_3|^2$
 $= 10 - (z\bar{z}_1 + \bar{z}z_1) + 10 - (z\bar{z}_2 + \bar{z}z_2) + 10 - (z\bar{z}_3 + \bar{z}z_3)$
 $= 30 - (z(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) + \bar{z}(z_1 + z_2 + z_3))$
 $= 30 - (z(\overline{z_1 + z_2 + z_3}) + \bar{z} \times 0)$
 $= 30 - z \times 0$
 $= 30$

QUESTION 3:

[3]

(a) $P(x) = x^5 + 2x^4 + ax^3 + bx^2$
 $P(1) = 1 + 2 + a + b = 0$
 $\Rightarrow a + b = -3$ ——— ①

$P'(x) = 5x^4 + 8x^3 + 3ax^2 + 2bx$
 $P'(1) = 5 + 8 + 3a + 2b = 0$
 $\Rightarrow 3a + 2b = -13$ ——— ②

② - 2 × ①: $a = -7$
 $\therefore b = 4$

(b) (i) Let $Q(x) = (x-d)^m \cdot P(x)$ where $P(d) \neq 0$ [8]

Then $Q'(x) = (x-d)^m \cdot P'(x) + P(x) \cdot m(x-d)^{m-1}$
 $= (x-d)^{m-1} [(x-d)P'(x) + mP(x)]$
 $= (x-d)^{m-1} R(x)$ where $R(d) = mP(d) \neq 0$

$\therefore Q'(x)$ has root of multiplicity $(m-1)$

(ii) $H(x) = x^n + px - q$ ——— ①
 $\Rightarrow H'(x) = nx^{n-1} + p$

now $H'(x) = 0 \Rightarrow x^{n-1} = -\frac{p}{n}$

and if there is to be a multiple root this must be it and hence it is a root of $H(x) = 0$

ie $x^n + px - q = 0$ where $x^{n-1} = -\frac{p}{n}$

ie $x^{n-1} + p - \frac{q}{x} = 0$

ie $-\frac{p}{n} + p = \frac{q}{x}$

$\frac{p(n-1)}{n} = \frac{q}{x}$

ie $\left[\frac{p(n-1)}{n}\right]^{n-1} = \frac{q^{n-1}}{x^{n-1}}$

$= \frac{q^{n-1}}{\left(-\frac{p}{n}\right)^{n-1}}$

$= -\frac{ng^{n-1}}{p}$

ie $\frac{p^{n-1} \cdot (n-1)^{n-1}}{n^{n-1}} = -\frac{ng^{n-1}}{p}$

ie $\frac{p^n}{n^n} = -\frac{q^{n-1}}{(n-1)^{n-1}}$

ie $\left(\frac{p}{n}\right)^n + \left(\frac{q}{n-1}\right)^{n-1} = 0$ QED

and the root is $x = \left(-\frac{p}{n}\right)^{\frac{1}{n-1}}$

(c) $x^3 + px^2 + q = 0$ ——— ①
 has roots α, β, γ

ie $\alpha + \beta + \gamma = -p$
 $\alpha\beta + \alpha\gamma + \beta\gamma = 0$
 $\alpha\beta\gamma = -q$

Then polynomial with roots $\alpha^2, \beta^2, \gamma^2$ is

$P(\sqrt{x}) = 0$

ie $(\sqrt{x})^3 + p(\sqrt{x})^2 + q = 0$

$x\sqrt{x} + px + q = 0$

$x\sqrt{x} = -px - q$

$x^3 = p^2x^2 + 2pqx + q^2$

ie $x^3 - p^2x^2 - 2pqx - q^2 = 0$

$$\therefore \left. \begin{aligned} a &= -p \\ b &= -2pq \\ c &= -q^2 \end{aligned} \right\}$$

Question 4

$$(a) \quad \frac{x^2}{25} + \frac{y^2}{16} = 1$$

$$a = 5 \quad b = 4$$

$$b^2 = a^2(1 - e^2)$$

$$16 = 25(1 - e^2)$$

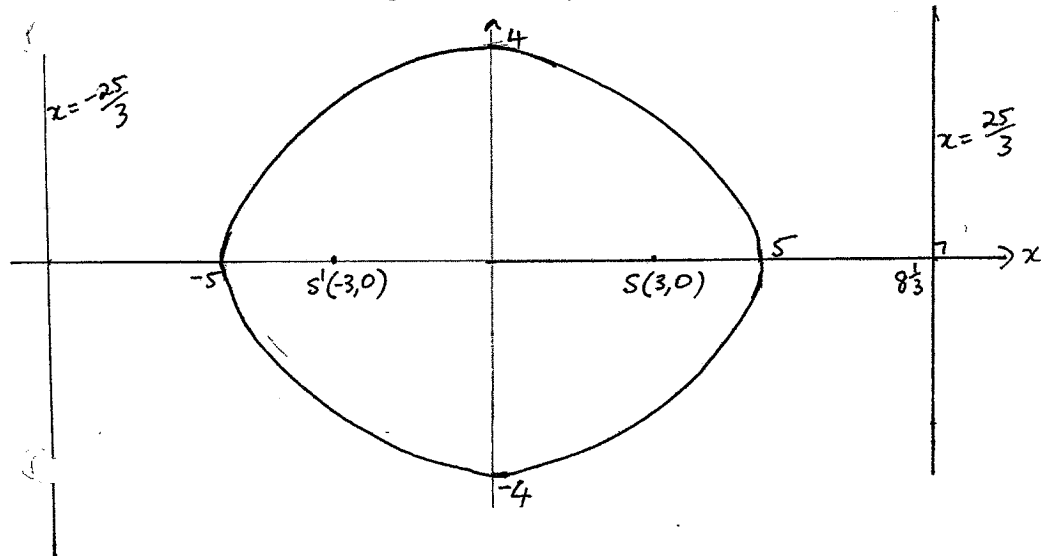
$$e^2 = 1 - \frac{16}{25}$$

$$= \frac{9}{25}$$

$$e = \frac{3}{5} \quad (e > 0)$$

$$ae = 5 \times \frac{3}{5} = 3$$

$$\frac{a}{e} = \frac{5}{\frac{3}{5}} = \frac{25}{3}$$



$$(b) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$(i) \quad \frac{2x}{a^2} + \frac{2y}{b^2} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x}{a^2} \times \frac{b^2}{2y}$$

$$= -\frac{b^2}{a^2} \cdot \frac{x}{y}$$

At $P(a \cos \theta, b \sin \theta)$

$$\frac{dy}{dx} = -\frac{b^2}{a^2} \cdot \frac{a \cos \theta}{b \sin \theta}$$

$$= -\frac{b \cos \theta}{a \sin \theta}$$

\therefore Eqⁿ of tangent at P is

$$y - b \sin \theta = -\frac{b \cos \theta}{a \sin \theta} (x - a \cos \theta)$$

$$ay \sin \theta - ab \sin^2 \theta = -bx \cos \theta + ab \cos^2 \theta$$

$$bx \cos \theta + ay \sin \theta = ab(\sin^2 \theta + \cos^2 \theta)$$

$$= ab$$

(ii) Grad of normal at P = $\frac{a \sin \theta}{b \cos \theta}$

Eqⁿ of normal is

$$y - b \sin \theta = \frac{a \sin \theta}{b \cos \theta} (x - a \cos \theta)$$

$$b \cos \theta y - b^2 \sin \theta \cos \theta = a \sin \theta x - a^2 \sin \theta \cos \theta$$

$$\therefore a \sin \theta x - b \cos \theta y = (a^2 - b^2) \sin \theta \cos \theta$$

(iii) For tangent in (i)

When $x = 0$

$$ay \sin \theta = ab$$

$$y = \frac{ab}{a \sin \theta} = \frac{b}{\sin \theta}$$

\therefore A has coords $(0, \frac{b}{\sin \theta})$

For normal in (ii)

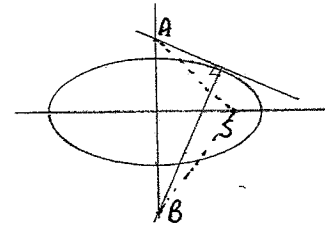
When $x = 0$

$$-b \cos \theta y = (a^2 - b^2) \sin \theta \cos \theta$$

$$y = \frac{a^2 - b^2}{-b} \sin \theta$$

\therefore B has coords $(0, \frac{b^2 - a^2}{b} \sin \theta)$

(iv)



If $S(ae, 0)$ lies on the circumference of the circle with AB as diameter then AB subtends a right angle at S

ie $AS \perp BS$

$$\text{Grad AS} = \frac{\frac{b}{\sin \theta} - 0}{0 - ae} = \frac{-b}{ae \sin \theta}$$

$$\text{Grad BS} = \frac{(\frac{b^2 - a^2}{b}) \sin \theta - 0}{0 - ae}$$

$$= \frac{(b^2 - a^2) \sin \theta}{-abe}$$

$$\text{Grad AS} \times \text{Grad BS} = \frac{-b}{ae \sin \theta} \times \frac{(b^2 - a^2) \sin \theta}{-abe}$$

$$= \frac{-(b^2 - a^2)}{-a^2 e^2}$$

$$= \frac{b^2 - a^2}{a^2 e^2}$$

$$= \frac{-a^2 e^2}{a^2 e^2}$$

$$= -1$$

$$b^2 = a^2(1 - e^2)$$

$$= a^2 - a^2 e^2$$

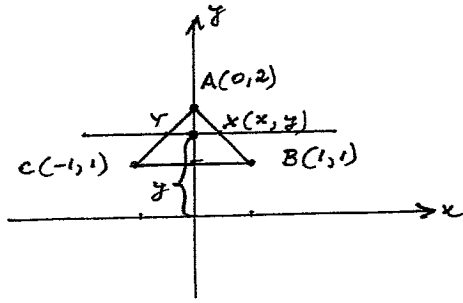
$$b^2 - a^2 = -a^2 e^2$$

$\therefore AS \perp BS$

Hence S lies on circle with AB as diameter

QUESTION 5:

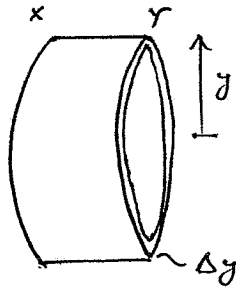
(a) 12



Equation AB: $y = mx + b$
 $y = -x + 2$
 $\therefore x = 2 - y$

(i) By symmetry $XY = 4 - 2y$

(ii) By shells



Volume of shell

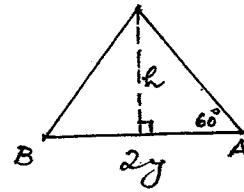
$$\Delta V = 2\pi y (4 - 2y) \Delta y$$

\therefore Volume of solid is

$$\begin{aligned} V &= \lim_{\Delta y \rightarrow 0} \sum_{y=1}^2 2\pi y (4 - 2y) \Delta y \\ &= 2\pi \int_1^2 y(4 - 2y) dy \\ &= 2\pi \int_1^2 (4y - 2y^2) dy \\ &= 2\pi \left[2y^2 - \frac{2}{3}y^3 \right]_1^2 \\ &= 8\pi \end{aligned}$$

(b) Equation of ellipse is $\frac{x^2}{5^2} + \frac{y^2}{4^2} = 1$ ——— (1) 13

(i)



To find h

$$\tan 60^\circ = \frac{h}{y}$$

$$\therefore h = y\sqrt{3}$$

\therefore area of Δ

$$\begin{aligned} A(y) &= \frac{1}{2} \cdot 2y \cdot y\sqrt{3} \\ &= y^2\sqrt{3} \end{aligned}$$

(ii) Volume of slice is

$$\Delta V = y^2\sqrt{3} \Delta x$$

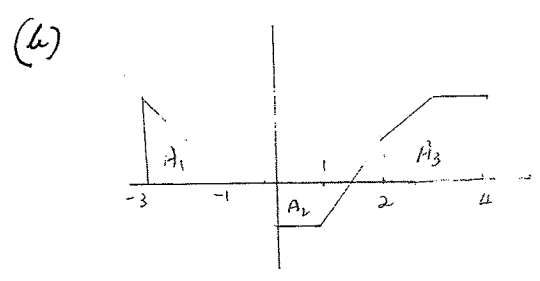
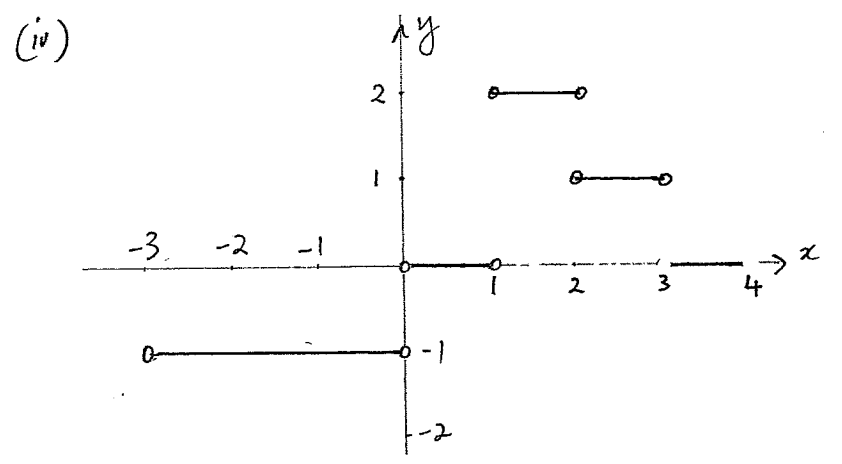
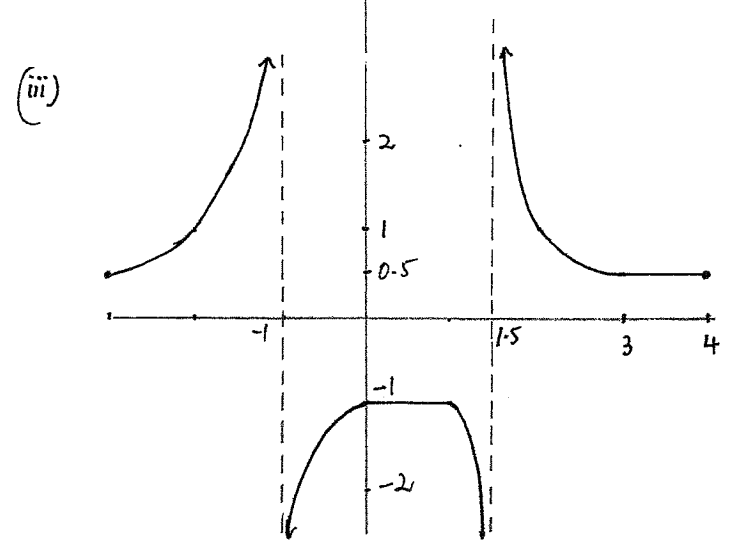
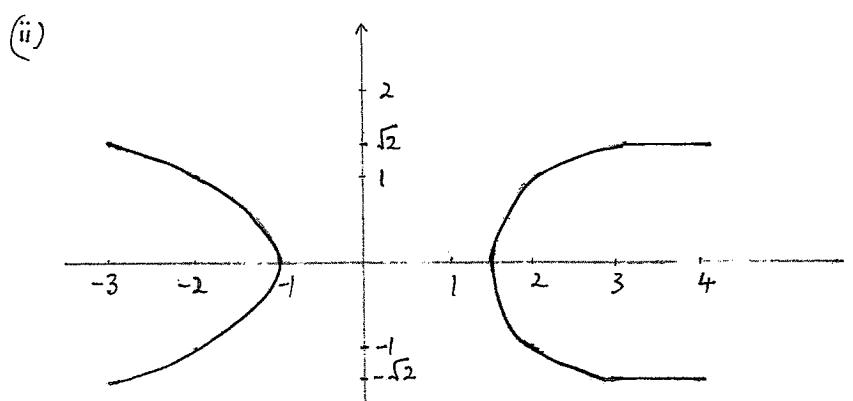
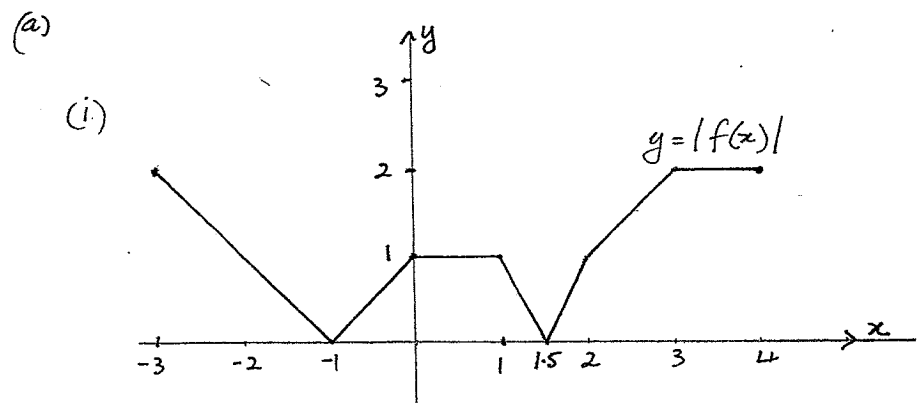
$$= \frac{16}{25} (25 - x^2) \cdot \sqrt{3} \Delta x \text{ from (1)}$$

(iii) Then

$$\begin{aligned} V &= \lim_{\Delta x \rightarrow 0} \sum_{x=-5}^5 \frac{16\sqrt{3}}{25} (25 - x^2) \Delta x \\ &= \frac{32\sqrt{3}}{25} \int_0^5 (25 - x^2) dx \\ &= \frac{32\sqrt{3}}{25} \left[25x - \frac{x^3}{3} \right]_0^5 \\ &= \frac{32\sqrt{3}}{25} \left[125 - \frac{125}{3} \right] \\ &= \frac{32\sqrt{3}}{25} \cdot \frac{250}{3} \\ &= \frac{320\sqrt{3}}{3} \text{ units}^3 \end{aligned}$$

(c) # of seating arrangements = $\binom{n}{k} \times (k-1)! \times (n-k-1)!$

18 Question 6



$$\int_{-3}^4 f(x) dx = A_1 - A_2 + A_3$$

$$A_1 = \frac{1}{2} \times 2 \times 2 = 2$$

$$A_2 = \frac{1}{2} (1 + 2.5) \times 1 = 1.75$$

$$A_3 = \frac{1}{2} \times \frac{1}{2} \times 1 + \frac{1}{2} (1 + 2) \times 1 + 1 \times 2 = 0.25 + 1.5 + 2 = 3.75$$

$$\therefore \int_{-3}^4 f(x) dx = 2 - 1.75 + 3.75 = 4$$

$$(c) \quad x^3 + y^3 - 3xy = 3$$

Differentiate wrt x

$$3x^2 + 3y^2 \frac{dy}{dx} - 3(1 \cdot y + x \frac{dy}{dx}) = 0$$

$$(3x^2 - 3y) + (3y^2 - 3x) \frac{dy}{dx} = 0$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{3(y-x^2)}{3(y^2-x)} \\ &= \frac{y-x^2}{y^2-x} \end{aligned}$$

When $x=2$ $y=1$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1-4}{1-2} \\ &= 3 \end{aligned}$$

\therefore Eqⁿ of tangent at (2,1) is

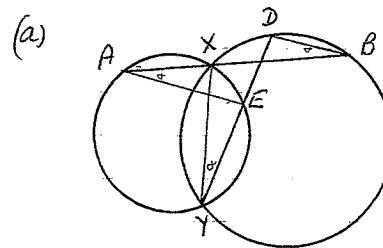
$$\begin{aligned} y-1 &= 3(x-2) \\ y &= 3x-5 \end{aligned}$$

$$\begin{aligned} (d) \quad (i) \quad 42000 &= 7 \times 6 \times 10^3 \\ &= 7 \times 3 \times 2 \times (5 \times 2)^3 \\ &= 2^4 \times 3 \times 5^3 \times 7 \end{aligned}$$

$$(ii) \quad \# \text{ of factors} = 5 \times 2 \times 4 \times 2 = 80$$

(can include 0, 1, 2, 3, or 4 factors of 2; 0 or 1 factor of 3, 0, 1, 2 or 3 factors of 5; 0 or 1 factors of 7)

Question 7



Join AE, DB and XY

$$\hat{XAE} = \hat{XYE} \quad (\text{angles subtended by arc } XE \text{ at circumf are equal})$$

$$\hat{XAD} = \hat{XBD} \quad (\text{angles subtended by arc } XD \text{ at circumf are equal})$$

$$\hat{XAD} = \hat{XYE} \quad (\text{same angle})$$

$$\therefore \hat{XAE} = \hat{XBD} \quad (= \hat{XAD})$$

AE \parallel DB (since alt \angle s are equal)

$$(h) \quad y = \frac{1}{2}(e^x + e^{-x})$$

$$\frac{dy}{dx} = \frac{1}{2}(e^x - e^{-x})$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{4}(e^{2x} - 2e^x e^{-x} + e^{-2x})$$

$$= \frac{4 + e^{2x} - 2 + e^{-2x}}{4}$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4}$$

$$= \frac{(e^x + e^{-x})^2}{4}$$

$$\therefore L = \int_0^1 \sqrt{\frac{(e^x + e^{-x})^2}{4}} dx = \frac{1}{2} \{ (e - e^{-1}) - (e^0 - e^0) \}$$

$$= \frac{e}{2} - \frac{1}{2e}$$

$$= \frac{1}{2} [e^x - e^{-x}]_0^1$$

- (c) (i) Starting from LH digit
 # of choices for 1st digit = 9 (cannot choose 0)
 # of choices for 2nd ~~and~~ = 9 (cannot choose previous digit but now 0 can be chosen)

of choices for each subsequent digit = 9 (")

$$\therefore B(n) = 9 \times 9 \times \dots \times 9 \quad (n \text{ times}) \\ = 9^n$$

- (ii) A blue integer with $k+1$ digits can be formed by attaching a digit at the RH end of a k -digit blue integer. If the k -digit integer is even then there are 4 choices for the final digit.

\therefore There are $4 \times C(k)$ even blue integers of this type. If the k -digit integer is odd then there are 5 choices for this final digit.

\therefore There are $5 \times D(k)$ blue integers of this type. Hence,

$$C(k+1) = 4 \times C(k) + 5 \times D(k) \\ (\text{since the } k\text{-digit integers must be either even or odd})$$

- (iii) Aim to prove that $C(n) = \frac{9^n + (-1)^n}{2}$

For $n=1$

The even blue 1-digit integers are 2, 4, 6, 8

$$\therefore C(1) = 4 \\ \frac{9^1 + (-1)^1}{2} = \frac{9-1}{2} = 4$$

\therefore Proposition is true for $n=1$

Let k be a value for which proposition is true

$$\text{i.e. } C(k) = \frac{9^k + (-1)^k}{2}$$

Aim to show that proposition is true for $n=k+1$ whenever it is true for $n=k$

$$\text{i.e. } C(k+1) = \frac{9^{k+1} + (-1)^{k+1}}{2}$$

$$\begin{aligned} \text{Now } C(k+1) &= 4C(k) + 5D(k) \\ &= 4C(k) + 5(B(k) - C(k)) \\ &= 5B(k) - C(k) \\ &= 5 \times 9^k - \frac{9^k + (-1)^k}{2} \quad (\text{by inductive assumption}) \\ &= \frac{10 \times 9^k - 9^k - (-1)^k}{2} \\ &= \frac{(10-1) \times 9^k + (-1)(-1)^k}{2} \\ &= \frac{9^{k+1} + (-1)^{k+1}}{2} \end{aligned}$$

which is of the required form

\therefore Proposition is true for $n=k+1$ whenever it is true for $n=k$

Since it is true for $n=1$ it is also true for $n=2$ and hence, by induction, it is true for all positive integers

$$\begin{aligned} \text{(iv) } D(5) &= B(5) - C(5) \\ &= 9^5 - \frac{9^5 + (-1)^5}{2} \\ &= \frac{2 \times 9^5 - 9^5 - (-1)}{2} \\ &= \frac{9^5 + 1}{2} \\ &= 29525 \end{aligned}$$

$$\begin{aligned}
 T &= \sqrt{\frac{p}{8k}} \int_p^0 (p-2x)(px-x^2)^{\frac{1}{2}} - \frac{p}{\sqrt{px-x^2}} dx \\
 &= \sqrt{\frac{p}{8k}} \left[2(px-x^2)^{\frac{3}{2}} \right]_p^0 - p \sqrt{\frac{p}{8k}} \int_p^0 \frac{1}{\sqrt{\frac{p^2}{4} - (x-\frac{p}{2})^2}} dx \\
 &= 0 - \sqrt{\frac{p^3}{8k}} \left[\sin^{-1} \left(\frac{x-\frac{p}{2}}{\frac{p}{2}} \right) \right]_p^0 \\
 &= -\sqrt{\frac{p^3}{8k}} (\sin^{-1}(-1) - \sin^{-1}(1)) \\
 &= \sqrt{\frac{p^3}{8k}} (\sin^{-1}(1) - \sin^{-1}(-1)) \\
 &= \sqrt{\frac{p^3}{8k}} \left(\frac{\pi}{2} + \frac{\pi}{2} \right) \\
 &= \frac{\pi \sqrt{p^3}}{\sqrt{8k}}
 \end{aligned}$$

OR $T = - \int_p^0 \sqrt{\frac{p}{2k}} \sqrt{\frac{x}{p-x}} dx$

Let $x = p \sin^2 \theta$ $\theta = \sin^{-1} \sqrt{\frac{x}{p}}$
 $dx = 2p \sin \theta \cos \theta d\theta$
 $p-x = p - p \sin^2 \theta$
 $= p(1 - \sin^2 \theta)$
 $= p \cos^2 \theta$

When $x = p$ $\theta = \frac{\pi}{2}$
 $x = 0$ $\theta = 0$

$$\begin{aligned}
 \therefore T &= - \int_{\frac{\pi}{2}}^0 \sqrt{\frac{p}{2k}} \sqrt{\frac{p \sin^2 \theta}{p \cos^2 \theta}} \cdot 2p \sin \theta \cos \theta d\theta \\
 &= \int_0^{\frac{\pi}{2}} \sqrt{\frac{p}{2k}} \cdot 2p \cdot \frac{\sin \theta}{\cos \theta} \cdot \sin \theta \cos \theta d\theta \\
 &= p \sqrt{\frac{p}{2k}} \int_0^{\frac{\pi}{2}} 2 \sin^2 \theta d\theta \\
 &= p \sqrt{\frac{p}{2k}} \int_0^{\frac{\pi}{2}} 1 - \cos 2\theta d\theta
 \end{aligned}$$

$$\begin{aligned}
 T &= p \sqrt{\frac{p}{2k}} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{2}} \\
 &= \sqrt{\frac{p^3}{2k}} \left\{ \left(\frac{\pi}{2} - 0 \right) - (0 - 0) \right\} \\
 &= \pi \sqrt{\frac{p^3}{8k}}
 \end{aligned}$$

OR (a) (iii)

$$v \frac{dv}{dx} = g - kv$$

$$\frac{dv}{dx} = \frac{g - kv}{v}$$

$$\frac{dx}{dv} = \frac{v}{g - kv}$$

$$= -\frac{1}{k} \frac{(g - kv - g)}{g - kv}$$

$$= -\frac{1}{k} \left(1 - \frac{g}{g - kv}\right)$$

$$X = \int_0^{\frac{1}{2}v_T} -\frac{1}{k} \left(1 - \frac{g}{g - kv}\right) dv$$

$$= -\frac{1}{k} \left[v + \frac{g}{k} \ln(g - kv) \right]_0^{\frac{1}{2}v_T}$$

$$= -\frac{1}{k} \left\{ \frac{v_T}{2} + \frac{g}{k} \ln\left(g - k \cdot \frac{v_T}{2}\right) - \frac{g}{k} \ln g \right\}$$

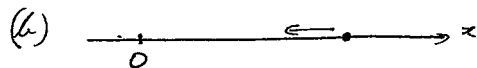
$$= -\frac{1}{k} \left\{ \frac{g}{2k} + \frac{g}{k} \ln\left(g - k \cdot \frac{g}{2k}\right) - \frac{g}{k} \ln g \right\}$$

$$= -\frac{1}{k} \left\{ \frac{g}{2k} + \frac{g}{k} \ln\left(\frac{g}{2}\right) - \frac{g}{k} \ln g \right\}$$

$$= -\frac{g}{2k^2} + \frac{g}{k^2} \ln\left(g \div \frac{g}{2}\right)$$

$$= \frac{g}{k^2} \left(\ln 2 - \frac{1}{2}\right)$$

$$= \frac{g}{2k^2} (2 \ln 2 - 1)$$



$$F = -\frac{k}{x^2}$$

$$m\ddot{x} = -\frac{k}{x^2}$$

$$\ddot{x} = -\frac{k}{x^2} \quad (m=1)$$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = -\frac{k}{x^2}$$

$$\frac{1}{2}v^2 = \frac{k}{x} + C$$

When $t=0$ $v=0$ $x=p$

$$0 = \frac{k}{p} + C$$

$$C = -\frac{k}{p}$$

$$\therefore \frac{1}{2}v^2 = \frac{k}{x} - \frac{k}{p}$$

$$v^2 = 2k \left(\frac{p-x}{px}\right)$$

$$= \frac{2k}{p} \left(\frac{p-x}{x}\right)$$

$$v = -\sqrt{\frac{2k}{p}} \frac{\sqrt{p-x}}{\sqrt{x}} \quad (\text{since directed towards } 0 \text{ from +ve } x \text{ axis})$$

$$\frac{dx}{dt} = -\sqrt{\frac{2k}{p}} \frac{\sqrt{p-x}}{\sqrt{x}}$$

$$\frac{dt}{dx} = -\sqrt{\frac{p}{2k}} \frac{\sqrt{x}}{\sqrt{p-x}}$$

Let T be the time taken to reach $x=0$ from $x=p$

$$T = \int_p^0 -\sqrt{\frac{p}{2k}} \cdot \frac{\sqrt{x}}{\sqrt{p-x}} dx$$

$$= -\sqrt{\frac{p}{2k}} \int_p^0 \frac{x}{\sqrt{x(p-x)}} dx$$

$$= -\sqrt{\frac{p}{2k}} \int_p^0 \frac{-2x+p-p}{\sqrt{px-x^2}} dx$$