St George Girls High School

Year 12

Common Test 3

May 2010

Mathematics

General Instructions

- Time allowed – 70 minutes
- Write using blue or black pen
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.
- Write on one side of the page only.
- Start each question on a new page.

Total marks – 60

- Attempt Questions 1 – 5
- All questions are of equal value

Question 1 – Start a New Page – (12 marks)

Marks

a) (i) Evaluate $\log_e 4$ correct to 3 decimal places. 1

(ii) Evaluate e^3 correct to 3 significant figures. 1

b) (i) Convert $63°$ to radians correct to 1 decimal place. 1

(ii) Convert $\frac{3\pi}{5}$ radians to degrees. 1

c) Differentiate

(i) $e^{(5x-7)}$ 1

(ii) $3\log_e x$ 1

(iii) x^3e^{3x} 2

d) Simplify

(i) $(4 \times 7^{3x}) \times (5 \times 7^{5x})$ 1

(ii) $\frac{7^{1+x}}{7^{1-x}}$ 1

e) Solve for x: 2

$$3x\log_a 64 - 3x\log_a 4 = 5\log_a 2$$

Question 2 – Start a New Page – (12 marks)

Marks

a) Find $\frac{dy}{dx}$ for the following:

(i) $y = \frac{x^2}{e^{2x}}$ 2

(ii) $y = 3\log_e(x^2 - 3x + 6)$ 2

(iii) $y = \log_e\left(\frac{3x+1}{2x-5}\right)$ 2

(iv) $y = (e^{3x} - 1)^3$ 2

b) For the curve $y = e^x - x$ show that there is a stationery point at $(0, 1)$ and determine its nature. 2

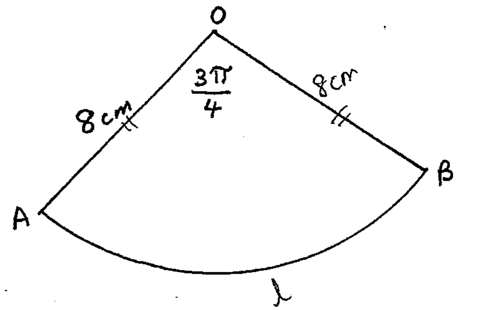
c) Find the primitive of e^{-5x} 1

d) Find the exact value of $\sin\frac{\pi}{6}$ 1

Question 3 – Start a New Page – (12 marks)

Marks

a) A piece of paper is cut into the shape of a sector of a circle (as shown in the diagram). The radius of the sector is 8 cm and the angle at the centre is $\frac{3\pi}{4}$ radians.



(i) Find the length of the arc AB. 1

(ii) Find the area of the sector AOB. 1

(iii) The straight edges of the sector are placed together to form a cone.

(α) Show that the base of this cone has a radius of 3 cm. 1

(β) Show that the cone has a perpendicular height of $\sqrt{55}$ cm. 1

b) Solve $2\cos x = \sqrt{3}$ for $0 \le x \le 2\pi$ 2

c) Evaluate $\int_0^1 e^{2x} + 1 \; dx$ 2

d) Find:

(i) $\int \frac{dx}{2x+3}$ 2

(ii) $\int \frac{2x^3 - 3x - 4}{x^2} \; dx$ 2

Question 4 – Start a New Page – (12 marks)

Marks

a) Using the function $y = 3\sin 2x$

(i) State the period and amplitude 2

(ii) Sketch the graph for $0 \leq x \leq 2\pi$ 3

b) The curve $y = \dfrac{2}{x}$ and the line $x + 2y - 5 = 0$ intersect at A and B.

(i) Find the x-coordinates of A and B. 2

(ii) Find the area between the curves $y = \dfrac{2}{x}$ and $x + 2y - 5 = 0$ 3

c) Differentiate e^{-x^2} and hence find the primitive of $2xe^{-x^2}$ 2

Question 5 – Start a New Page – (12 marks)

Marks

a) For the curve $y = e^{3x-6}$

(i) Explain why every tangent to the curve has a positive gradient. 2

(ii) Find the coordinates of the y-intercept of the curve. 2

(iii) Find the equation of the tangent to the curve at the y-intercept. 2

b) A solid is formed by rotating the curve $y = \dfrac{1}{x^2}$ about the y-axis between $y = 1$ and $y = 6$. Evaluate the volume of this solid. 3

c) Use Simpson's Rule with five function values to approximate the area between $y = e^{-x^2}$ and the x-axis from $x = 0$ to $x = 4$. Give your answer correct to 3 decimal places. 3

End of Paper

$\frac{12}{12}$

Question 1

ai) $\log_e 4 = \ln 4$
$= 1.386294\ldots$
$= 1.386$ (3dp)

ii) $e^3 = 20.0855\ldots$
$= 20.1$ (3.s.f)

bi) $63° \times \frac{\pi}{180°} = \frac{7\pi}{20}$ Radians
$\cancel{\approx 1.22}$ $= 1.09955\ldots$
$= 1.1$ Radian (1dp)

ii) $\frac{3\pi}{5} \times \frac{180°}{\pi} = 108°$

ci) $\frac{dy}{dx} e^{5x-7} = 5e^{5x-7}$

ii) $\frac{dy}{dx} 3\log_e x = \frac{dy}{dx} 3 . \ln x$
$= 3 . \frac{1}{x}$
$= \frac{3}{x}$.

iii) $\frac{dy}{dx} \overset{u}{x^3} \overset{v}{e^{3x}} = vu' + uv'$ $\quad u = x^3 \quad v = e^{3x}$
$u' = 3x^2 \quad v' = 3e^{3x}$
$= e^{3x} . 3x^2 + x^3 . 3e^{3x}$
$= \cancel{x^2 e^{3x}}\ 3x^2 e^{3x} + 3x^3 e^{3x}$
$= 3x^2 e^{3x}(1+x)$

di) $(4 \times 7^{3x}) \times (5 \times 7^{5x})$
$= 4 \times 7^{3x} \times 5 \times 7^{5x}$
$= 20 \times 7^{3x+5x}$
$= 20 \times 7^{8x}$

ii) $\frac{7^{1+x}}{7^{1-x}} = 7^{1+x-(1-x)}$
$= 7^{1+x-1+x}$
$= 7^{2x}$

e) $3x\log_a 64 - 3x\log_a 4 = 5\log_a 2 \rightarrow \quad x = ?$

LHS: $3x\log_a 64 - 3x\log_a 4$
$= 3x\log_a 2^6 - 3x\log_a 2^2$
$= \log_a 2^{6.3x} - \log_a 2^{2.3x}$
$= \log_a 2^{18x} - \log_a 2^{6x}$
$= \log_a \left(\frac{2^{18x}}{2^{6x}}\right)$
$= \log_a \left(2^{18x-6x}\right)$
$= \log_a \left(2^{12x}\right)$
$= 12x\log_a 2$.

As RHS = LHS ; $12x\cancel{\log_a 2} = 5\cancel{\log_a 2}$
$12x = 5$
$\therefore x = \frac{5}{12}$.

Question 2

a i) $\frac{dy}{dx}\left(\frac{x^2}{e^{2x}}\right)^{u}_{v} = \frac{vu' - uv'}{v^2}$ $\quad u = x^2 \quad v = e^{2x}$

$u' = 2x \quad v' = 2e^{2x}$

$= \frac{e^{2x} \cdot 2x - x^2 \cdot 2e^{2x}}{(e^{2x})^2 \cdot (e^{2x})}$

$= \frac{2xe^{2x} - 2x^2e^{2x}}{e^{4x}}$

$= \frac{2x\cancel{e^{2x}}(1-x)}{\cancel{e^{2x}} \cdot e^{2x}}$

$= \frac{2x(1-x)}{e^{2x}}$ 2

ii) $\frac{dy}{dx}\left(3\log_e(x^2 - 3x + 6)\right)$ $\quad * \frac{dy}{dx}\log(ax+b) = \frac{a}{ax+b}$

$= 3\frac{dy}{dx}\log_e(x^2 - 3x + 6)$

$= 3 \cdot \frac{2x - 3}{x^2 - 3x + 6}$

$= \frac{3(2x-3)}{x^2 - 3x + 6}$ 2

iii) $\frac{dy}{dx}\log_e\left(\frac{3x+1}{2x-5}\right)$.

As $\ln\left(\frac{3x+1}{2x-5}\right) = \ln(3x+1) - \ln(2x-5)$

$\therefore \frac{dy}{dx}\ln\left(\frac{3x+1}{2x-5}\right) = \frac{dy}{dx}\ln(3x+1) - \ln(2x-5)$

$= \frac{3}{3x+1} - \frac{2}{2x-5}$

$= \frac{3(2x-5) - 2(3x+1)}{(3x+1)(2x-5)}$

$= \frac{\cancel{6x} - 15 - \cancel{6x} - 2}{(3x+1)(2x-5)}$

$= \frac{-17}{(3x+1)(2x-5)}$ 2

iv) $\frac{dy}{dx}(e^{3x} - 1)^3 = 3(e^{3x} - 1)^2 \cdot 3e^{3x}$

$= 9e^{3x}(e^{3x} - 1)^2$ 2

b) $y = e^x - x$.

$y' = e^x - 1$ $\quad$ For stat pt $y' = 0$.

$0 = e^x - 1$

$1 = e^x$

$\therefore x = 0$. $\quad$ When $x = 0$, $y = e^0 - 0$

$= 1 - 0$

$= 1$ $\quad \therefore$ Stat pt at $P(0,1)$

$y'' = e^x$.

$y''(0) = e^0$

$= 1$

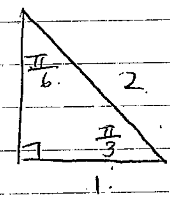
minimum turning point

$> 0 \therefore$ is a min t.p at $(0,1)$ 2

c) $\int e^{-5x}\,dx = -\frac{1}{5} \cdot e^{-5x} + C$

$= \frac{-1}{5e^{5x}} + C$

d) $\sin\frac{\pi}{6} = \frac{1}{2}$



12

Question 3

a) i) length : $l = r\theta^c$

$= 8 \cdot \frac{3\pi}{4}$

$= 6\pi$ cm

ii) Area $= \frac{1}{2}r^2\theta^c$

$= \frac{1}{2} \cdot 64 \cdot \frac{3\pi}{4}$

$\therefore$ area $= 24\pi\, cm^2$

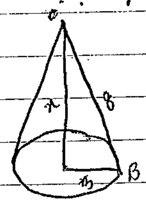
iii) length AB becomes circumference.

α) $C = 2\pi r$.

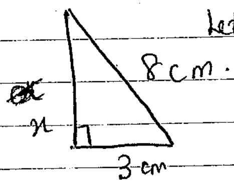
$2\pi r = 6\pi$

$r = \frac{6}{2}$

$\therefore r = 3$ cm.

β) 

Let perpendicular height be x



$8^2 - 3^2 = x^2$.

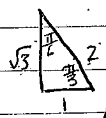
$x^2 = 55$

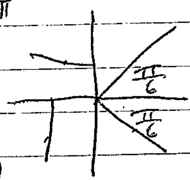
$\therefore x = \pm\sqrt{55}$ cm , as is a length, $x > 0$

$\therefore x = \sqrt{55}$ cm

b) $2\cos x = \sqrt{3}$ $\quad 0 \le x \le 2\pi$

$\cos x = \frac{\sqrt{3}}{2}$





$x = \frac{\pi}{6}, \left(2\pi - \frac{\pi}{6}\right)$

2

$= \frac{\pi}{6}, \frac{11\pi}{6}$ as $0 \le x \le 2\pi$, and $\cos x = \frac{\sqrt{3}}{2} > 0$

c) $\int_0^1 e^{2x} + 1\, dx = \left[\frac{1}{2}e^{2x} + x\right]_0^1$

$= \left[\frac{e^{2x}}{2} + x\right]_0^1$

$= \left(\frac{e^2}{2} + 1\right) - \left(\frac{e^0}{2} + 0\right)$

$= \frac{e^2}{2} + 1 - \frac{1}{2}$

$= \frac{e^2}{2} + \frac{1}{2}$ 2

$= \frac{1}{2}(e^2 + 1)$

d) i) $\int \frac{dx}{2x+3} = \int \frac{1}{2x+3}\,dx$ $\quad * \int \frac{1}{ax+b}\,dx = \frac{1}{a}\log(ax+b) + C$

$= \frac{1}{2}\log(2x+3) + C$

$= \frac{\log(2x+3)}{2} + C$ 2

ii) $\int \frac{2x^3 - 3x - 4}{x^2}\,dx$

$= \int \frac{2x^{\cancel{3}}}{\cancel{x^2}} - \frac{3x}{x^{\cancel{2}}} - \frac{4}{x^2}\,dx$

$= \int 2x - \frac{3}{x} - \frac{4}{x^2}\,dx$ $\quad \frac{4}{x^2} = 4 \cdot x^{-2} = 4x^{-2}$

$= \frac{2x^2}{2} - 3\log x - \int 4x^{-2}\,dx$

$= x^2 - 3\log x - \frac{-4x^{-1}}{1} + c$ 2

$= x^2 - 3\log x + \frac{4}{x} + c$

Question 4.

ai) $y = 3\sin 2x$

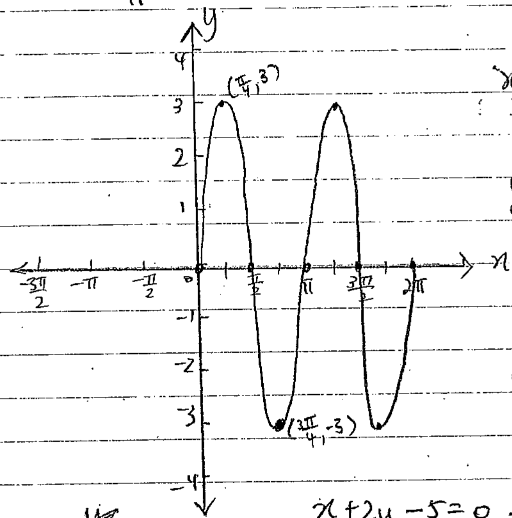
amplitude $= 3$

period $= \frac{2\pi}{2}$

$= \pi$

2

ii)



x scale to small

$y = 3\sin 2x$,

$D: \{0 \leq x \leq 2\pi\}$

3

b) $y_1 = \frac{2}{x}$ ~~y~~

$x + 2y - 5 = 0$.

$2y = 5 - x$

$y_2 = \frac{5-x}{2}$

12

For int, $y^1 = y^2$ $\therefore$ $\frac{2}{x} = \frac{5-x}{2}$

$4 = x(5-x)$

$4 = 5x - x^2$

$x^2 - 5x + 4 = 0$

$p = 4$ $s = -5$

$(-4, -1)$.

$(x-4)(x-1) = 0$ $\therefore$ $\boxed{x = 1, 4}$

2

when $x = 1$,

$y = \frac{2}{1}$

$= 2$

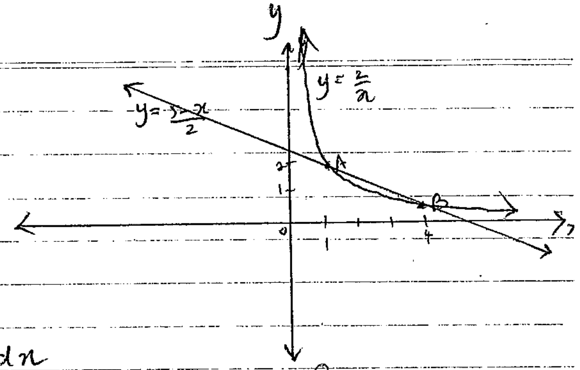
$\therefore A(1, 2)$

when $x = 4$

$y = \frac{2}{4}$

$= \frac{1}{2}$.

$\therefore B\left(4, \frac{1}{2}\right)$



ii) Area $= \int_1^4 y_2 - y_1 \, dx$

$= \int_1^4 \frac{5}{2} - \frac{x}{2} - \frac{2}{x} \, dx$

~~$\int \frac{1}{x} = \log x$~~

$= \left[\frac{5x}{2} - \frac{x^2}{4} - 2\log x\right]_1^4$

$= \left(10 - 4 - 2\log 4\right) - \left(\frac{5}{2} - \frac{1}{4} - 2\log 1\right)$

$= \left(6 - 2\log 4\right) - \left(\frac{9}{4} - 0\right)$

$= 6 - 2\log 4 - \frac{9}{4}$

$= \frac{15}{4} - 2\log 2^2$

$= \left(\frac{15}{4} - 4\log 2\right) u^2$, exact form

3

c) $\frac{dy}{dx} e^{-x^2} = -2xe^{-x^2}$

$-\int 2xe^{-x^2} dx = -e^{-x^2} + c$

2

Question 5.

a i) $y = e^{3x+6}$

$y' = 3e^{3x+6}$, as $e^{3x+6} > 0$, and $3 > 0$

$y' = +ve \times +ve$

$= +ve$

> 0 ∴ all gradients to curve are positive

2

ii) y-int. at $x = 0$.

$y = e^{3x+6}$

$y = e^{0+6}$

$y = e^{6}$ ∴ Coordinates are $(0, e^6)$

2

iii) $y' = 3e^{3x+6}$ at y-intercept $(0, e^6)$

$x = 0$.

$y'(0) = 3e^{3(0)+6}$

$= 3e^6$ ∴ $m = 3e^6$ $P(0, e^6)$

$y - y_1 = m(x - x_1)$

$y - e^6 = 3e^6(x - 0)$

$y - e^6 = 3e^6 x$

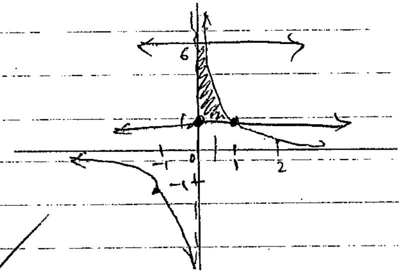
∴ $y = 3e^6 x + e^6$

$y = e^6(3x+1)$ or in general form,

$y - 3e^6 x - e^6 = 0$.

2

b) $y = \dfrac{1}{x^2}$



$V = \pi \int_6^9 x^2 \, dy$.

$= \pi$. $y = \dfrac{1}{x^2}$

$x^2 y = 1$

$\boxed{x^2 = \dfrac{1}{y}}$, $\cancel{x^2 = y^{-1}}$

$\cancel{x = \pm\sqrt{\frac{1}{y}} = y^{-\frac{1}{2}}}$

$\cancel{V = \pi. \int_1^6 y^{-\frac{1}{2}} \, dy}$

$\cancel{= \pi. [2y^{\frac{1}{2}}]_1^6 = \pi. (2.6^{\frac{1}{2}}) - (2.1^{\frac{1}{2}})}$

$V = \pi \int_1^6 \frac{1}{y} \, dy$.

$= \pi. [\log_e y]_1^6$

$= \pi. \log_e 6 - \log_e 1$

$= \pi. \log_e(3 \times 2) - 0$ ∴ Volume $= (\pi \log_e 6) u^3$

3

$\cancel{= \pi. \ln_2 3 + \ln_2 2}$

$\cancel{= (\pi(\ln_2 3 + 1)) u^3}$ $\cancel{\text{Volume} = \pi(\ln_2 3 + 1) u}$

c) Simp. Rule : $A \doteqdot \frac{h}{3}\left[y_0+y_n+4(y_1+y_3)+2(y_2+y_4)\right]$

$y = e^{-x^2}$

x	0	1	2	3	4
y	1	$\frac{1}{e}$	$\frac{1}{e^4}$	$\frac{1}{e^9}$	$\frac{1}{e^{16}}$
	y_0	y_1	y_2	y_3	y_n

$h=1$

$e^{-3^2} = e^{-9} = \frac{1}{e^9}$

3.

$A \doteqdot \frac{1}{3}\left[1+\frac{1}{e^{16}}+4\left(\frac{1}{e}+\frac{1}{e^9}\right)+2\left(\frac{1}{e^4}\right)\right]$

$= 0.8362\ldots$

$\therefore$ Area $= 0.836u^2$ (3dp).