St George Girls High School

Trial Higher School Certificate Examination

2008

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Write your Student Number on every page
- All questions may be attempted.
- Begin each question in a new booklet.
- All necessary working must be shown.
- Marks may be deducted for careless or poorly presented work.
- Board-approved calculators may be used.
- A list of standard integrals is included at the end of this paper.
- The mark allocated for each question is listed at the side of the question.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Question 1 – (15 marks) – Start a new booklet Marks

a) Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\, dx$ 2

b) Find $\int \frac{e^{2x}}{e^x+1}\, dx$ 2

c) If $z = 3 - 3i$ and $w = 1 + i$, express $\frac{z^4}{w^3}$ in the form $a + ib$ 4

d) For the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$

(i) Calculate the eccentricity of the ellipse. 1

(ii) Sketch the ellipse showing the co-ordinates of the foci and the equation of the directrices. 2

e) The equation $2x^3 + 5x - 3 = 0$ has roots α, β and γ.

(i) Find the polynomial equation with roots α^2, β^2 and γ^2. 2

(ii) Evaluate $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$ 2

Question 2 – (15 marks) – Start a new booklet

Marks

a) (i) If α is a double zero of a polynomial $P(x)$, show that α is a single zero of $P'(x)$ **2**

(ii) Find integers m and n such that $(x+1)^2$ is a factor of x^5+2x^2+mx+n **3**

b) (i) Find the real numbers A, B and C such that **2**

$$\frac{2x^2+7x-1}{(x-2)(x^2+x+1)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+x+1}\ dx$$

(ii) Hence find: **3**

$$\int\frac{2x^2+7x-1}{(x-2)(x^2+x+1)}\ dx$$

c) Given $P(a\cos\theta,\ b\sin\theta)$ is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, show that the equation of the tangent and the equation of the normal to the ellipse at P are given by **5**

(i) $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

(ii) $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$

Prove that $OR\times OQ=a^2e^2$ where R and Q are the x intercepts in (i) and (ii) respectively.

Question 3 – (15 marks) – Start a new booklet

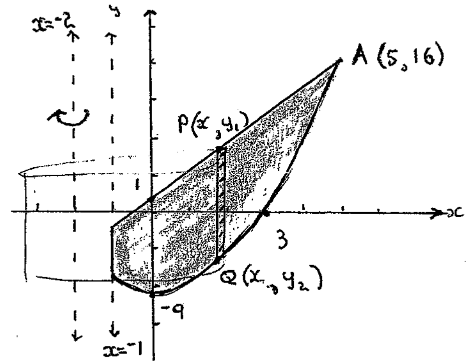
Marks

a) Consider the function defined by $x=\theta+\frac{(\sin 2\theta)}{2}$ and $y=\theta-\frac{(\sin 2\theta)}{2}$

(i) Show that $\frac{dy}{dx}=\tan^2\theta$ **2**

(ii) Show that $\frac{d^2y}{dx^2}=\tan\theta\sec^4\theta$ **2**

b) The region bounded by the curve $y=x^2-9$, the line $3x-y+1=0$ and the line $x=-1$ is rotated about the line $x=-2$ to form a solid.



(i) Using the method of cylindrical shells show that the volume of an elemental shell is given by **3**

$$\delta V=2\pi(x+2)(10+3x-x^2)\ \delta x$$

(ii) Find the volume of the solid formed. **2**

c) The velocity, v m/s, of a particle of mass m kg moving along the x-axis is given by $v=v_0\,e^{\frac{-kx}{m}}$ where v_0 is positive. Initially the particle is at the origin.

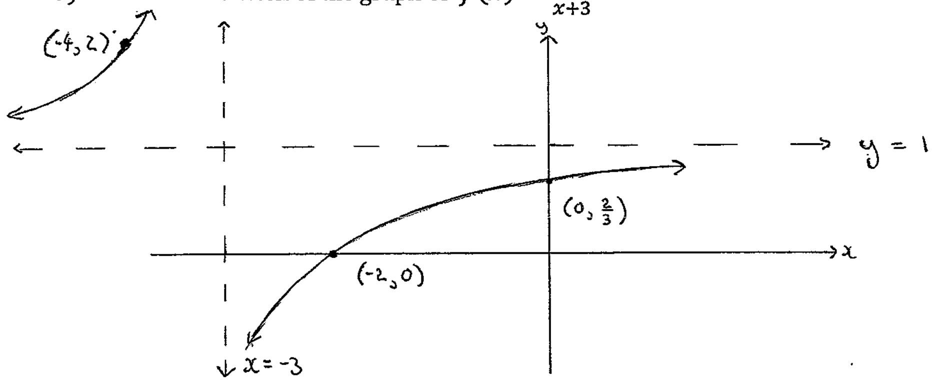
(i) Find the displacement, x m, as a function of time. **3**

(ii) Find the resultant force acting on this particle as a function of x **2**

(iii) Carefully describe the motion. **1**

Question 4 – (15 marks) – Start a new booklet Marks

a) Given the sketch of the graph of $f(x) = \frac{x+2}{x+3}$



Use the graph of $f(x) = \frac{x+2}{x+3}$ above to

(i) find the largest possible domain of the function $y = \sqrt{\frac{x+2}{x+3}}$ 1

(ii) find the set of values of x for which the function $y = x - \log_e(x+3)$ is increasing. 1

(iii) Use the graph of $f(x) = \frac{x+2}{x+3}$ above to sketch on separate axes (provided)

(α) the graph of $y = [f(x)]^2$ 2

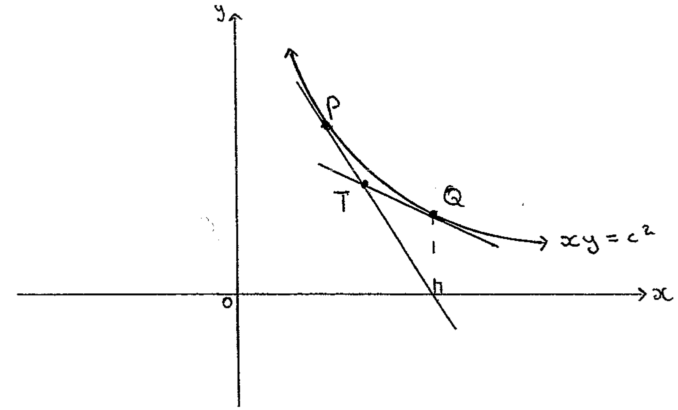
(β) the graph of $y^2 = f(x)$ 2

(γ) the graph of $y = e^{f(x)}$ 2

Question 4 continued next page

Question 4 – (cont'd) Marks

b) The distinct points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ are on the same branch of the hyperbola with equation $xy = c^2$. The tangents at P and Q meet at the point T.



(i) Show that the equation of the tangent at P is $x + p^2y = 2cp$ 2

(ii) Show that T has co-ordinates $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$ 2

(iii) Let P and Q move so that the tangent at P intersects the x-axis at $(cq, 0)$. Show that the locus of T is a hyperbola and state its eccentricity. 3

Question 5 – (15 marks) – Start a new booklet Marks

a) A mass of m kg falls from a stationary balloon at height h metres above the ground. It experiences air resistance of mkv^2 during its fall where v is its speed in metres per second and k is a positive constant.

The equation of motion of the mass is $\ddot{x} = g - kv^2$ where g is the acceleration due to gravity.

(i) Show that $v^2 = \frac{g}{k}(1 - e^{-2kx})$ 3

(ii) Find the velocity V when the mass hits the ground. 1

(iii) Find x when $v = \frac{V}{2}$ 3

(iv) Find V if air resistance is neglected. 1

b) For the curve $y^2 = x^2(6 + x)$

(i) By implicit differentiation show that $\frac{dy}{dx} = \frac{3x^2+12x}{2y}$ 1

(ii) Find any stationary points for the curve and discuss their nature. 3

(iii) Using at least $\frac{1}{3}$ of a page, sketch the curve $y^2 = x^2(6 + x)$ showing all essential features. 2

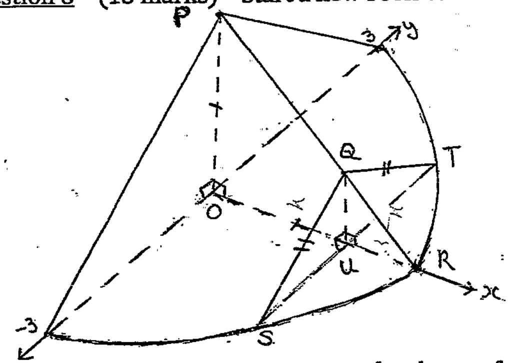
(iv) To calculate the area bounded by the loop, use the expression 1

$$A = 2\int_{-6}^{0} x\,(6 + x)^{\frac{1}{2}}\;dx$$

This provides the answer $\frac{-96\sqrt{6}}{5}$. Explain why the negative sign appears on this numerical outcome.

Question 6 – (15 marks) – Start a new booklet Marks

a)



A solid figure has a semi-circular base of radius 3cm. Cross sections taken perpendicular to the x-axis are isosceles triangles.

The vertical cross section containing the radius OR of the base of the solid is a right isosceles triangle ORP where $OR = OP$.

(i) Show that the area of triangle SQT $[SQ = QT]$ is given by $A = (3 - x)(9 - x^2)^{\frac{1}{2}}$ where $x = OU$ 2

(ii) Show that the volume of this solid is $\frac{1}{4}(27\pi - 36)\ cm^3$ 4

b) The polynomial $P(x)$ is defined by $P(x) = x^4 + Ax^2 + B$ where A and B are real positive numbers.

(i) Explain why $P(x)$ has no real zeros. 3

(ii) If two of the zeros of $P(x)$ are ib and id where b and d are real, show that $b^4 + d^4 = A^2 - 2B$ 3

c) If $I_n = \int_0^1 (1 - x^2)^n\,dx$ show that $I_n = \frac{2n}{2n+1} I_{n-1}$ for all positive integers $n \geq 1$

[Hint: Let $I_n = \int_0^1 (1 - x^2)(1 - x^2)^{n-1}\,dx$] 3

Question 7 – (15 marks) – Start a new booklet Marks

a) Use integration by parts to evaluate $\int_1^2 x^2 \log_e x \, dx$ 3

b) A sprinkler is watering part of the school oval. As the water leaves the sprinkler with velocity V m/s it makes an angle θ with the ground. This angle varies continuously from 30° to 60°.

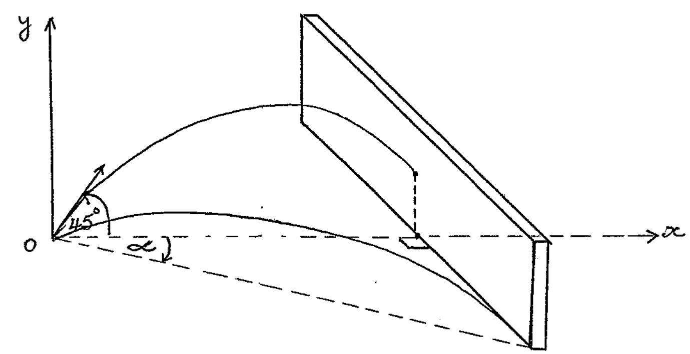
(i) Show that the water reaches a horizontal distance R from the sprinkler where $V^2 \frac{\sqrt{3}}{2g} \leq R \leq \frac{V^2}{g}$ 4

(ii) If this sprinkler rotates through 360°, find the area watered by the sprinkler. 1

(iii) θ is fixed at 45°. If the sprinkler is still free to rotate through 360° and it is placed $V^2 \frac{\sqrt{3}}{2g}$ from a wall, as shown, find:

(α) the angle of rotation, α, if the water lands exactly at the base of the wall. 2

(β) the maximum height that the water can reach up the wall. 2



c) Solve for x: $\frac{|x|-2}{4+3x-x^2} > 0$ 3

$-(x^2-3x-4)$
$-(x-4)(x+1)$

Question 8 – (15 marks) – Start a new booklet Marks

a) Show that $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$ 1

b) $z = \cos\theta + i\sin\theta$ is a root of $z^5 = 1$ where $z \neq 1$

(i) Show that $z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} = 0$ 2

(ii) Let $x = z + \frac{1}{z}$. Show that $x^2 + x - 1 = 0$ 2

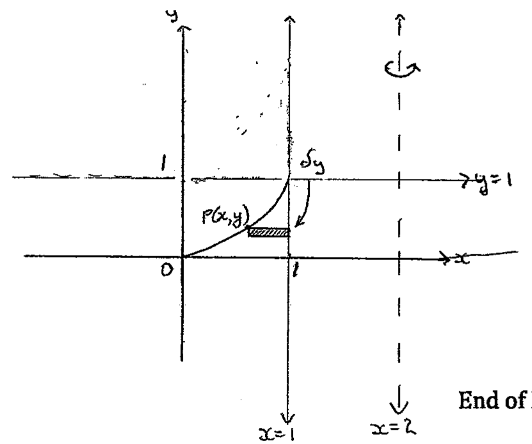
(iii) Show that $z + \frac{1}{z} = 2\cos\frac{2\pi}{5}$ or $-2\cos\frac{\pi}{5}$ 3

(iv) Hence show that $\cos\frac{2\pi}{5} \cdot \cos\frac{\pi}{5} = \frac{1}{4}$ 1

(v) Find the exact value of $\cos\frac{2\pi}{5}$ 2

c) The area bounded by $x = 1$, $y = 0$ and $y = x^2$ is rotated about the line $x = 2$.

The volume of the solid formed is to be determined by taking slices perpendicular to the axis of rotation.



(i) Show that the area of the annulus for an elemental slice is $A = \pi[3 - 4x + x^2]$ 2

(ii) Find the volume of the solid formed. 2

End of Paper

TRIAL HSC 2008

EXTENSION 2

SOLUTIONS

SOLUTIONS

QUESTION 1:

(a) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx$ $\qquad u = \cos x$, $du = -\sin x\,dx$

$= -\int_1^0 \frac{du}{1+u^2}$

$= \int_0^1 \frac{du}{1+u^2}$

$= \tan^{-1} u \Big]_0^1$

$= \frac{\pi}{4}$

(b) $\int \frac{e^{2x}}{e^x+1}\,dx$ $\qquad u = e^x$, $du = e^x\,dx$

$= \int \frac{e^x \cdot e^x\,dx}{e^x+1}$

$= \int \frac{u}{u+1}\,du$

$= \int \left(1 - \frac{1}{u+1}\right) du$

$= u - \ln|u+1| + c$

$= e^x - \ln(e^x+1) + c$

(c) $3-3i = 3\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$ $\qquad 1+i = \sqrt{2}\,\text{cis}\,\frac{\pi}{4}$

$\therefore \frac{z^4}{w^3} = \frac{324\,\text{cis}(-\pi)}{2\sqrt{2}\,\text{cis}\,\frac{3\pi}{4}}$

$= 81\sqrt{2}\,\text{cis}\left(-\frac{7\pi}{4}\right)$

$= 81\sqrt{2}\,\text{cis}\,\frac{\pi}{4}$

$= 81\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

$= 81 + 81i$

(d) (i) $a^2 = 25 \quad b^2 = 9$

$b^2 = a^2(1-e^2)$

$9 = 25(1-e^2)$

$\therefore 1-e^2 = \frac{9}{25}$

$e^2 = \frac{16}{25}$

$\therefore e = \frac{4}{5} \quad (e>0)$

(ii) x intercepts $= \pm 5$

y intercepts $= \pm 3$

foci $= (\pm 4, 0)$

directrices : $x = \pm \frac{25}{4}$

(e) $2x^3 + 5x - 3 = 0 \qquad \alpha, \beta, \gamma$

(i) $P(\sqrt{x}) = 2(\sqrt{x})^3 + 5\sqrt{x} - 3 = 0 \qquad \alpha^2, \beta^2, \gamma^2$

ie $\sqrt{x}(2x+5) = 3$

$x(4x^2 + 20x + 25) = 9$

ie $4x^3 + 20x^2 + 25x - 9 = 0$ has roots $\alpha^2, \beta^2, \gamma^2$

(ii) $\Rightarrow \quad 4\left(\frac{1}{x}\right)^3 + 20\left(\frac{1}{x}\right)^2 + 25\left(\frac{1}{x}\right) - 9 = 0$ has roots $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$

$\Rightarrow \quad 4 + 20x + 25x^2 - 9x^3 = 0$

ie $9x^3 - 25x^2 - 20x - 4 = 0$

$\therefore \sum \frac{1}{\alpha^2} = \frac{25}{9}$

QUESTION 2:

(a) (i) Let $P(x) = (x-\alpha)^2 Q(x)$ where $Q(\alpha) \neq 0$

$\Rightarrow P'(x) = Q(x).2(x-\alpha) + (x-\alpha)^2.Q'(x)$

$= (x-\alpha)\underbrace{[2Q(x) + (x-\alpha).Q'(x)]}_{R(x)}$

where $R(\alpha) = 2Q(\alpha) + 0$

$\neq 0$ since $Q(\alpha) \neq 0$

$\therefore x = \alpha$ is a single zero of $P'(x)$

(ii) $P(x) = x^5 + 2x^2 + mx + n$

$P(-1) = 0 \Rightarrow -1 + 2 - m + n = 0$

ie $n - m = -1$ ——— (1)

$P'(x) = 5x^4 + 4x + m$

$P'(-1) = 0 \Rightarrow 5 - 4 + m = 0$

$\therefore m = -1$ sub in (1)

$\therefore n + 1 = -1$

$n = -2$

$\therefore \left.\begin{array}{l} m = -1 \\ n = -2 \end{array}\right\}$

(b) (i) $2x^2 + 7x - 1 = A(x^2 + x + 1) + (Bx + C)(x-2)$

$x = 2 \Rightarrow \quad 21 = 7A$

$A = 3$

co-eff $x^2 \Rightarrow \quad 2 = A + B$

$\therefore B = -1$

constant $\Rightarrow \quad -1 = A - 2C$

$2C = 4$

$C = 2$

$\therefore \int \frac{2x^2 + 7x - 1}{(x-2)(x^2+x+1)}\,dx = \int \left(\frac{3}{x-2} + \frac{-x+2}{x^2+x+1}\right) dx$

$$= 3\ln|x-2| - \frac{1}{2}\int\frac{2x+1}{x^2+x+1} + \frac{5}{2}\int\frac{dx}{(x+\frac{1}{2})^2+(\frac{\sqrt3}{2})^2}$$

$$= 3\ln|x-2| - \frac{1}{2}\ln|x^2+x+1| + \frac{5}{2}\cdot\frac{2}{\sqrt3}\tan^{-1}\left(\frac{2x+1}{\sqrt3}\right)+c$$

$$= 3\ln|x-2| - \frac{1}{2}\ln(x^2+x+1) + \frac{5\sqrt3}{3}\tan^{-1}\left(\frac{2x+1}{\sqrt3}\right)+c$$

(c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow \frac{d}{dx}\left(\frac{x^2}{a^2}\right) + \frac{d}{dx}\left(\frac{y^2}{b^2}\right) = 0$

$\therefore \frac{2x}{a^2} + \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{b^2x}{a^2y}$

at $P(a\cos\theta, b\sin\theta)$

$\frac{dy}{dx} = -\frac{b^2 \cdot a\cos\theta}{a^2 \cdot b\sin\theta}$

$= -\frac{b\cos\theta}{a\sin\theta}$

(i) Tangent is:

$y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$(a\sin\theta)y - ab\sin^2\theta = (-b\cos\theta)x + ab\cos^2\theta$

ie $(b\cos\theta)x + (a\sin\theta)y = ab$

$\Rightarrow \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ —— (1)

(ii) Normal is:

$y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$

$(b\cos\theta)y - b^2\sin\theta\cos\theta = (a\sin\theta)x - a^2\sin\theta\cos\theta$

$\therefore (a\sin\theta)x - (b\cos\theta)y = \sin\theta\cos\theta(a^2-b^2)$

$\Rightarrow \frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$ —— (2)

$y=0$ in (1) $\Rightarrow x = a\sec\theta$

$y=0$ in (2) $\Rightarrow x = \frac{(a^2-b^2)\cos\theta}{a}$

$\therefore OR.OQ = a\sec\theta \times \frac{a^2-b^2}{a}\cdot\cos\theta$

$= a^2 - b^2$

$b^2 = a^2(1-e^2)$

$\Rightarrow b^2 = a^2 - a^2e^2$

$\therefore a^2 - b^2 = a^2e^2$

$= a^2e^2$

QUESTION 3)

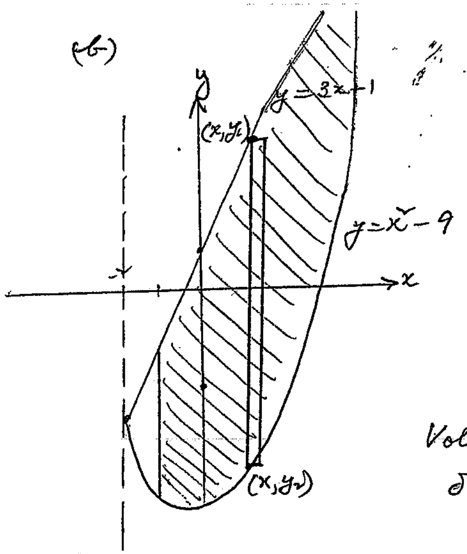
(a) (i) $x = \theta + \frac{1}{2}\sin 2\theta \qquad y = \theta - \frac{1}{2}\sin 2\theta$

$\frac{dx}{d\theta} = 1 + \cos 2\theta \qquad \frac{dy}{d\theta} = 1 - \cos 2\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$
$$= \frac{1-\cos 2\theta}{1+\cos 2\theta}$$
$$= \frac{2\sin^2\theta}{2\cos^2\theta}$$
$$= \tan^2\theta$$

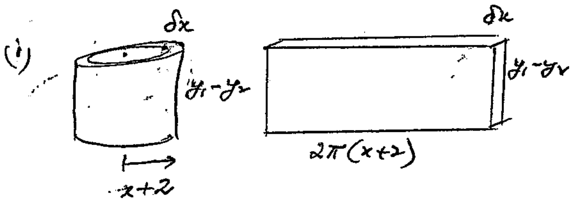
(ii) $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$
$$= \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \cdot \frac{d\theta}{dx}$$
$$= 2\tan\theta\sec^2\theta \times \frac{1}{2\cos^2\theta}$$
$$= \tan\theta\sec^4\theta$$

(b)



$$x^2 - 9 = 3x + 1$$
$$\Rightarrow x^2 - 3x - 10 = 0$$
$$(x-5)(x+2) = 0$$
$$x = 5, -2$$

(i)



Volume of slice is

$$\delta V = 2\pi(x+2)(y_1 - y_2)$$
$$= 2\pi(x+2)[3x+1-(x^2-9)]$$
$$= 2\pi(x+2)(10+3x-x^2)$$

Volume of solid is

$$V = \lim_{\delta x \to 0} \sum_{x=-1}^{5} 2\pi(x+2)(10+3x-x^2)\,\delta x$$
$$= 2\pi\int_{-1}^{5}(x+2)(10+3x-x^2)\,dx$$
$$= 2\pi\int_{-1}^{5}(10x+3x^2-x^3+20+6x-2x^2)\,dx$$
$$= 2\pi\int_{-1}^{5}(20+16x+x^2-x^3)\,dx$$
$$= 2\pi\left[20x+8x^2+\frac{x^3}{3}-\frac{x^4}{4}\right]_{-1}^{5}$$
$$= 2\pi\left[\left(100+200+\frac{125}{3}-\frac{625}{4}\right)-\left(-20+8-\frac{1}{3}-\frac{1}{4}\right)\right]$$
$$= 396\pi$$

$\therefore$ Volume is 396π units^3

(c) $v = v_0 e^{-\frac{kx}{m}} \qquad v_0 > 0$

(i) $$\frac{dx}{dt} = v_0 e^{-\frac{kx}{m}}$$
$$= \frac{v_0}{e^{kx/m}}$$
$$\therefore \frac{dt}{dx} = \frac{e^{\frac{kx}{m}}}{v_0}$$
$$\Rightarrow t = \int \frac{e^{\frac{kx}{m}}}{v_0}\,dx$$
$$= \frac{1}{v_0}\cdot\frac{e^{\frac{kx}{m}}}{\left(\frac{k}{m}\right)} + c$$
$$= \frac{m}{kv_0}e^{\frac{kx}{m}} + c$$

$\left.\begin{matrix} t=0 \\ x=0 \end{matrix}\right\} \Rightarrow 0 = \frac{m}{kv_0} + c$

$\therefore c = -\frac{m}{kv_0}$

$$\therefore t = \frac{m}{kv_0}\left(e^{\frac{kx}{m}} - 1\right)$$

$$\Rightarrow \frac{kv_0 t}{m} + 1 = e^{\frac{kx}{m}}$$

$$\therefore \frac{kx}{m} = \ln\left(\frac{kv_0 t + m}{m}\right)$$

$$\therefore x = \frac{m}{k}\ln\left(\frac{kv_0 t + m}{m}\right) \quad \text{——— (1)}$$

(ii)
$$R = m\ddot{x}$$
$$= mv\frac{dv}{dx}$$
$$= \cancel{m}\, v_0 e^{-\frac{kx}{m}} \cdot -\frac{kv_0}{\cancel{m}} \cdot e^{-\frac{kx}{m}}$$
$$= -kv_0^2 e^{-\frac{2kx}{m}}$$

(iii) at $t=0$, $x=0$, $v>0$ (since $v_0>0$)
$\therefore$ Particle starts at the origin and moves to the right under a retarding force. From (1) we see that $x \to \infty$ as $t \to \infty$, also $v \to 0$ as $x \to \infty$

QUESTION 4:

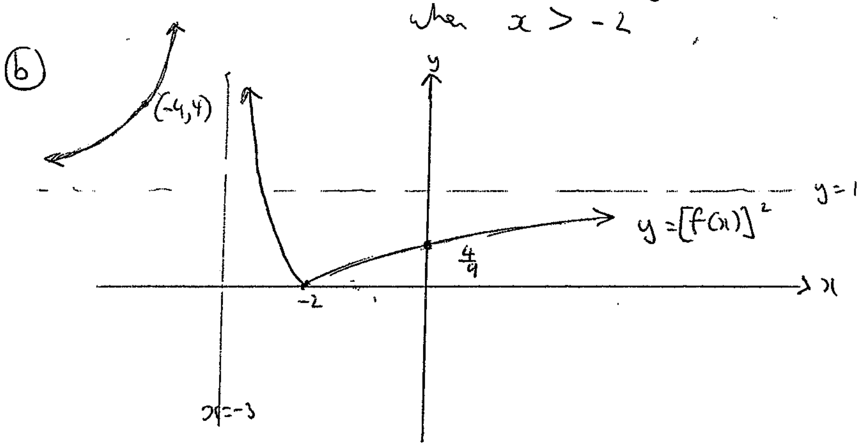
Graphs.

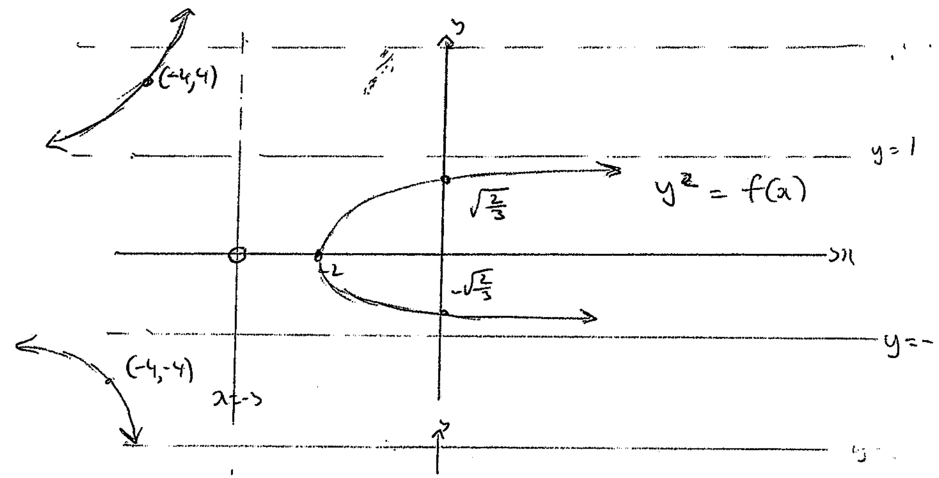
(a) (i) only have $\sqrt{\ }$ of positive values
$\therefore$ Domain $x<-3$ and $x \geq -2$

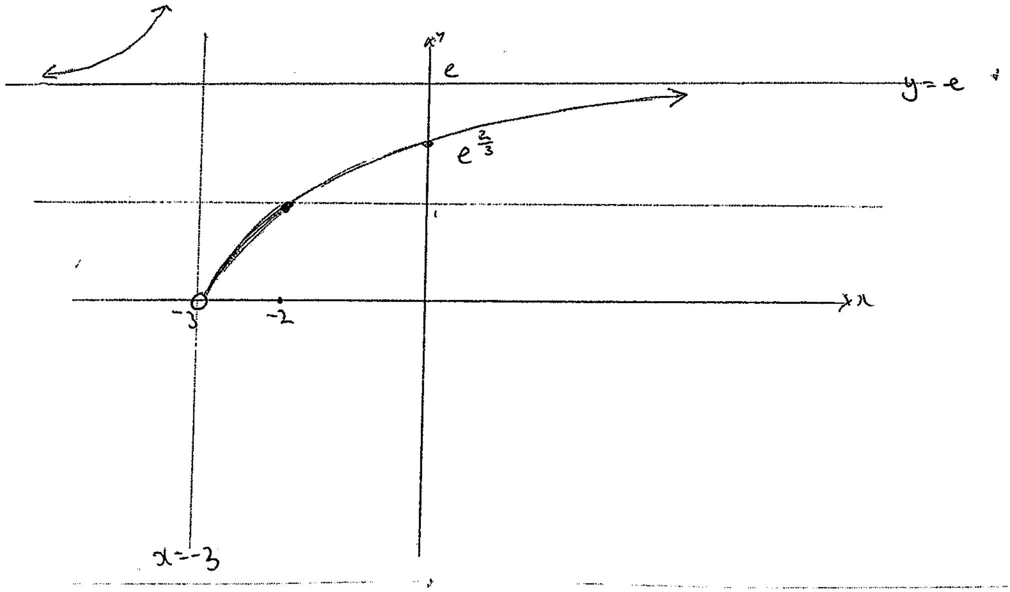
(ii) Note: $\int \frac{x+3-1}{x+3}\,dx = \int 1\,dx - \int \frac{dx}{x+3}$
$= x - \ln(x+3)$

From graph area increasing when $x > -2$

(b)







(ii) Similarly, tangent at Q can be shown to by $x + q^2y = 2cq$

Solving $x + p^2y = 2cp$ .. I

$x + q^2y = 2cq$ .. II

(α) I − II $\Rightarrow (p^2-q^2)y = 2c(p-q)$

$(p-q)(p+q)y = 2c(p-q)$

$y = \frac{2c}{p+q}$

(β) $(q^2 \times \text{I}) - (p^2 \times \text{II})$

$q^2x + p^2q^2y = 2cpq^2$

$p^2x + p^2q^2y = 2cp^2q$

$(q^2-p^2)x = 2cpq(q-p)$

$\therefore x = \frac{2cpq}{p+q}$

$\therefore T\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$

(iii) $X = \frac{2cpq}{p+q}$ ... A

$Y = \frac{2c}{p+q}$ ... B

Given $(cq, 0)$ lies on tangent at P.

Then $cq = 2cp$

$q = 2p$ ... C

From C, $pq = 2p^2$

From (A) $X = \frac{4cp^2}{3p}$

$= \frac{4cp}{3}$

From (B) $Y = \frac{2c}{3p}$

Then $XY = \frac{4cp}{3} \times \frac{2c}{3p}$

$xy = \frac{8c^2}{9}$

Since $\frac{8c^2}{9}$ is a constant, the $xy = \frac{8c^2}{9}$ represent a rectangular hyperbola eccentricity $\underline{e = \sqrt{2}}$

(b) (i) Given $y = \frac{c^2}{x}$

$\frac{dy}{dx} = -\frac{c^2}{x^2}$

or $y + x\frac{dy}{dx} = 0$

$x\frac{dy}{dx} = -y$

$x\frac{dy}{dx} = -\frac{c^2}{x}$

$\frac{dy}{dx} = -\frac{c^2}{x^2}$

at $P(cp, \frac{c}{p})$ gradient of tangent $m = \frac{-c^2}{(cp)^2}$

$= -\frac{1}{p^2}$

$\therefore$ Equation of tangent $y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$

$p^2y - cp = -x + cp$

$x + p^2y = 2cp$

QUESTION 5:

(a)

$\downarrow mg \quad \uparrow mkv^2$

(i) $R = m\ddot{x}$

$\Rightarrow m\ddot{x} = mg - mkv^2$

$\Rightarrow \ddot{x} = g - kv^2$

(ii) $v\dfrac{dv}{dx} = g - kv^2$

$\Rightarrow \dfrac{dv}{dx} = \dfrac{g - kv^2}{v}$

$\Rightarrow \dfrac{dx}{dv} = \dfrac{v}{g - kv^2}$

$\Rightarrow x = \displaystyle\int_0^v \frac{v}{g - kv^2}\, dv$

$= -\dfrac{1}{2k}\left[\ln(g - kv^2)\right]_0^v$

$x = -\dfrac{1}{2k}\left[\ln\left(\dfrac{g - kv^2}{g}\right)\right]$

$\Rightarrow -2kx = \ln\left(\dfrac{g - kv^2}{g}\right)$

$\Rightarrow e^{-2kx} = \dfrac{g - kv^2}{g}$

$ge^{-2kx} = g - kv^2$

$\therefore kv^2 = g(1 - e^{-2kx})$

$\therefore v^2 = \dfrac{g}{k}(1 - e^{-2kx})$

(iii) when $x = h$, $v^2 = \dfrac{g}{k}(1 - e^{-2kh})$

$\therefore V = \sqrt{\dfrac{g}{k}(1 - e^{-2kh})}$

(iv) If $v = \dfrac{V}{2}$

$v^2 = \dfrac{V^2}{4}$

$= \dfrac{g}{4k}(1 - e^{-2kh})$

$\therefore \dfrac{g}{k}(1 - e^{-2kx}) = \dfrac{g}{4k}(1 - e^{-2kh})$

$4 - 4e^{-2kx} = 1 - e^{-2kh}$

$3 + e^{-2kh} = 4e^{-2kx}$

$\dfrac{3 + e^{-2kh}}{4} = e^{-2kx}$

$\therefore -2kx = \ln\left(\dfrac{3 + e^{-2kh}}{4}\right)$

$\therefore x = -\dfrac{1}{2k}\ln\left(\dfrac{3 + e^{-2kh}}{4}\right)$

(v) If air resistance is neglected

$\ddot{x} = g$

$\therefore \dot{x} = gt + c \qquad \left.\begin{matrix} t=0 \\ \dot{x}=0 \end{matrix}\right\} \Rightarrow c = 0$

$x = \frac{1}{2}gt^2 + c \qquad \left.\begin{matrix} t=0 \\ x=0 \end{matrix}\right\} \Rightarrow c = 0$

$\therefore x = \frac{1}{2}gt^2$

$\dfrac{d}{dx}\left(\frac{1}{2}v^2\right) = g$

$\frac{1}{2}v^2 = gx + c \qquad \left.\begin{matrix} x=0 \\ v=0 \end{matrix}\right\} \Rightarrow c = 0$

$\therefore v^2 = 2gx$

at $x = h$, $v^2 = 2gh$

$\therefore V = \sqrt{2gh}$

(b) (i)

$$2y\frac{dy}{dx} = 2x(6+x) + 1.x^2$$

$$2y\frac{dy}{dx} = 12x + 3x^2$$

$$\frac{dy}{dx} = \frac{12x+3x^2}{2y}$$

(ii) Stat. points $\frac{dy}{dx} = 0$

$$12x+3x^2 = 0$$

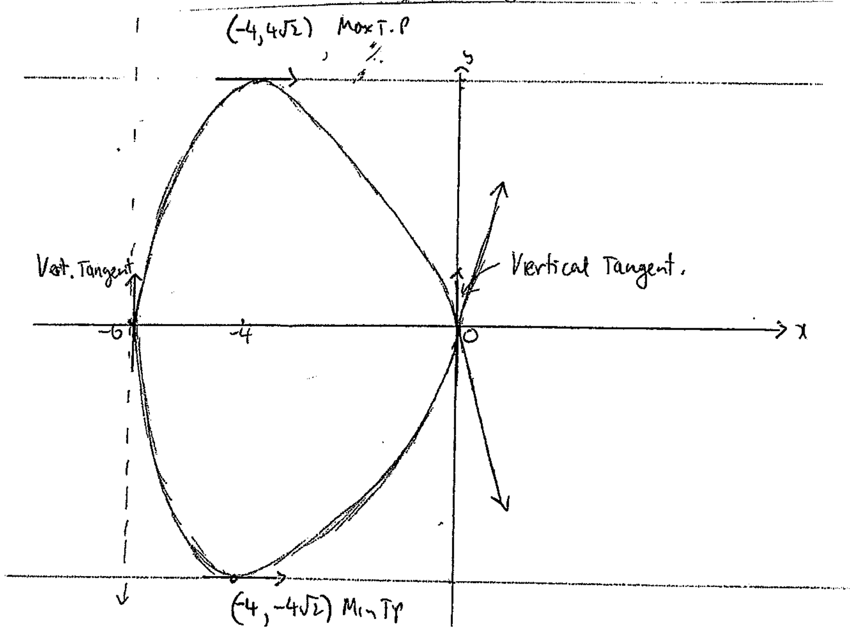
$$3x(4+x) = 0$$

(α) at $\left.\begin{array}{l} x=0 \\ x=-6 \end{array}\right\} y=0$ gives gradient function as undefined. Vertical tangent at $(0,0)$ and $(-6,0)$

(β) at $x=-4$, $y^2 = 32$

$\therefore y = \pm 4\sqrt{2}$

(iii)



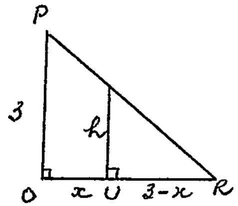
(iv) If we consider $y^2 = x^2(6+x)$

then $y = \pm\sqrt{x^2(6+x)} = \pm x(6+x)^{\frac{1}{2}}$

The area calculated is part of the loop below the x-axis.

QUESTION 6 :

(a) (i)



By similar triangles

$$\frac{h}{3} = \frac{3-x}{3}$$

$$\therefore h = 3-x$$

$\therefore$ Area of ΔSQT is $\frac{1}{2}\cdot(3-x)\cdot 2\sqrt{9-x^2}$

$$= (3-x)(9-x^2)^{\frac{1}{2}}$$

(ii) Volume of solid is

$$V = \lim_{\delta x \to 0} \sum_{x=0}^{3} (3-x)(9-x^2)^{\frac{1}{2}}\,\delta x$$

$$= \int_0^3 (3-x)(9-x^2)^{\frac{1}{2}}\,dx$$

$$= 3\underbrace{\int_0^3 \sqrt{9-x^2}\,dx}_{\text{quadrant of circle}} - \int_0^3 x\underbrace{\sqrt{9-x^2}}_{\substack{u=9-x^2 \\ du=-2x\,dx}}\,dx$$

$$= 3\times\frac{1}{4}\times\pi\cdot 9 + \frac{1}{2}\int_0^3 -2x\sqrt{9-x^2}\,dx$$

$$= \frac{27\pi}{4} + \frac{1}{2}\int_9^0 u^{\frac{1}{2}}\,du$$

$$= \frac{27\pi}{4} + \frac{1}{2}\cdot\frac{2}{3}\cdot\left[u^{\frac{3}{2}}\right]_9^0$$

$$= \frac{27\pi}{4} + \frac{1}{3}\left[0 - 27\right]$$

$$= \frac{27\pi}{4} - 9$$

$$= \frac{1}{4}(27\pi - 36)$$

$\therefore$ Volume is $\frac{1}{4}(27\pi - 36)$ units^3

(b) $P(x) = x^4 + Ax^2 + B$

(i) $P'(x) = 4x^3 + 2Ax$

stationary points at $P'(x) = 0$

ie $4x^3 + 2Ax = 0$

$\Rightarrow 2x(2x^2 + A) = 0$

$\therefore x = 0$ since $2x^2 + A \neq 0$ $(A > 0)$

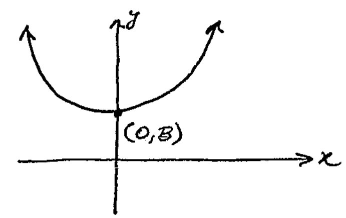
$P(0) = B \ (>0)$

$P''(x) = 12x^2 + 2A$

> 0

$\therefore$ Only one stationary point at $(0, B)$

Hence sketch of $y = P(x)$ must be



$\therefore$ $P(x) = 0$ has no real zeros.

(ii) zeros of $P(x)$ are $ib, -ib, id$ and $-id$ since all co-efficients of $P(x)$ are real

$$\sum\alpha = 0$$

$$\sum\alpha\beta = A = (ib)(-ib) + (ib)(id) + ib(-id)$$
$$(-ib)(id) + (-ib)(-id) + (id)(-id)$$

$$\Rightarrow b^2 - bd + bd + bd - bd + d^2 = A$$

$$\text{ie } b^2 + d^2 = A \quad \text{—— (1)}$$

Product of roots $\Rightarrow (ib)(-ib)(id)(-id) = B$

$$\therefore b^2d^2 = B \quad \text{—— (2)}$$

Then $b^4 + d^4 = (b^2 + d^2)^2 - 2b^2d^2$

$= A^2 - 2B$

(c) $I_n = \int_0^1 (1-x^2)^n \, dx$

$= \int_0^1 (1-x^2)(1-x^2)^{n-1} \, dx$

$= \int_0^1 (1-x^2)^{n-1} dx - \int_0^1 x^2 (1-x^2)^{n-1} \, dx$

$= I_{n-1} - \int_0^1 \underbrace{x}_{v} \cdot \underbrace{x(1-x^2)^{n-1}}_{du} \, dx$

$= I_{n-1} - \left[\left[-\frac{1}{2n}(1-x^2)^n \cdot x \right]_0^1 - \int_0^1 -\frac{1}{2n} \cdot (1-x^2)^n \, dx \right]$

$I_n = I_{n-1} - \frac{1}{2n} \cdot I_n$

$\therefore I_n + \frac{1}{2n} I_n = I_{n-1}$

$\left(\frac{2n+1}{2n}\right) \cdot I_n = I_{n-1}$

$\therefore I_n = \left(\frac{2n}{2n+1}\right) \cdot I_{n-1}$

QUESTION 7:

(a) let $u = \log_e x$ and $dv = x^2$ $\qquad \int u\,dv\,dx = uv - \int v\,du\,dx$

then $du = \frac{1}{x}$ $\qquad v = \frac{x^3}{3}$

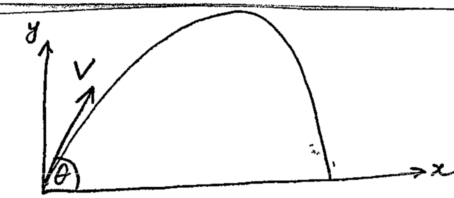
So $\int_1^2 x^2 \log_e x \, dx = \left[\frac{x^3}{3} \cdot \log_e x\right]_1^2 - \int_1^2 \frac{x^2}{3} \, dx$

$= \left[\frac{8}{3} \cdot \log_e 2 - 0\right] - \left[\frac{x^3}{9}\right]_1^2$

$= \frac{8}{3} \log_e 2 - \left(\frac{8}{9} - \frac{1}{9}\right)$

$= \frac{8}{3} \log_e 2 - \frac{7}{9}$

(b)



(i) $\ddot{y} = -g$

$\dot{y} = -gt + c$

$\ddot{x} = 0$

$\Rightarrow \dot{x} = V\cos\alpha$

$x = Vt\cos\alpha$ —— (1)

$\left.\begin{matrix} t=0 \\ \dot{y} = V\sin\alpha \end{matrix}\right\} \Rightarrow c = V\sin\alpha$

$\therefore \dot{y} = V\sin\alpha - gt$

$y = Vt\sin\alpha - \frac{1}{2}gt^2 + c$

$\left.\begin{matrix} t=0 \\ y=0 \end{matrix}\right\} \Rightarrow c = 0$

$\therefore y = Vt\sin\alpha - \frac{1}{2}gt^2$ —— (2)

Water hits ground at $y = 0$

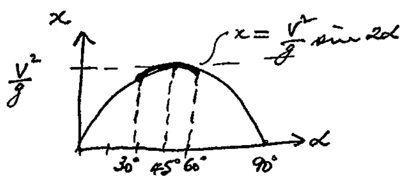
ie $Vt\sin\alpha - \frac{1}{2}gt^2 = 0$

$\frac{1}{2}t(2V\sin\alpha - gt) = 0$

$t = \frac{2V\sin\alpha}{g}$ sub in (1)

$x = V\cos\alpha \cdot \frac{2V\sin\alpha}{g}$

$= \frac{V^2 \sin 2\alpha}{g}$

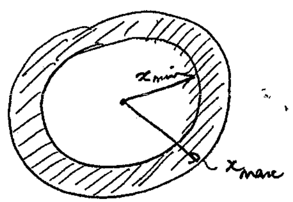


Hence $x_{max} = \frac{V^2}{g}$ when $\alpha = 45°$

$$x_{min} = \frac{V^2}{g}\sin 60°$$
$$= \frac{V^2\sqrt{3}}{2g}$$

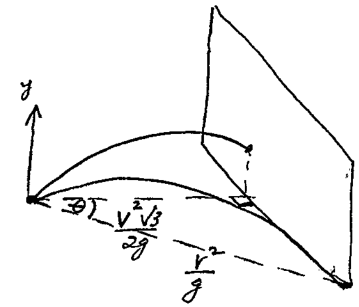
$$\therefore \frac{V^2\sqrt{3}}{2g} \le x \le \frac{V^2}{g}$$

(ii)



$$\text{Area watered} = \pi\left(\frac{V^2}{g}\right)^2 - \pi\left(\frac{V^2\sqrt{3}}{2g}\right)^2$$
$$= \pi\left[\frac{V^4}{g^2} - \frac{3V^4}{4g^2}\right]$$
$$= \pi \cdot \frac{V^4}{4g^2} \ m^2$$

(iii)



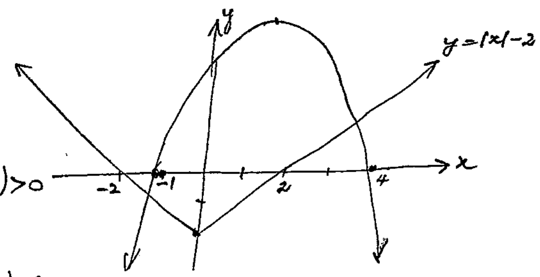
$$(\alpha) \quad \cos\theta = \frac{\frac{V^2\sqrt{3}}{2g}}{\frac{V^2}{g}}$$
$$= \frac{V^2\sqrt{3}}{2g} \times \frac{g}{V^2}$$
$$= \frac{\sqrt{3}}{2}$$
$$\therefore \theta = 30°$$

(β) from ① $x = Vt\cos\alpha$

$$\Rightarrow Vt\cos\alpha = \frac{V^2\sqrt{3}}{2g}$$
$$\therefore t = \frac{V\sqrt{3}}{2g\cos\alpha} \quad \text{where } \alpha = 45°$$
$$= \frac{V\sqrt{3}}{g\sqrt{2}} \quad \text{sub in ②}$$

$$\therefore y = V \cdot \frac{1}{\sqrt{2}} \cdot \frac{V\sqrt{3}}{g\sqrt{2}} - \frac{g}{2} \cdot \frac{3V^2}{2g^2}$$
$$= \frac{V^2\sqrt{3}}{2g} - \frac{3V^2}{4g}$$
$$= \frac{2V^2\sqrt{3}}{4g} - \frac{3V^2}{4g}$$
$$= \frac{V^2}{4g}\left(2\sqrt{3} - 3\right)$$

(c) $\dfrac{|x|-2}{(4-x)(1+x)} > 0$



when

(i) $|x|-2>0$ and $(4-x)(1+x)>0$

ie $2<x<4$

(ii) $|x|-2<0$ and $(4-x)(1+x)<0$

$-2<x<-1$

$\therefore -2<x<-1$ or $2<x<4$

QUESTION 8:

(a) $(z-1)(z^4+z^3+z^2+z+1) = z^5+z^4+z^3+z^2+z$
$\quad -z^4-z^3-z^2-z-1$

$= z^5-1$

(b) (i) $z^5=1 \Rightarrow z^5-1=0$

ie $(z-1)(z^4+z^3+z^2+z+1)=0$

$z\neq 1 \Rightarrow z^4+z^3+z^2+z+1=0$

$\Rightarrow z^2+z+1+\frac{1}{z}+\frac{1}{z^2}=0$ —— (1)

(ii) $\left(z^2+\frac{1}{z^2}\right)+\left(z+\frac{1}{z}\right)+1=0$

$\Rightarrow \left(z+\frac{1}{z}\right)^2-2+\left(z+\frac{1}{z}\right)+1=0$

$x=z+\frac{1}{z} \Rightarrow x^2-2+x+1=0$

$\therefore x^2+x-1=0$

(iii) Since $z^5=1$

$z=1\operatorname{cis}\frac{2k\pi}{5} \qquad \frac{1}{z}=\operatorname{cis}\left(-\frac{2k\pi}{5}\right)$

$\therefore$ Roots are 1, $z=\operatorname{cis}\frac{2\pi}{5}$, $z^2=\operatorname{cis}\frac{4\pi}{5}$

$z^3=\operatorname{cis}\frac{6\pi}{5}$, $z^4=\operatorname{cis}\frac{8\pi}{5}$

Then $z+\frac{1}{z} = \operatorname{cis}\frac{2\pi k}{5}+\left(\operatorname{cis}\frac{2\pi k}{5}\right)^{-1}$

$= \operatorname{cis}\frac{2\pi k}{5}+\operatorname{cis}\left(-\frac{2\pi k}{5}\right)$

$= 2\cos\frac{2\pi k}{5}$

$k=1 \Rightarrow z+\frac{1}{z} = 2\cos\frac{2\pi}{5}$

$k=2 \Rightarrow z+\frac{1}{z} = 2\cos\frac{4\pi}{5} = -2\cos\frac{\pi}{5}$

$k=3 \Rightarrow z+\frac{1}{z} = 2\cos\frac{6\pi}{5} = -2\cos\frac{\pi}{5}$

$k=4 \Rightarrow z+\frac{1}{z} = 2\cos\frac{8\pi}{5} = 2\cos\frac{2\pi}{5}$

Hence the values of $z+\frac{1}{z}$ are $2\cos\frac{2\pi}{5}$, $-2\cos\frac{\pi}{5}$

ie $x = 2\cos\frac{2\pi}{5}, -2\cos\frac{\pi}{5}$

(iv) $\therefore$ $x^2+x-1=0$ has roots $2\cos\frac{2\pi}{5}, -2\cos\frac{\pi}{5}$

Product of roots $\Rightarrow -4\cos\frac{2\pi}{5}\cos\frac{\pi}{5} = -1$

$\therefore \cos\frac{2\pi}{5}\cos\frac{\pi}{5} = \frac{1}{4}$

(v) $x^2 + x - 1 = 0$

$\Rightarrow x = \frac{-1\pm\sqrt{5}}{2}$ $\qquad \frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}$

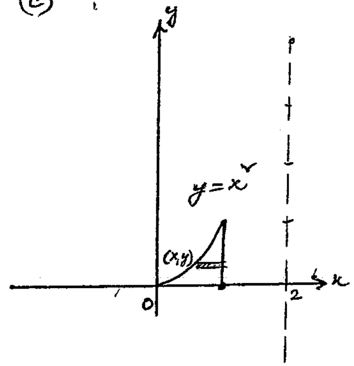
since $\cos\frac{\pi}{5} > \cos\frac{2\pi}{5}$

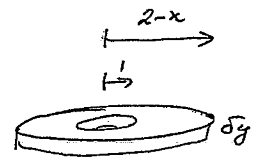
we see that $-2\cos\frac{\pi}{5} = \frac{-1-\sqrt{5}}{2}$

$2\cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{2}$

hence $\cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}$

(c)





(i) Volume of disc is $\delta V = \pi(2-x)^2\delta y - \pi\cdot 1^2\delta y$

$= \pi(4-4x+x^2-1)\,\delta y$

$= \pi(3-4x+x^2)\,\delta y$

$= \pi(3-4y^{\frac{1}{2}}+y)\,\delta y$

(ii) $\therefore$ Volume of solid is

$$V = \lim_{\delta y\to 0}\sum_{y=0}^{1} \pi(3-4y^{\frac{1}{2}}+y)\,\delta y$$

$$= \pi\int_0^1 (3-4y^{\frac{1}{2}}+y)\,dy$$

$$= \pi\left[3y - 4\cdot\frac{2}{3}y^{\frac{3}{2}} + \frac{y^2}{2}\right]_0^1$$

$$= \pi\left(3-\frac{8}{3}+\frac{1}{2}-0\right)$$

$$= \frac{5\pi}{6}$$

$\therefore$ Volume is $\frac{5\pi}{6}$ units^3