St George Girls High School

Year 11

End of Preliminary Course Examination

2007

Mathematics Extension 1

Time Allowed: 2 hours
(plus 5 minutes reading time)

Instructions

1. Attempt all 5 questions.
2. All necessary working must be shown.
3. Begin each question on a **new page**.
4. Marks will be deducted for careless work or poorly presented solutions.
5. All sketches are to be at least ⅓ page.

Question 1 (13 marks) – Start a New Page **Marks**

a) Simplify $\dfrac{4}{3x-x^2}+\dfrac{3}{x^2-x-6}$ 3

b) If $\tan\theta=3$ and $180^\circ<\theta<270^\circ$, find the exact value of $\sin\theta$. 2

c) A cannonball is fired. Its height above ground level t seconds after being fired is $2+10t-5t^2$ metres

(i) When does the ball reach maximum height? 2

(ii) What is the maximum height reached? 1

d) Give two different methods to prove that $3x-y-8=0$ is a tangent to $(x-1)^2+(y-5)^2=10$ 5

Question 2 (13 marks) – Start a New Page **Marks**

a) Solve:

(i) $\dfrac{6+3x}{2x} \geq 1$ 3

(ii) $|2x+1| > 5$ 2

b) If α, β are the roots of $7x^2+9x+3=0$, without finding α and β, find the values of:

(i) $\alpha+\beta$ 1

(ii) $\alpha\beta$ 1

(iii) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$ 1

(iv) $\alpha^2+\beta^2$ 1

(iv) $\alpha^3+\beta^3$ 2

c) Simplify $\log_b a \times \log_a c \times \log_c b$ 2

Question 3 (13 marks) – Start a New Page **Marks**

a) By considering the effect of replacing x by $\dfrac{x}{4}$ and y by $2y$, sketch the curve $\dfrac{x^2}{16}+4y^2=1$. (Hint: Firstly, sketch $x^2+y^2=1$). 3

b) Consider the function: $y=f(x)=\dfrac{x}{x^2-1}$

(i) Is the function odd, even or neither? (give reasons) 1

(ii) What is the domain? 1

(iii) Find the equations of all asymptotes. 2

(iv) Sketch the curve: $y=\dfrac{x}{x^2-1}$, showing all important features. 2

(v) Hence, solve $f(x) \geq 0$ 1

c) By using an appropriate Pythagorean trigonometric identity, solve $\sec^2\theta - 2\tan\theta - 4 = 0$ for $0^\circ \leq \theta \leq 360^\circ$ 3

(give answers to nearest minute).

Question 4 (13 marks) – Start a New Page **Marks**

a) Differentiate these expressions with respect to x:

(i) $7x^3 - 6x + \frac{3}{x}$ 1

(ii) $x\sqrt[3]{2x^2 - 7}$ 2

(iii) $\frac{5x^2+1}{2x-3}$ 2

b) For the parabola $x^2 = -16y$, find:

(i) the vertex. 1

(ii) the coordinates of the focus. 1

(iii) the equation of the directrix. 1

c) Water is flowing into a container which expands so as to be always spherical. Its volume is increasing at constant rate of 100cm^3/s. When its radius is 5cm,

(i) find the rate of change of the radius. 2

(ii) find the rate of change of the surface area. 1

d) Find the equation of the normal to the curve $x^2 - y^2 = 4$ at the point $\left(3, \sqrt{5}\right)$ 2

Question 5 (13 marks) – Start a New Page **Marks**

a) Prove by mathematical induction that $1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2}{4}(n+1)^2$ for all positive integers n. 6

b) Find the equation of the locus of a point which moves so that it is equidistant from $(-1, 2)$ and $(3, 4)$. 3

c) For the function

$$f(x) = \begin{cases} x^2 + 1, & x \le 2 \\ 3x - 1, & x > 2 \end{cases}$$

(i) state whether the function is continuous at $x = 2$ (give full reasons). 2

(ii) state whether the function is differentiable at $x = 2$ (give full reasons). 2

End of Paper

YEAR 11 EXTENSION 1 YEARLY EXAMINATION 2007 SOLUTIONS.

QUESTION 1:

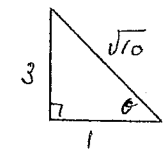
(a) $\dfrac{4}{3x-x^2} + \dfrac{3}{x^2-x-6}$

$= \dfrac{4}{x(3-x)} + \dfrac{3}{(x-3)(x+2)}$

$= \dfrac{4x+8-3x}{x(3-x)(2+x)}$

$= \dfrac{x+8}{x(3-x)(2+x)}$

(b) Base angle



θ is in 3rd quadrant

Hence $\sin\theta = -\dfrac{3}{\sqrt{10}}$

(c) (i) $h = 2+10t-5t^2$

$= 2-5(t^2-2t)$

$= 2-5[(t^2-2t+1)-1]$

$= 2-5(t-1)^2+5$

$= 7-5(t-1)^2$

$\therefore$ maximum height occurs at $t=1$

(ii) maximum height reached is 7m

(d) METHOD A: Solving simultaneously.

$3x-y-8=0$ —— ①

$(x-1)^2+(y-5)^2=10$ —— ②

from ①: $y = 3x-8$ sub in ②

$\Rightarrow (x-1)^2+(3x-13)^2 = 10$

$10x^2-80x+170 = 10$

$x^2-8x+16 = 0$

$(x-4)^2 = 0$

$\therefore x = 4$

Since there is only one point of intersection the line must be tangential to the circle.

METHOD B: $y = 3x - 8$ —— ①

$(x-1)^2 + (y-5)^2 = 10$ —— ②

Circle has centre $(1,5)$ and radius $\sqrt{10}$.

If the line $3x - y - 8 = 0$ is $\sqrt{10}$ units from $(1,5)$ then the line must be tangential to the circle.

$3x - y - 8 = 0 \qquad (1,5)$

$$d = \frac{|3(1) - 5 - 8|}{\sqrt{3^2 + (-1)^2}}$$

$$= \frac{10}{\sqrt{10}}$$

$$= \sqrt{10}$$

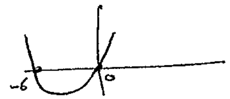
$\therefore$ Line is tangential to the circle.

(a) (i) $\dfrac{6+3x}{2x} \geq 1 \qquad x \neq 0$

$\times 2x^2]: \Rightarrow \quad x(6+3x) \geq 2x^2$

$3x^2 + 6x \geq 2x^2$

$x^2 + 6x \geq 0$



$\therefore x \leq -6$ or $x \geq 0$

but $x \neq 0 \Rightarrow x \leq -6$ or $x > 0$.

(ii) $|2x+1| > 5$

$\Rightarrow 2x + 1 > 5$ or $2x + 1 < -5$

$2x > 4 \qquad 2x < -6$

$x > 2 \qquad x < -3$

ie $x < -3$ or $x > 2$

(b) $7x^2 + 9x + 3 = 0$

(i) $\alpha + \beta = -\frac{9}{7}$

(ii) $\alpha\beta = \frac{3}{7}$

(iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\beta + \alpha}{\alpha\beta}$

$$= \frac{-\frac{9}{7}}{\frac{3}{7}}$$

$$= -3$$

(iv) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$$= \left(-\frac{9}{7}\right)^2 - 2\left(\frac{3}{7}\right)$$

$$= \frac{39}{49}$$

(v) $\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)$

$$= \left(-\frac{9}{7}\right)\left[\frac{39}{49} - \frac{3}{7}\right]$$

$$= \frac{-162}{343}$$

(c) $\log_b a \times \log_a c \times \log_c b$

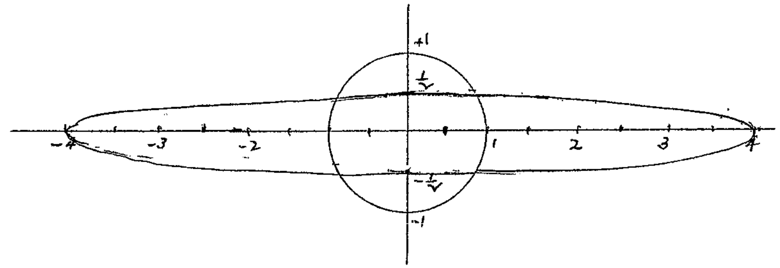
$= \frac{\cancel{\log a}}{\cancel{\log b}} \times \frac{\cancel{\log c}}{\cancel{\log a}} \times \frac{\cancel{\log b}}{\cancel{\log c}}$

$= 1$

QUESTION 3:

(a) $x \to \frac{x}{4}$ expands in x direction by factor of 4

$y \to \frac{y}{2}$ contracts in y direction by factor of 2



(b) (i) $f(x) = \frac{x}{x^2-1}$

$\Rightarrow f(-x) = \frac{(-x)}{(-x)^2-1}$

$= -\frac{x}{x^2-1}$

$= -f(x)$

$\therefore f(x)$ is an ODD function.

(ii) Domain: $x \in R,\ x \neq 1,\ x \neq -1$

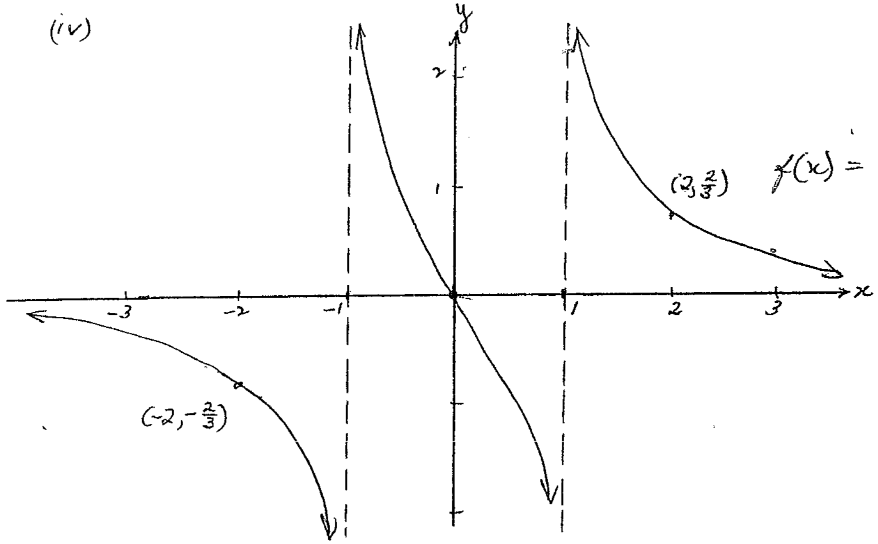
(iii) Vertical asymptotes: $x = 1,\ x = -1$

Horizontal asymptote: $\lim_{x \to \infty} \frac{x}{x^2-1}$

$= 0$

$\therefore y = 0$ is asymptote

(iv)



(v) $f(x) \geq 0 \implies -1 < x \leq 0$ or $x > 1$

(c) $\sec^2\theta - 2\tan\theta - 4 = 0 \qquad 0^\circ \leq \theta \leq 360^\circ$

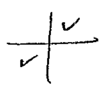
$\implies 1 + \tan^2\theta - 2\tan\theta - 4 = 0$

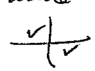
$\implies \tan^2\theta - 2\tan\theta - 3 = 0$

$(\tan\theta - 3)(\tan\theta + 1) = 0$

$\therefore \tan\theta = 3, -1$

(i) $\tan\theta = 3 \qquad \theta_{acute} = 71^\circ 34'$ (ii) $\tan\theta = -1 \qquad \theta_{acute} = 45^\circ$





$\theta = 71^\circ 34', 251^\circ 34' \qquad \theta = 135^\circ, 315^\circ$

$\therefore \theta = 71^\circ 34', 135^\circ, 251^\circ 34', 315^\circ$

QUESTION 4:

(a) (i) $y = 7x^3 - 6x + 3x^{-1}$

$\implies \frac{dy}{dx} = 21x^2 - 6 - \frac{3}{x^2}$

(ii) $y = \underbrace{x}_{u}\underbrace{(2x^2-7)^{\frac{1}{3}}}_{v}$

$\frac{dy}{dx} = vu' + uv'$

$= (2x^2-7)^{\frac{1}{3}} \cdot 1 + x \cdot \frac{1}{3}(2x^2-7)^{-\frac{2}{3}} \cdot 4x$

$= (2x^2-7)^{\frac{1}{3}} + \frac{4x^2}{3(2x^2-7)^{2/3}}$

$= \frac{3(2x^2-7) + 4x^2}{3(2x^2-7)^{2/3}}$

$= \frac{10x^2 - 21}{3(2x^2-7)^{2/3}}$

(iii) $y = \frac{5x^2+1}{2x-3} \quad \frac{u}{v}$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$= \frac{(2x-3) \cdot 10x - (5x^2+1) \cdot 2}{(2x-3)^2}$

$= \frac{20x^2 - 30x - 10x^2 - 2}{(2x-3)^2}$

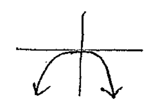
$= \frac{10x^2 - 30x - 2}{(2x-3)^2}$

(b) (i) Vertex $= (0, 0)$

(ii) $x^2 = -16y$

$= -4(4)y$

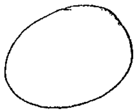
$\therefore a = 4$



$\therefore$ Focus $= (0, -4)$

(iii) Directrix : $y = 4$

c)



Let radius, surface area and volume at any instant be r, S and V.

(i) $V = \frac{4}{3}\pi r^3$

$\Rightarrow \frac{dV}{dr} = 4\pi r^2 \qquad \frac{dV}{dt} = 100 \text{ cm}^3/s \qquad \frac{dr}{dt} = ?$

Then $\frac{dr}{dt} = \frac{dr}{dV} \cdot \frac{dV}{dt}$ cm/s

$= \frac{1}{4\pi r^2} \times 100$ cm/s

$= \frac{25}{\pi r^2}$ cm/s

when $r = 5$: $\frac{dr}{dt} = \frac{1}{\pi}$ cm/s

(ii) when $r = 5$, $\frac{dr}{dt} = \frac{1}{\pi}$ cm/s

$S = 4\pi r^2$

$\Rightarrow \frac{dS}{dr} = 8\pi r$

Then $\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$ cm²/s

$= 8\pi r \times \frac{1}{\pi}$ cm²/s

when $r = 5$: $\frac{dS}{dt} = 40$ cm²/s

d)

$x^2 - y^2 = 4$

$\Rightarrow \frac{d}{dx}(x^2) - \frac{d}{dx}(y^2) = \frac{d}{dx}(4)$

$\therefore 2x - 2y\,y' = 0$

ie $y' = \frac{x}{y}$

at $(3, \sqrt{5})$ $\qquad y' = \frac{3}{\sqrt{5}}$

$\therefore$ Normal is $y - \sqrt{5} = -\frac{\sqrt{5}}{3}(x - 3)$

$3y - 3\sqrt{5} = -x\sqrt{5} + 3\sqrt{5}$

ie $x\sqrt{5} + 3y - 6\sqrt{5} = 0$

QUESTION 5:

(a) The assertion is that

$1^3 + 2^3 + \dots + n^3 = \frac{n^2}{4}(n+1)^2 \qquad n \geq 1$ —— ①

(i) Test the assertion for $n = 1$

LHS of ① $= 1$

RHS of ① $= \frac{1}{4}(1+1)^2$

$= 1$

$\therefore$ Assertion is true for $n = 1$

(ii) Assume the assertion is true for some integer $n = k$

ie $1^3 + 2^3 + \dots + k^3 = \frac{k^2}{4}(k+1)^2$ —— ②

[We now aim to prove the assertion is true for $n = k+1$

ie $1^3 + 2^3 + \dots + k^3 + (k+1)^3 = \frac{(k+1)^2}{4}(k+2)^2$ —— ③]

LHS of ③ $= 1^3 + 2^3 + \dots + k^3 + (k+1)^3$

$= \frac{k^2}{4}(k+1)^2 + (k+1)^3$ from ②

$= \frac{(k+1)^2}{4}\left[k^2 + 4(k+1)\right]$

$= \frac{(k+1)^2}{4}(k^2 + 4k + 4)$

$= \frac{(k+1)^2}{4}(k+2)^2$

$=$ RHS of ③

$\therefore$ If the assertion is true for $n = k$ then it must also be true for $n = k+1$

But true for $n = 1 \Rightarrow$ true for $n = 2$

true for $n = 2 \Rightarrow$ true for $n = 3$

and hence, by the Principle of Mathematical Induction the assertion is true for all integers $n \geq 1$

(b) Let $P(x, y)$ be a point on the locus.
A is $(-1, 2)$ B is $(3, 4)$

CONDITION : $PA = PB$

$\therefore \sqrt{(x+1)^2 + (y-2)^2} = \sqrt{(x-3)^2 + (y-4)^2}$

squaring $\Rightarrow \cancel{x^2} + 2x + 1 + \cancel{y^2} - 4y + 4 = \cancel{x^2} - 6x + 9 + \cancel{y^2} - 8y + 16$

ie $8x + 4y - 20 = 0$

ie $2x + y - 5 = 0$

(c) $f(x) = \begin{cases} x^2 + 1 & x \le 2 \\ 3x - 1 & x > 2 \end{cases}$

(i)
- $f(2) = 5$
- $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 + 1) = 5$
- $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (3x - 1) = 5$

Since these are all equal, $f(x)$ is continuous at $x = 2$.

(ii)
- $f(2)$ exists and $f(x)$ is continuous at $x = 2$
- for $x \le 2$ $f'(x) = 2x$

$\therefore \lim_{x \to 2^-} f'(x) = 2(2) = 4$

- for $x > 2$, $f'(x) = 3$

Since these limits are different $f(x)$ is not differentiable at $x = 2$