St George Girls High School

Year 11

End of Preliminary Course Examination

2008

Mathematics Extension 1

Time Allowed: 2 hours
(plus 5 minutes reading time)

Instructions

1. Attempt all 6 questions.
2. All necessary working must be shown.
3. Begin each question on a **new page**.
4. Marks will be deducted for careless work or poorly presented solutions.
5. All sketches are to be at least ⅓ page.

Question 1 (14 marks) – Start a New Page **Marks**

a) Find 'a' and 'b' if $(2+\sqrt{2})^3 = a + \sqrt{b}$ 3

b) Solve for x:

(i) $|2x-3| = x-3$ 3

(ii) $\frac{2x}{x-1} \geq 1$ 3

c) Solve for $0° \leq \theta \leq 360°$: 3

$$2\sin^2\theta + \sin\theta = 0$$

d) Sketch any function which is continuous for all x in the interval $0 \leq x \leq 2$ but is not differentiable at $x = 1$. 2

Question 2 (14 marks) – Start a New Page **Marks**

a) m and n are the roots of $5x^2 - 3x - 1 = 0$

(i) Evaluate each of the following:

(α) $m + n$ 1

(β) mn 1

(γ) $m^2 + n^2$ 2

(ii) Find the quadratic equation with roots of m^2 and n^2 2

b) Simplify $\cot\theta \, . \sec(180° - \theta)$ 2

c) Eliminate θ from the parametric equations 2

$$\left.\begin{aligned} x &= 1 - 2\sin\theta \\ y &= \cos\theta + 2 \end{aligned}\right\}$$

d) (i) Show that $(2^x)^2 = 4^x$ 1

(ii) Solve for x:

$$4^x - 3(2^x) - 4 = 0$$ 3

Question 3 (14 marks) – Start a New Page **Marks**

a) Prove, by Mathematical Induction, that $5^n - 1$ is divisible by 4 for all integers n where $n \geq 1$ 3

b) Consider the series $100 + 50 + 25 + \cdots$. The sum of the first n terms is S_n and the limiting sum is S.

(i) Show that $S - S_n = 200(\frac{1}{2})^n$ 3

(ii) Find the smallest integer n such that $S - S_n < 0.000\,02$ 2

c) Consider the function $f(x) = \frac{x^2 - 1}{x^2}$

(i) Determine whether $f(x)$ is an even function or an odd function. 1

(ii) For the graph of $y = f(x)$, determine each of the following:

(α) the x intercepts 1

(β) the vertical asymptote 1

(γ) the horizontal asymptote 1

(iii) Sketch the curve $y = f(x)$ clearly showing each of the above. 2

Question 4 (14 marks) – Start a New Page **Marks**

a) Find the vertex, focus and directrix of the parabola

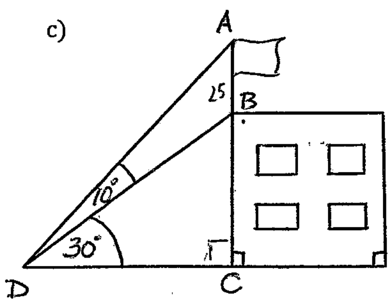
$$y^2 + 2y + 8x - 23 = 0$$ 3

b) Find the derivatives of each of the following:

(i) $\frac{3}{5x^2}$ 2

(ii) $y = \frac{\sqrt{x}-1}{\sqrt{x}}$ 2

c)



The diagram shows a flagpole AB on top of a building.

If $A\hat{D}B = 10°$, $B\hat{D}C = 30°$ and $AB = 25$ metres find the height of the building BC. 4

d) (i) Find the co-ordinates of the point $P(x, y)$ which divides the join of $A(-1, -4)$ to $B(1, -2)$ in the ratio $3:2$ externally. 2

(ii) In what ratio does B divide AP? 1

Question 5 (14 marks) – Start a New Page **Marks**

a) A spherical balloon is filling at the rate of $10\pi\ m^3$/hour. When the radius is 2m, find the rate of increase of:

(i) the radius 3

(ii) the surface area 2

b) Find the distance between the parallel lines $x - y - 2 = 0$ and $x - y + 1 = 0$ 3

c) Find $\frac{d^2y}{dx^2}$ if $y = (x^2 + 1)^5$. Express your answer in factored form. 4

d) Sketch the graph of $y = 1 - 2^x$ 2

Question 6 (14 marks) – Start a New Page **Marks**

a) The gradient function of a curve is given by $\frac{dy}{dx} = 2x - 1$ 3

Find the equation of the curve if it passes through $(2, 5)$

b) Consider the function $f(x) = x^3 - 3x$. For the graph of $y = f(x)$

(i) find the x-intercepts 1

(ii) find any stationary points and their nature 4

(iii) sketch $y = f(x)$ clearly showing all of the above. 2

c) Use your graph in (b) above to determine the number of solutions of

(i) $x^3 - 3x = -1$ 1

(ii) $x^3 + 2x + 3 = 0$. (justify your answer) 3

End of Paper

Year 11 2008

Mathematics Extension 1 End of Preliminary Course

Question 1

Marks

3 (a) $a+\sqrt{b} = (2+\sqrt{3})^3$

$= 2^3 + 3\times 2^2 \times\sqrt{3} + 3\times 2\times(\sqrt{3})^2 + (\sqrt{3})^3$

$= 8 + 12\sqrt{3} + 6\times 3 + 3\sqrt{3}$

$= 26 + 15\sqrt{3}$

$= 26 + \sqrt{15^2\times 3}$

$\therefore\ a = 26 \quad b = 675$

3 (b) (i) $|2x-3| = x-3$

Since LHS ≥ 0 $\quad x-3 \geq 0$

ie $x \geq 3$

$2x-3 = x-3$ or $-(2x-3) = x-3$

$x = 0$ or $-2x+3 = x-3$

$6 = 3x$

$x = 2$

Since $x \geq 3$ there are no solutions

OR Solve as above then check solutions in original equation

When $x=0$ LHS $= |2\times 0-3| = 3$

RHS $= 0-3 = -3 \neq$ LHS

When $x=2$ LHS $= |2\times 2-3| = 1$

RHS $= 2-3 = -1 \neq$ LHS

$\therefore$ The equation has no solutions

3 (ii) $\dfrac{2x}{x-1} \geq 1 \qquad x \neq 1$

$\dfrac{2x}{x-1} - 1 \geq 0$

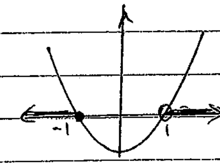
$\dfrac{2x-(x-1)}{x-1} \geq 0$

$\dfrac{x+1}{x-1} \geq 0$

$(x-1)^2 \times \dfrac{(x+1)}{x-1} \geq (x-1)^2 \times 0$

$(x-1)(x+1) \geq 0 \qquad (x \neq 1)$

$x \leq -1$ or $x > 1$



3 (c) $2\sin^2\theta + \sin\theta = 0 \qquad 0^\circ \leq \theta \leq 360^\circ$

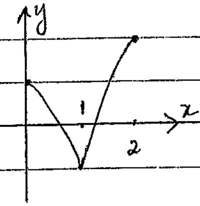
$\sin\theta\,(2\sin\theta + 1) = 0$

$\sin\theta = 0$ or $\sin\theta = -\frac{1}{2} \qquad \theta_{acute} = 30^\circ$

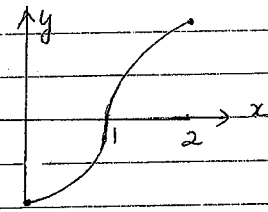
$\theta = 0^\circ, 180^\circ, 360^\circ$ or $\theta = 180^\circ+30^\circ, 360^\circ-30^\circ$

$\therefore\ \theta = 0^\circ, 180^\circ, 210^\circ, 330^\circ, 360^\circ$

2 (d)



OR



$\therefore$ The graph of any function that is continuous over $0 \leq x \leq 2$ and has a sharp turn or a vertical tangent at $x=1$

Question 2

(a) $5x^2 - 3x - 1 = 0$

1, 1 (i) (α) $m+n = \frac{3}{5}$ (β) $mn = -\frac{1}{5}$

2 (γ) $m^2+n^2 = (m+n)^2 - 2mn$

$= \left(\frac{3}{5}\right)^2 - 2\times\left(-\frac{1}{5}\right)$

$= \frac{19}{25}$

2 (ii) Equation is $(x-m^2)(x-n^2) = 0$

$x^2 - (m^2+n^2)x + m^2n^2 = 0$

$x^2 - \frac{19}{25}x + \left(-\frac{1}{5}\right)^2 = 0$

$25x^2 - 19x + 1 = 0$

2 (b) $\cot\theta \sec(180^\circ - \theta) = \frac{\cos\theta}{\sin\theta} \times \frac{1}{\cos(180^\circ - \theta)}$

$= \frac{\cos\theta}{\sin\theta} \times \frac{1}{-\cos\theta}$

$= -\frac{1}{\sin\theta}$

$(= -\operatorname{cosec}\theta)$

2 (c) $x = 1 - 2\sin\theta$ $y = \cos\theta + 2$

$\sin\theta = \frac{1-x}{2}$ $\cos\theta = y - 2$

$\left(\frac{1-x}{2}\right)^2 + (y-2)^2 = \sin^2\theta + \cos^2\theta$

$=$

$\frac{(x-1)^2}{4} + (y-2)^2 = 1$

1 (d) (i) $(2^x)^2 = 2^{2x}$

$= (2^2)^x$

$= 4^x$

3 (ii) $4^x - 3(2^x) - 4 = 0$

Let $m = 2^x$

$m^2 - 3m - 4 = 0$

$(m-4)(m+1) = 0$

$m = 4, -1$

$2^x = 4, -1$

But $2^x > 0$ $(\neq -1$

$2^x = 2^2$

$x = 2$

Question 3

3 (a) Rtp $5^n - 1 = 4A$ (where A is an integer) for $n \geq 1$

When $n=1$ $\quad LHS = 5^1 - 1 = 4 = 4 \times 1$

$\therefore$ Proposition is true for $n=1$

Let k be a positive integer for which proposition is true

ie $\quad 5^k - 1 = 4M$ where M is an integer

Aim to show that proposition is then true for $n=k+1$

ie $\quad 5^{k+1} - 1 = 4Y$ where Y is a positive integer

Now $\quad 5^{k+1} - 1 = 5^k \times 5 - 1$

$$= 5(4M+1) - 1$$
$$= 5 \times 4M + 5 - 1$$
$$= 4 \times 5M + 4$$
$$= 4(5M+1)$$
$$= 4Y \quad (Y \text{ is an integer})$$

$\therefore$ If proposition is true for integer $n=k$ it is also true for the next integer $n=k+1$.

Since it is true for $n=1$ it is also true for $n=2$ and hence by induction for all positive integers

(b) $100 + 50 + 25 + \ldots$

Geometric series: $\quad a = 100 \quad r = \frac{1}{2}$

3 (i) $\quad S - S_n = \dfrac{a}{1-r} - \dfrac{a(1-r^n)}{1-r}$

$$= \frac{a - a + ar^n}{1-r}$$
$$= \frac{ar^n}{1-r}$$
$$= \frac{100 \times \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}}$$
$$= 200 \times \left(\frac{1}{2}\right)^n.$$

2 (ii) $\quad S - S_n < 0.00002$

$200\left(\frac{1}{2}\right)^n < 0.00002$

$\left(\frac{1}{2}\right)^n < \dfrac{0.00002}{200}$ $\qquad$ OR $\quad 2^n > \dfrac{200}{0.00002}$

$\left(\frac{1}{2}\right)^n < 0.0000001$ $\qquad$ $2^n > 10^7$

$n \log\left(\frac{1}{2}\right) < \log\left(10^{-7}\right)$ $\qquad$ $n \log 2 > 7$

$n > \dfrac{-7}{\log\left(\frac{1}{2}\right)}$ $\qquad$ $n > \dfrac{7}{\log 2}$

$n > 23.25\ldots$ $\qquad$ $n > 23.25\ldots$

Smallest integer is 24

1 (c) $\quad f(x) = \dfrac{x^2 - 1}{x^2}$

$f(-x) = \dfrac{(-x)^2 - 1}{(-x)^2}$

$$= \frac{x^2 - 1}{x^2}$$

$= f(x)$ $\qquad \therefore f(x)$ is an even function

1 (ii) (α) If $y=0$ $\quad x^2-1=0$

$$x = 1, -1$$

$\therefore$ x-intercepts are $1, -1$

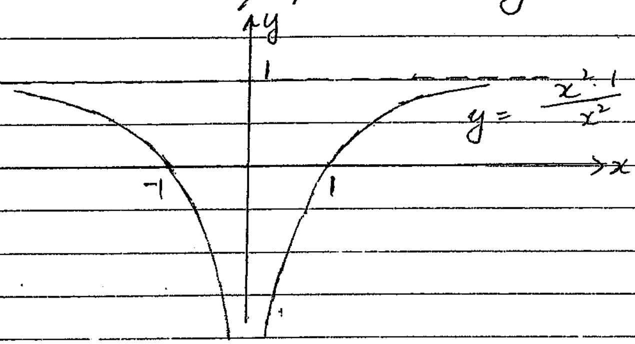
1 (β) Vertical asymptote is $x=0$ (ie y axis)

1 (γ) $\lim\limits_{x\to\infty} \dfrac{x^2-1}{x^2} = \lim\limits_{x\to\infty} \dfrac{1-\frac{1}{x^2}}{1}$

$$= 1$$

$\therefore$ Horizontal asymptote is $y = 1$

2 (iii)



Question 4

3 (a) $y^2 + 2y + 8x - 23 = 0$

$$y^2 + 2y + 1 = -8x + 23 + 1$$

$$(y+1)^2 = -8x + 24$$

$$= -8(x-3)$$

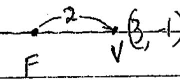
$$(y+1)^2 = -4 \times 2(x-3)$$

Vertex is $(3, -1)$

Focal length $= 2$

Hence Focus is $(1, -1)$

Directrix is $x = 5$



2 (b) (i) $f(x) = \dfrac{3}{5x^2}$

$$= \frac{3}{5} x^{-2}$$

$$f'(x) = -\frac{6}{5} x^{-3}$$

$$= \frac{-6}{5x^3}$$

2 (ii) $y = \dfrac{\sqrt{x}-1}{\sqrt{x}}$

$$= 1 - \frac{1}{\sqrt{x}}$$

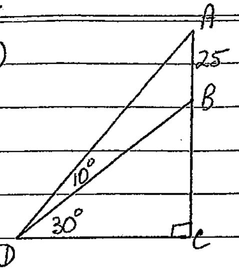
$$= 1 - x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2} x^{-\frac{3}{2}}$$

$$= \frac{1}{2x^{\frac{3}{2}}}$$

$$= \frac{1}{2x\sqrt{x}}$$

4 (c)



In ΔABD

$D\hat{A}B = 50°$

$\therefore \dfrac{BD}{\sin 50°} = \dfrac{25}{\sin 10°}$

$BD = \dfrac{25 \sin 50°}{\sin 10°}$

In ΔBCD $\quad \dfrac{BC}{BD} = \sin 30°$

$BC = BD \sin 30°$

$= \dfrac{25 \sin 50° \sin 30°}{\sin 10°}$

$= 55.143\ldots$

$\therefore$ Height of building is 55.14 m (2dp)

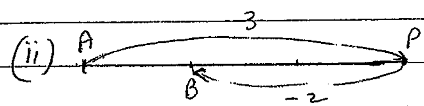
2 (d) (i) $A(-1,-4)$ $\quad B(1,-2)$

$3 : -2$

$P(x,y) \equiv \left(\dfrac{-2\times -1 + 3\times 1}{3+-2}, \dfrac{-2\times -4 + 3\times -2}{3+-2} \right)$

$(5, 2)$

1 (ii)



B divides AP in the ratio 1:2

Question 5

(a) $V = \frac{4}{3}\pi r^3$ $\qquad \dfrac{dV}{dt} = 10\pi$

3 (i) $\dfrac{dV}{dt} = \dfrac{dV}{dr} \cdot \dfrac{dr}{dt}$

$10\pi = 4\pi r^2 \cdot \dfrac{dr}{dt}$

$\dfrac{dr}{dt} = \dfrac{10\pi}{4\pi r^2} = \dfrac{5}{2r^2}$

When $r = 2$ $\quad \dfrac{dr}{dt} = \dfrac{5}{2\times 2^2} = \dfrac{5}{8}$

$\therefore$ Rate of increase of radius is $\frac{5}{8}$ m/h when $r = 2$

2 (ii) Let A be surface area $\qquad A = 4\pi r^2$

$\dfrac{dA}{dt} = \dfrac{dA}{dr} \dfrac{dr}{dt}$

$= 8\pi r \cdot \dfrac{5}{2r^2}$

When $r = 2$

$\dfrac{dA}{dt} = 8\pi \times 2 \times \dfrac{5}{8}$

$= 10\pi$

$\therefore$ Rate of increase of surface area is 10π m^2/h when $r = 2$

3 (b) $x - y - 2 = 0$ ①

When $x = 2$ $y = 0$

$\therefore (2,0)$ lies on ①

Hence find distance from $(2,0)$ to

$x - y + 1 = 0$

$$d = \frac{|2 - 0 + 1|}{\sqrt{1^2 + (-1)^2}}$$

$$= \frac{3}{\sqrt{2}}$$

$$\left(= \frac{3\sqrt{2}}{2}\right)$$

4 (c) $y = (x^2+1)^5$

$$\frac{dy}{dx} = 5(x^2+1)^4 \times 2x$$

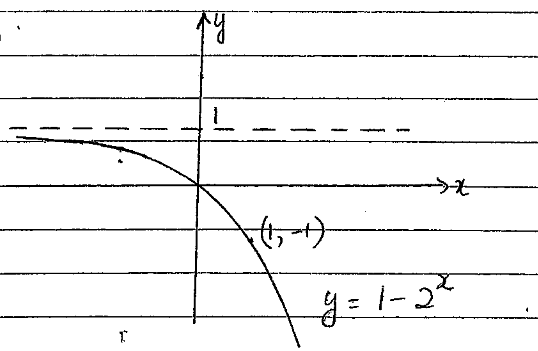
$$= 10x(x^2+1)^4$$

$$\frac{d^2y}{dx^2} = 10(x^2+1)^4 + 10x \cdot 4(x^2+1)^3 \times 2x$$

$$= 10(x^2+1)^3 \left[x^2 + 1 + 8x^2\right]$$

$$= 10(x^2+1)^3(9x^2+1)$$

2 (d)



Question 6

3 (a) $\frac{dy}{dx} = 2x - 1$

$$y = x^2 - x + c$$

When $x = 2$ $y = 5$

$$5 = 4 - 2 + c$$

$$3 = c$$

$\therefore$ Equation of curve is

$$y = x^2 - x + 3$$

(b) $y = f(x) = x^3 - 3x$

1 (i) When $y = 0$

$$x(x^2 - 3) = 0$$

$$x(x - \sqrt{3})(x + \sqrt{3}) = 0$$

$$x = 0, \sqrt{3}, -\sqrt{3}$$

ie x-intercepts are $0, \sqrt{3}, -\sqrt{3}$

4 (ii) $y = x^3 - 3x$

$$\frac{dy}{dx} = 3x^2 - 3$$

$$\frac{d^2y}{dx^2} = 6x$$

Stationary points occur when $\frac{dy}{dx} = 0$

$$3(x^2 - 1) = 0$$

$$(x-1)(x+1) = 0$$

$$x = 1, -1$$

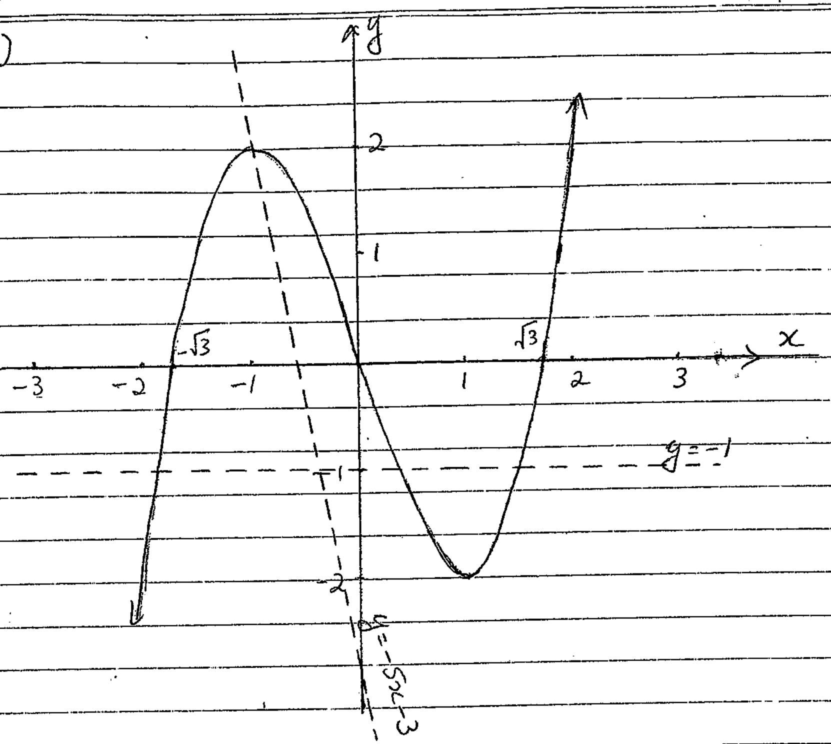
$$y = -2, 2$$

$$\frac{d^2y}{dx^2} = 6, -6$$

$(1, -2)$ is a minimum turning point since $\frac{d^2y}{dx^2} > 0$

$(-1, 2)$ is a maximum turning point since $\frac{d^2y}{dx^2} < 0$

2 (iii)



1 (c) (i) $x^3 - 3x = -1$ has 3 distinct (real) solutions since $y = -1$ meets $y = x^3 - 3x$ in 3 points

3 (ii)

$$
\begin{aligned}
x^3 + 2x + 3 &= 0 \\
x^3 &= -2x - 3 \\
x^3 - 3x &= -2x - 3 - 3x \\
x^3 - 3x &= -5x - 3
\end{aligned}
$$

$y = x^3 - 3x$ and $y = -5x - 3$ meet in only one point and so $x^3 + 2x + 3 = 0$ has one and only one (real) solution.