St George Girls High School

Year 12

Mid-HSC Course Examination

2007

Mathematics Extension 2

General Instructions

- Start each question on a new page.
- Attempt questions 1, 2, 3 and 4.
- Marks for each question are shown.
- All necessary working should be shown in every question.
- Marks may not be awarded for careless or poorly presented work.
- Board-approved calculators may be used.
- A table of standard integrals is provided.

Total marks – ~~75~~ 60

- Working time: 1½ hours
- Reading time: 5 minutes
- Write using blue or black pen.
- Do curve sketching questions on the number planes provided.

Question 1 – **(11 marks)** – (Start a new page) **Marks**

Let α, β and γ be the roots of $x^3+qx+r=0$

a) Find the polynomial equation with roots.

(i) $\alpha-1,\ \beta-1,\ \gamma-1$ 1

(ii) $2\alpha,\ 2\beta,\ 2\gamma$ 1

(iii) $\alpha^{-2},\ \beta^{-2},\ \gamma^{-2}$ 2

(iv) $\alpha+\beta,\ \beta+\gamma,\ \gamma+\alpha$ 2

b) Evaluate:

(i) $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}+\dfrac{1}{\gamma^2}$ 1

(ii) $\alpha^3+\beta^3+\gamma^3$ 2

(iii) $(\alpha+\beta-\gamma)(\beta+\gamma-\alpha)(\gamma+\alpha-\beta)$ 2

Question 2 – (9 marks) – (Start a new page) **Marks**

a) (i) Prove that if $x=\alpha$ is a root of $P(x)=0$ with multiplicity p then $x=\alpha$ is a root of $P'(x)=0$ with multiplicity $(p-1)$. 2

(ii) If $P(x)=x^3+5x^2+3x-9$ has a repeated zero, factor $P(x)$ over the real numbers. 2

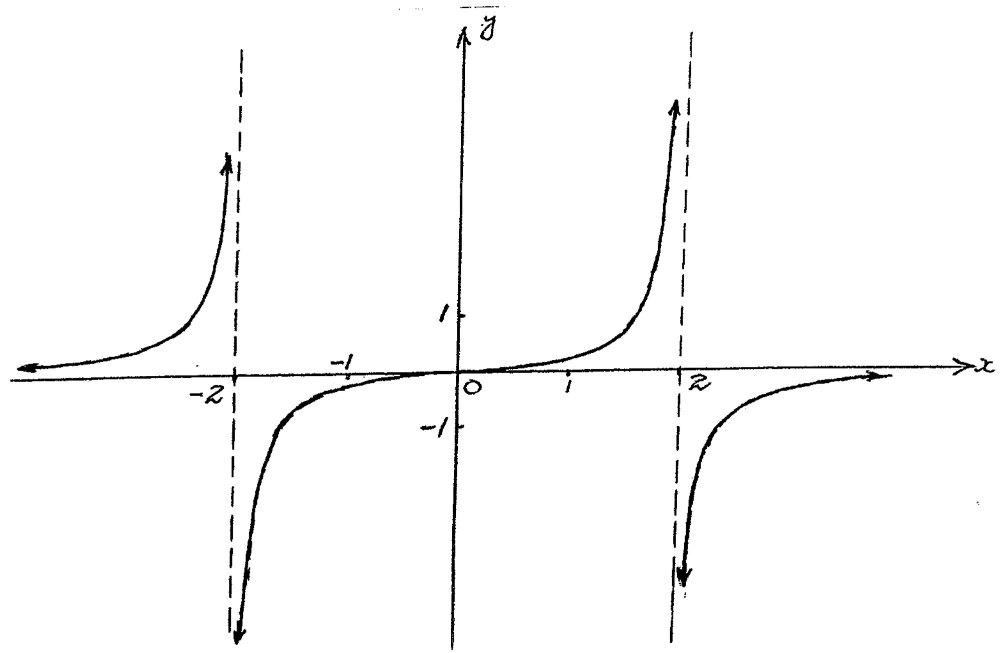
b) (i) Solve the equation $x^4+4x^3-6x^2-4x+1=0$ given that

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$$ 3

(ii) Hence show that $\tan\frac{3\pi}{16}+\tan\frac{7\pi}{16}-\tan\frac{\pi}{16}-\tan\frac{5\pi}{16}=4$ 2

Question 3 – (20 marks) – (Start a new page) **Marks**

a) The graph of $y=f(x)$ is shown below.



NOTES: 1. $y=0$ is a horizontal asymptote.
2. $x=2$ and $x=-2$ are vertical asymptotes.

On the graph paper provided draw separate sketches of each of the following clearly showing the relationships with $y=f(x)$

(i) $y=[f(x)]^2$ 2

(ii) $y^2=f(x)$ 2

(iii) $y=f(|x|)$ 2

(iv) $y=\frac{1}{f(x)}$ 2

(v) $y=f'(x)$ 2

(vi) $y=e^{f(x)}$ 2

Question 3 (cont'd) **Marks**

b) (i) On the same number plane sketch 2

$$y=|x|-2 \quad \text{and} \quad y=4+3x-x^2$$

(ii) Hence, or otherwise, solve for x: 2

$$\frac{|x|-2}{4+3x-x^2}>0$$

c) Consider the function $x^5+y^5=1$

(i) Find all stationary points and critical points. 2

(ii) Carefully explain the behaviour of the function as $x \to \pm\infty$ 1

(iii) Sketch the function. 1

<u>Question 4</u> – (20 marks) – (Start a new page) **Marks**

a) (i) Express $z=\dfrac{1+2i}{1-3i}$ in the form $a+ib$ 1

(ii) Hence find the modulus and argument of z. 1

b) (i) Find $\sqrt{-8+6i}$ in the form $a+ib$ where a, b are real numbers 2

(ii) Hence solve for z: $2z^2-(3+i)z+2=0$ 2

c) Find the complex number z such that $\text{Im}(z)=3$ and z^2+5 is imaginary. 2

d) The triangle OAB is isosceles with $OA=OB$ and angle OBA equal to 75°. O is the origin and A represents the complex number $-\sqrt{3}+i$.

(i) Carefully explain why B could represent the complex number $2\text{cis}\,\pi$. 2

(ii) Find one other complex number, in the form $a+ib$, which could be represented by B. 2

e) z_1, z_2 and z_3 are three complex numbers represented by the points Z_1, Z_2 and Z_3 respectively in the Argand diagram.

If $\arg z_1=\alpha$, $\arg z_2=\beta$ and $\arg z_3=\gamma$ and $z_1 z_2 = z_3{}^2$

(i) Show that $\alpha+\beta=2\gamma$ 1

(ii) Show that OZ_3 bisects $Z_1\hat{O}Z_2$ where O is the origin. 2

f) $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$ are complex numbers. If z_1+z_2 and $z_1 z_2$ are both negative real numbers.

(i) Show that $y_1=-y_2$ 1

(ii) prove that z_1 and z_2 must be real numbers. 4

End of Paper

QUESTION 1:

(a) $P(x) \equiv x^3 + qx + r = 0$

(i) $y = x - 1 \Rightarrow x = y + 1$

$\therefore P(x+1) = 0$

$\Rightarrow (x+1)^3 + q(x+1) + r = 0$

$x^3 + 3x^2 + x(3+q) + (1+q+r) = 0$

(ii) $y = 2x \Rightarrow x = \frac{y}{2}$

$\therefore P(\frac{x}{2}) = 0$

$\Rightarrow (\frac{x}{2})^3 + q(\frac{x}{2}) + r = 0$

$\frac{x^3}{8} + \frac{qx}{2} + r = 0$

$\Rightarrow x^3 + 4qx + 8r = 0$

(iii) $y = x^{-2} \Rightarrow x^2 = \frac{1}{y}$

$x = \frac{1}{\sqrt{y}}$

$\therefore P(\frac{1}{\sqrt{x}}) = 0$

$\Rightarrow (\frac{1}{\sqrt{x}})^3 + q(\frac{1}{\sqrt{x}}) + r = 0$

$\frac{1}{x\sqrt{x}} + \frac{q}{\sqrt{x}} + r = 0$

$1 + qx + rx\sqrt{x} = 0$

$rx\sqrt{x} = -1 - qx$

$r^2 x^3 = 1 + 2qx + q^2x^2$

ie $r^2x^3 - q^2x^2 - 2qx - 1 = 0$

(iv) α, β, γ are roots of $P(x) = 0$

$\therefore \sum \alpha = 0$

ie $\alpha + \beta + \gamma = 0$

$$\left.\begin{aligned} \therefore \alpha + \beta &= -\gamma \\ \beta + \gamma &= -\alpha \\ \gamma + \alpha &= -\beta \end{aligned}\right\} \Rightarrow \quad \begin{aligned} y &= -x \\ \therefore x &= -y \end{aligned}$$

ie $P(-x) = 0$

$\Rightarrow (-x)^3 + q(-x) + r = 0$

$-x^3 - qx + r = 0$

ie $x^3 + qx - r = 0$

(b) (i) from (a)(iii) above

$\sum \alpha^{-2} = \frac{q^2}{r^3}$

(ii) α is a root of $P(x) = 0 \Rightarrow \alpha^3 + q\alpha + r = 0$ — (1)

β " " " $\Rightarrow \beta^3 + q\beta + r = 0$ — (2)

γ " " " $\Rightarrow \gamma^3 + q\gamma + r = 0$ — (3)

(1) + (2) + (3) : $\sum \alpha^3 + q\sum\alpha + 3r = 0$

$\therefore \sum\alpha^3 = -q\sum\alpha - 3r$

ie $\alpha^3 + \beta^3 + \gamma^3 = -3r$

(iii) Since $\alpha + \beta + \gamma = 0$

then $\alpha + \beta = -\gamma$

$\beta + \gamma = -\alpha$

$\gamma + \alpha = -\beta$

$\therefore (\alpha + \beta - \gamma)(\beta + \gamma - \alpha)(\gamma + \alpha - \beta)$

$= (-2\gamma)(-2\alpha)(-2\beta)$

QUESTION 2:

(a) (i) Let $P(x) = (x-\alpha)^p . Q(x)$ where $Q(\alpha) \neq 0$

Then $P'(x) = Q(x) . p(x-\alpha)^{p-1} + (x-\alpha)^p . Q'(x)$

$$= (x-\alpha)^{p-1}\left[p\,Q(x) + (x-\alpha)\,Q'(x)\right]$$

$$= (x-\alpha)^{p-1} . R(x)$$

where $R(\alpha) = p\,Q(\alpha) + 0$

$\neq 0$ since $Q(\alpha) \neq 0$

$\therefore \alpha$ is a root of multiplicity $p-1$.

(ii) $P(x) = x^3 + 5x^2 + 3x - 9$

$P'(x) = 3x^2 + 10x + 3$

$P'(x) = 0 \Rightarrow 3x^2 + 10x + 3 = 0$

$(x+3)(3x+1) = 0$

$\therefore x = -3, -\frac{1}{3}$

$P(-3) = (-3)^3 + 5(-3)^2 + 3(-3) - 9$

$= -27 + 45 - 9 - 9$

$= 0$

$\therefore -3$ is the repeated zero

$\therefore P(x) = (x+3)^2(x-1)$ by observation.

(b) (i) $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$ ——— ①

$\Rightarrow 4x - 4x^3 = x^4 - 6x^2 + 1$

$$\Rightarrow \frac{4x - 4x^3}{1 - 6x^2 + x^4} = 1$$

let $x = \tan\theta$

$$\Rightarrow \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta} = 1$$

$\Rightarrow \tan 4\theta = 1$

ie $4\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \ldots$

$\therefore \theta = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}, \ldots$

Then $x = \tan\theta$

$$= \tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, \underbrace{\tan\frac{9\pi}{16}}_{\tan\left(\pi - \frac{7\pi}{16}\right) = -\tan\frac{7\pi}{16}}, \underbrace{\tan\frac{13\pi}{16}}_{\tan\left(\pi - \frac{3\pi}{16}\right) = -\tan\frac{3\pi}{16}}$$

(ii) Sum of roots of ① $= -4$

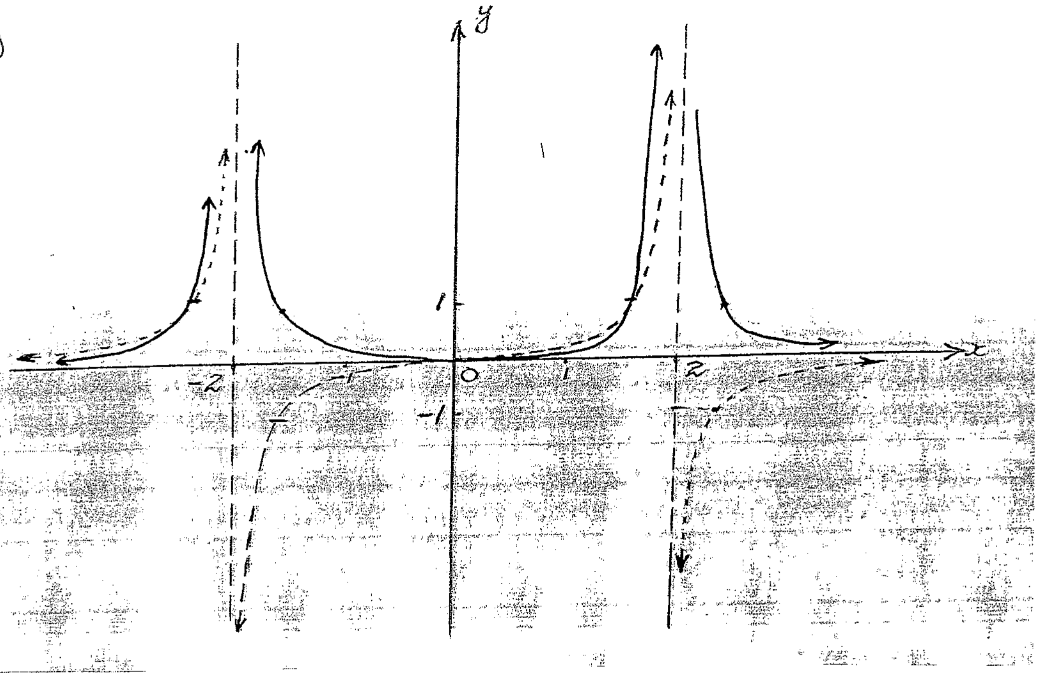
$\Rightarrow \tan\frac{\pi}{16} + \tan\frac{5\pi}{16} - \tan\frac{7\pi}{16} - \tan\frac{3\pi}{16} = -4$

$\Rightarrow \tan\frac{3\pi}{16} + \tan\frac{7\pi}{16} - \tan\frac{\pi}{16} - \tan\frac{5\pi}{16} = 4$

QUESTION 3:

(a)

(i)



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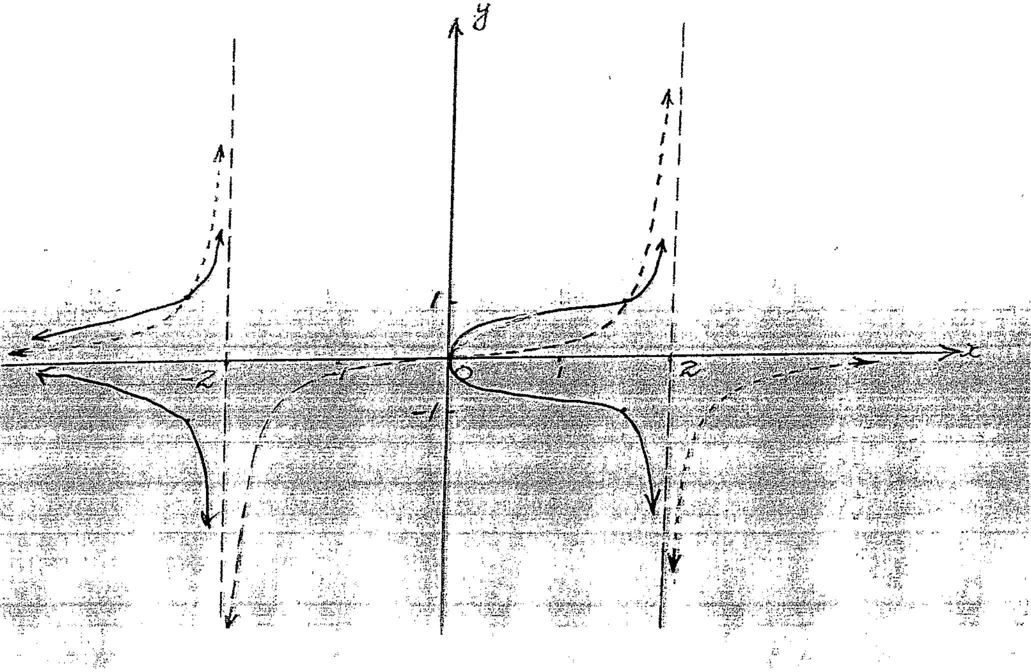

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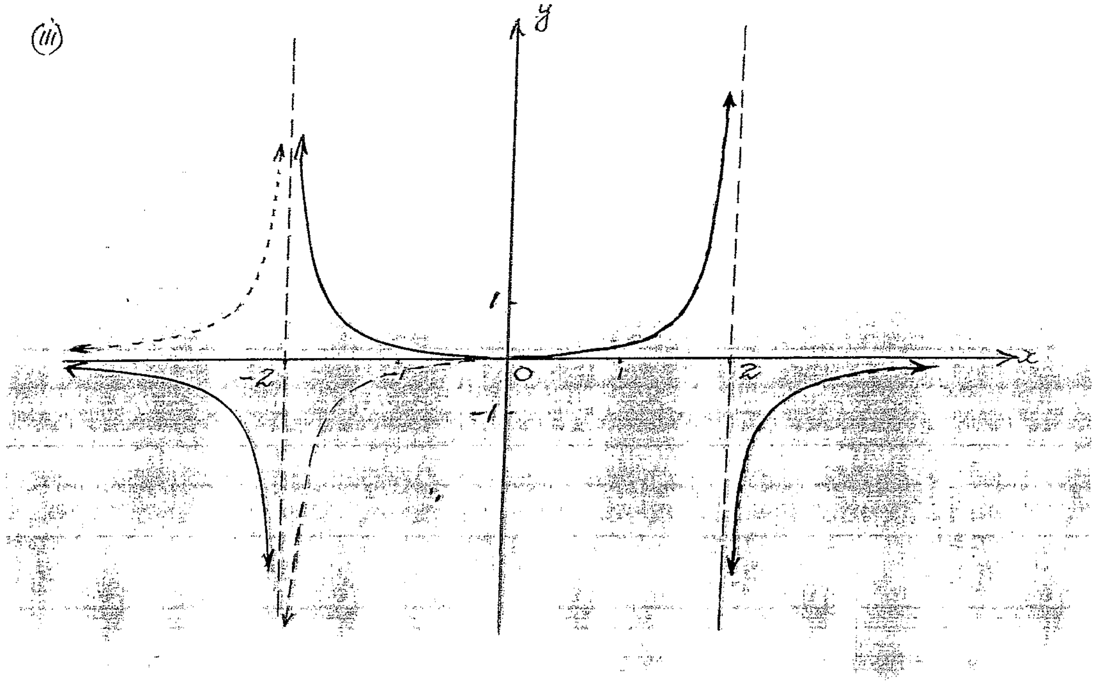

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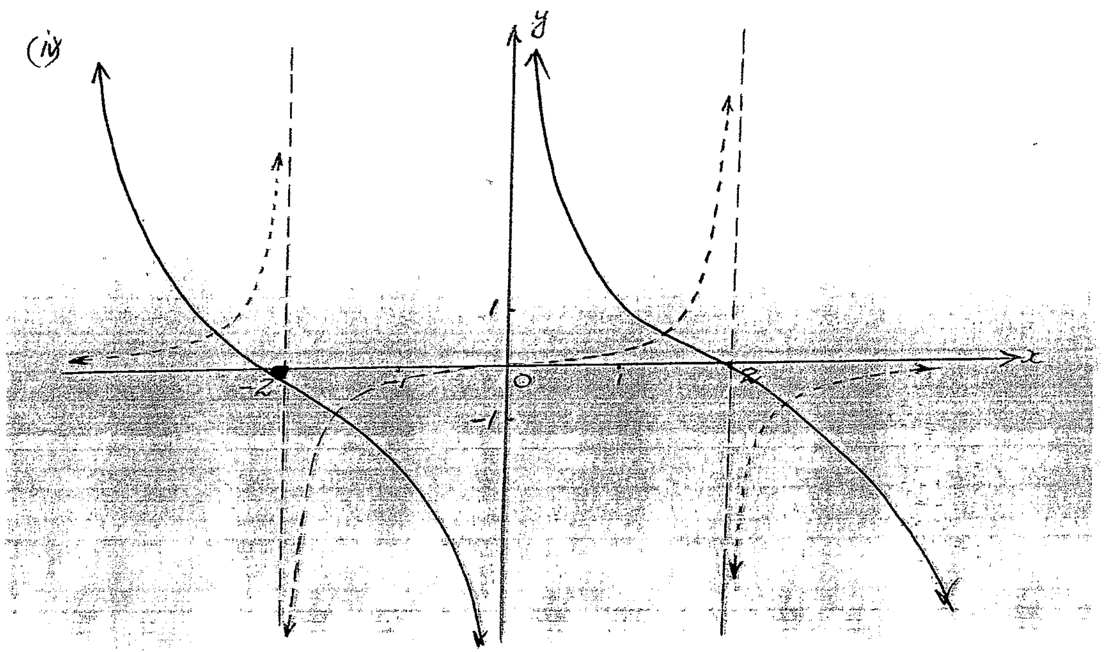

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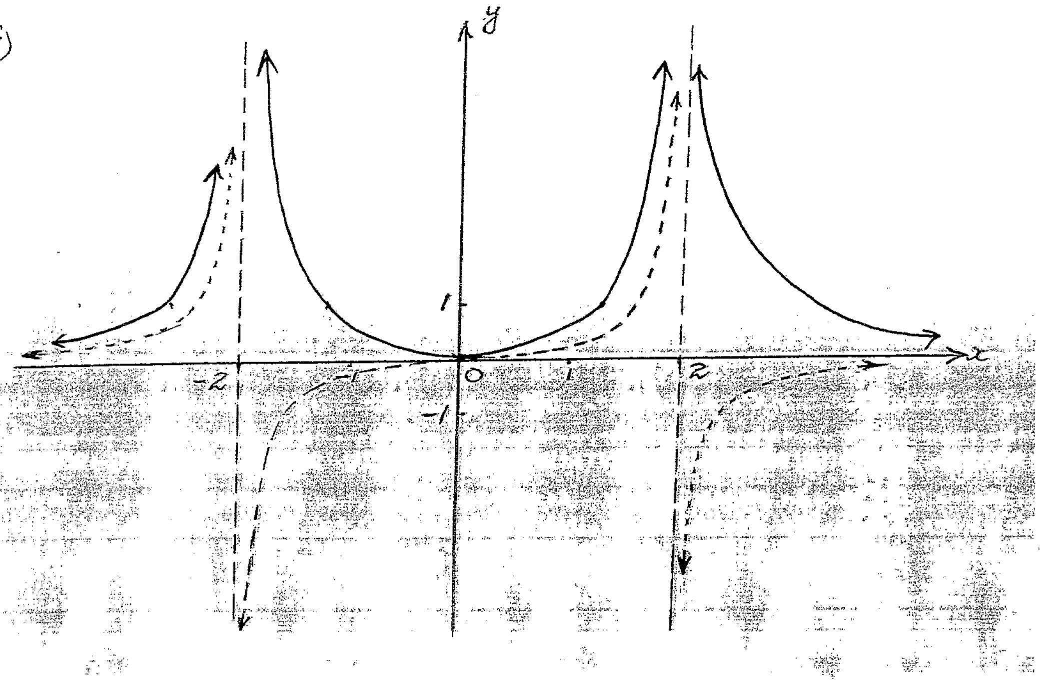
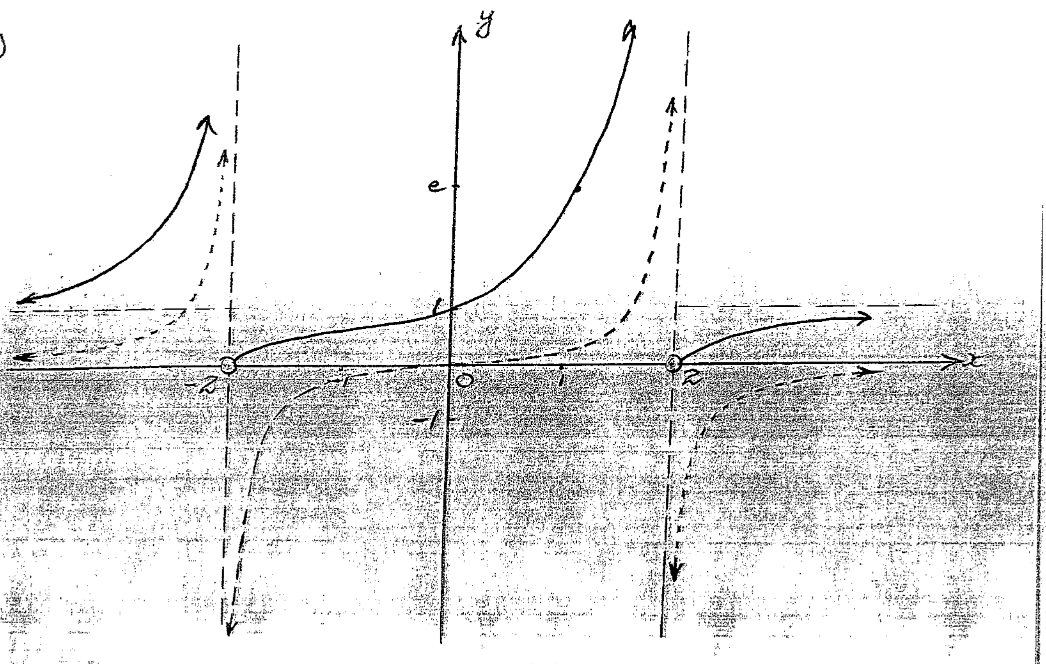
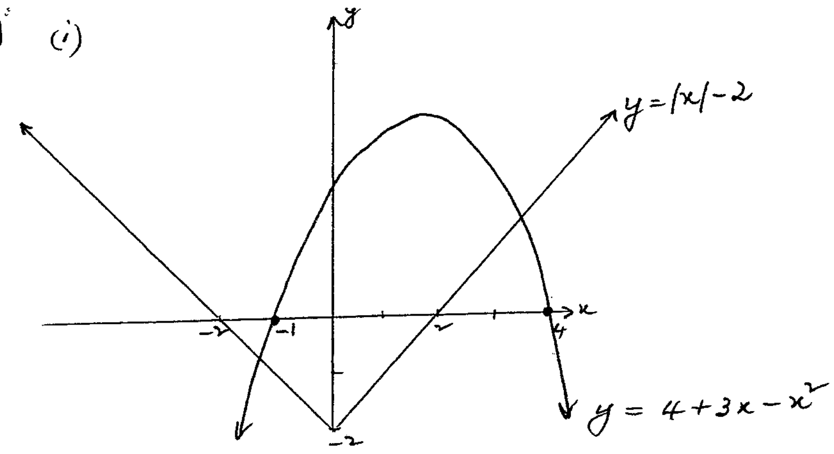


(vi)



(b) (i)



$y = 4 + 3x - x^2$

$\quad = (1+x)(4-x)$

(ii) $\dfrac{|x|-2}{4+3x-x^2} > 0$ when

- $|x|-2 > 0$ <u>and</u> $4+3x-x^2 > 0$

  ie $2 < x < 4$

- $|x|-2 < 0$ <u>and</u> $4+3x-x^2 < 0$

  ie $-2 < x < -1$ OR

ie $2 < x < 4$ OR $-2 < x < -1$

(c) (i) $x^5 + y^5 = 1$

$$\frac{d}{dx}(x^5) + \frac{d}{dx}(y^5) = 0$$

$$5x^4 + 5y^4\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x^4}{y^4}$$

$$= 0 \text{ at } (0,1)$$

is undefined at $(1,0)$

(ii) as $x \to \infty$

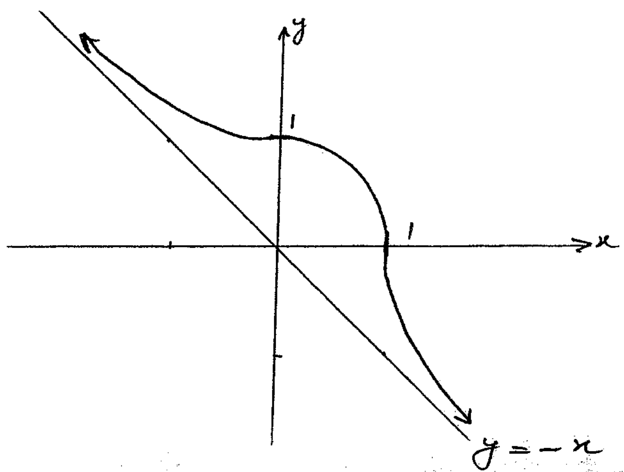
since $y^5 = 1 - x^5$

$\therefore y \to -x$

as $x \to -\infty$

$y \to -x$

(iii)



QUESTION 4:

(a) (i) $z = \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i}$

$$= \frac{1+5i-6}{10}$$

$$= -\frac{1}{2} + \frac{1}{2}i$$

(ii) $|z| = \sqrt{(-\frac{1}{2})^2 + (\frac{1}{2})^2}$

$$= \frac{1}{\sqrt{2}}$$

$$\arg z = \frac{3\pi}{4}$$

(b) (i) let $x + iy = \sqrt{-8+6i}$

$$(x^2-y^2) + 2xyi = -8+6i$$

$\therefore x^2 - y^2 = -8$ ——— (1)

$xy = 3$ ——— (2)

from (2) $y = \frac{3}{x}$ sub in (1)

$$x^2 - \frac{9}{x^2} = -8$$

$$x^4 + 8x^2 - 9 = 0$$

$$(x^2+9)(x^2-1) = 0$$

$\therefore x = 1, -1$ sub in (2)

$y = 3, -3$

$\therefore \sqrt{-8+6i} = 1+3i, -1-3i$

(ii) $z = \frac{3+i \pm \sqrt{(3+i)^2 - 4(2)(2)}}{4}$

$$= \frac{3+i \pm \sqrt{-8+6i}}{4}$$

$$= \frac{3+i \pm (1+3i)}{4}$$

(c) let $z = x + 3i$

Then $z^2 + 5 = (x+3i)^2 + 5$

$= x^2 + 6xi - 9 + 5$

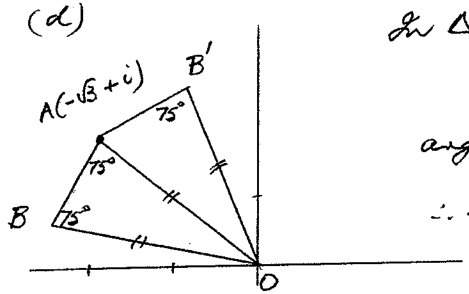
$= (x^2 - 4) + 6x\,i$

If imaginary then $x^2 - 4 = 0$

$x = \pm 2$

$\therefore z = 2 + 3i, \ -2 + 3i$

(d)



In $\triangle OBA$ $\quad O\hat{A}B = 75^\circ$ $\quad (OA = OB)$

$\therefore B\hat{O}A = 30^\circ$

$\arg(-\sqrt{3}+i) = \frac{5\pi}{6}$ $\qquad |-\sqrt{3}+i| = 2$

$\therefore \arg B = \frac{5\pi}{6} + \frac{\pi}{6}$

$= \pi$

and $OB = |OA|$

$= 2$

$\therefore B \equiv 2\operatorname{cis}\pi$

also $\qquad B' \equiv 2\operatorname{cis}\left(\frac{5\pi}{6} - \frac{\pi}{6}\right)$

$= 2\operatorname{cis}\frac{2\pi}{3}$

$= 2\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]$

$= 2\left[-\frac{1}{2} + i\cdot\frac{\sqrt{3}}{2}\right]$

$= -1 + \sqrt{3}\,i$

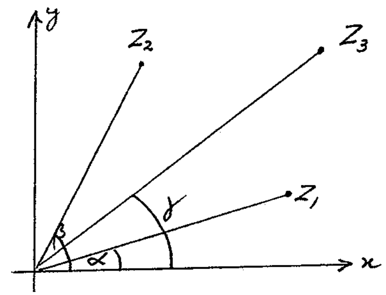
(e) (i) $z_1 z_2 = z_3^2$

$\therefore \arg(z_1 z_2) = \arg z_3^2$

$\arg z_1 + \arg z_2 = 2\arg z_3$

$\therefore \alpha + \beta = 2\gamma$

(ii)



$Z_2\hat{O}Z_3 = \beta - \gamma$

$Z_3\hat{O}Z_1 = \gamma - \alpha$

But $\quad 2\gamma = \alpha + \beta$

$\therefore \gamma + \gamma = \alpha + \beta$

ie $\gamma - \alpha = \beta - \gamma$

$\therefore Z_2\hat{O}Z_3 = Z_3\hat{O}Z_1$

ie Z_3 bisects $Z_1\hat{O}Z_2$

(f) (i) $z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

$z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$

as $z_1 + z_2$ is real then $y_1 + y_2 = 0$

ie $y_1 = -y_2$

(ii) $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= x_1 x_2 + i x_1 y_2 + i y_1 x_2 - y_1 y_2$

$= x_1 x_2 - y_1 y_2 + i(x_1 y_2 + y_1 x_2)$

as $z_1 z_2$ is real $x_1 y_2 + y_1 x_2 = 0$

as $y_1 = -y_2$ $x_1 y_2 + y_1 x_2 = -x_1 y_1 + y_1 x_2 = 0$

$y_1(-x_1 + x_2) = 0$

Hence $y_1 = 0$ or $-x_1 + x_2 = 0$

ie $x_1 = x_2$

If $y_1 = -y_2$ then $x_1 x_2 - y_1 y_2 = x_1 x_2 + y_1^2$

if $x_1 = x_2$ then $x_1 x_2 + y_1^2 = x_1^2 + y_1^2$

But $z_1 z_2$ is a negative real number ie $x_1 x_2 - y_1 y_2 < 0$

This means $x_1^2 + y_1^2 < 0$ which is impossible

Hence $x_1 \neq x_2$

Thus for z_1 and z_2 $x_1 \neq x_2$ and $y_1 = y_2 = 0$

$\therefore$ $z_1 = x_1$ and $z_2 = x_2$

ie z_1 and z_2 are both real numbers