St George Girls High School

Year 12

Mid-HSC Course Examination

2008

Mathematics Extension 1

General Instructions

- Working time – 1½ hours
- Reading time – 5 minutes
- Write using blue or black pen
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.
- Write on one side of the page only.
- Start each question in a **new booklet**.

Total marks – 72

- Attempt Questions 1 – 6
- All questions are of equal value

Question 1 (12 marks) **Marks**

a) Find the second derivate of e^{x^2} 3

b) Find the equation of the tangent to the curve $y = 2e^{3x}$ at the point where $x = 0$ 3

c) The size of an insect colony is given by the equation $P = 3000e^{kt}$ where P is the population after t days. 6

(i) Write down the initial population.

(ii) If there are 3600 insects after one day, find the value of the k, correct to 2 decimal places.

(iii) When will the colony double its initial population (answer correct to the nearest day)?

(iv) What is the growth rate of the population after 2 days?

Question 2 (12 marks) **Marks**

a) Solve the inequality $3x^2 - 4x - 4 > 0$ 2

b) If $x^2 = a\,x\,(x-1) + bx$ for more than two values of x, find a and b 2

c) Show that for all values of a, b, c, the equation $(x-a)(x-b) = c^2$ has real roots. 4

d) If α, β are the roots of $\frac{1}{x+m} = x$ find in terms of m 4

(i) $\alpha + \beta$

(ii) $\alpha\beta$

(iii) $\frac{1}{\alpha} + \frac{1}{\beta}$

Question 3 (12 marks) **Marks**

a) (i) Solve $\sin\left(2x + \frac{\pi}{3}\right) = 0$ for $0 \le x \le \pi$

(ii) Draw a neat sketch of $y = \frac{1}{2}\sin\left(2x + \frac{\pi}{3}\right)$ for $0 \le x \le \pi$ 4

b) Find the following: 6

(i) $\int \cos 3x \, dx$

(ii) $\int \cos^2 x \, dx$

(iii) $\int \cos^2 x \sin x \, dx$

(iv) $\int \tan^2 x \, dx$

c) If $y = \sin 3x$ show that $\frac{d^2y}{dx^2} + 9y = 0$ 2

Question 4 (12 marks) **Marks**

a) Differentiate $\log_e \sqrt{\frac{x^2-1}{2x+1}}$ 2

b) Show that $\frac{1}{x-2} - \frac{1}{x-1} = \frac{1}{(x-1)(x-2)}$ and hence show that 3

$\int_3^5 \frac{dx}{x^2-3x+2} = \ln\frac{3}{2}$

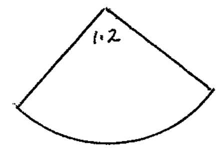
c) The region between the curve $y = 1 - \frac{2}{x}$ and the x-axes from $x = 1$ to $x = 2$ is rotated about the x-axis. Find the volume generated. 4

d) (i) Differentiate $x^3 \ln x$ 3

(ii) Hence evaluate $\int_1^2 x^2 \ln x \; dx$

Question 5 (12 marks) **Marks**

a) The sector given has a radius of 8 cm and an angle of 1.2 radians. 2



Find:

(i) its perimeter

(ii) its area

b) A and B are acute angles such that $\cos A = \frac{3}{5}$ and $\sin B = \frac{1}{\sqrt{5}}$. 4
Without finding the size of either angle:

(i) Show that $A = 2B$

(ii) Hence find the exact value of $\sin 3B$

c) Given that $\log_b a = 5$ and $\log_c b = 2$ find the value of $\log_a c$ 2

d) Using 2 applications of Simpsons Rule evaluate $\int_2^4 \ln 2x \, dx$ 4

Question 6 (12 marks) **Marks**

a) Express $4x - 5 - 4x^2$ in the form $p(x+q)^2 + r$ 3

(i) decide if the expression has a minimum or maximum value. Give reasons.

(ii) Find the minimum or maximum value.

b) Solve the equation $4^x - 6 \times 2^x + 8 = 0$ 2

c) The gradient of a curve is given as $\frac{x^2+1}{x^3+3x}$. If the point $(1, 0)$ lies on the curve find its equation. 2

d) For the curve $y = \ln(2 - x)$

(i) Sketch the curve clearly showing the values of the x- and y- intercepts. 3

(ii) Calculate the area bounded by the curve $y = \ln(2-x)$ and the x and y- axes. 2

End of Paper

SOLUTIONS Yr12 Ext 1 2008 - Mid Course

Q1 (a) $\frac{d}{dx}\left(e^{x^2}\right) = 2x.e^{x^2}$ — (1)

$\frac{d}{dx}\left[2x.e^{x^2}\right] = 2.e^{x^2} + 2xe^{x^2}.2x$ (1) (1)

$= 2e^{x^2}\left[1 + 2x^2\right]$

(b) $\frac{dy}{dx} = 6e^{3x}$ (1) at $x = 0$ $\Big\}$ $y = 2$, $m = 6$ — (1)

So equ. of tangent $y - 2 = 6(x - 0)$ — (1)

$6x - y + 2 = 0$

(c) (i) Let $t = 0$, Then $P_0 = 3000e^{k \times 0}$

$P_0 = 3000$

Initial pop. 3000 — (1)

(ii) let $t = 1$ and $P = 3600$, Then $3600 = 3000\,e^{k}$

$\frac{6}{5} = e^{k}$ — (1)

$\therefore k = \ln\left(\frac{6}{5}\right)$

$\doteq 0.18$ [2 dec pl.] — (1)

(iii) Require pop $= 2P$

Then $2P = Pe^{kt}$ (1)

$2 = e^{kt}$

$\ln 2 = kt$

$t = \ln 2 / \ln\left(\frac{6}{5}\right)$

$t = 3.8$ days. — (1)

It will take 4 days [nearest day]

(iv) $\frac{dP}{dt} = k\,3000e^{kt}$

$= \ln\left(\frac{6}{5}\right).3000.e^{\ln\left(\frac{6}{5}\right)^2}$

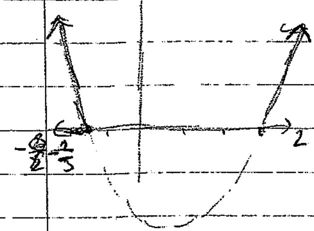
$= \ln\left(\frac{6}{5}\right).3000.\frac{36}{25}$ — (1)

$= 787.6$ mice per day.

Total 12

Question 2

(a) $(3x + 2)(x - 2) > 0$



$x < -\frac{2}{3}$ — (1)

&

$x > 2$ — (1)

(b) If = for more than two values, then, equivalent for all x.

$\therefore x^2 = ax^2 - ax + bx$ — (1)

$x^2 = ax^2 + (b-a)x$

Then $a = 1$, $b - a = 0$ $\Big\}$ equating coeff., $b = 1$

So $a = 1$, $b = 1$ — (1)

or let $x = 0$; $0 = 0 + 0$

$x = 1$; $1 = 0 + b \Rightarrow b = 1$

$x = 2$; $4 = 2a + 2b$

$4 = 2a + 2b$

$2 = a + b \Rightarrow a = 1$

(c) $x^2-(a+b)x+ab=c^2$

$x^2-(a+b)x+ab-c^2=0$

Real roots if $\Delta > 0$ — (1)

$\Delta = [-(a+b)]^2 - 4\times 1\times (ab-c^2)$

$= a^2+2ab+b^2-4ab+4c^2$ — (1)

$= a^2-2ab+b^2+4c^2$

$= (a-b)^2+4c^2$ — (1)

Sum of squares is >0 for all a, b and c. $\therefore$ Has real roots for all a, b, c — (1)

(d) $x \neq -m$; $1 = x^2+mx$

$\Rightarrow x^2+mx-1=0$

(i) $\alpha+\beta = -\frac{m}{1}$ — (1)

(ii) $\alpha\beta = \frac{-1}{1}$ — (1)
$= -1$

(iii) $\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\beta+\alpha}{\alpha\beta}$ — (1)
$= \frac{-m}{-1}$
$= m$, — (1)

Q3

(a) (i) $\sin\left(2x+\frac{\pi}{3}\right)=0$ for $0 \le x \le \pi$

$0 \le 2x \le 2\pi$

$\frac{\pi}{3} \le 2x+\frac{\pi}{3} \le 2\pi+\frac{\pi}{3}$

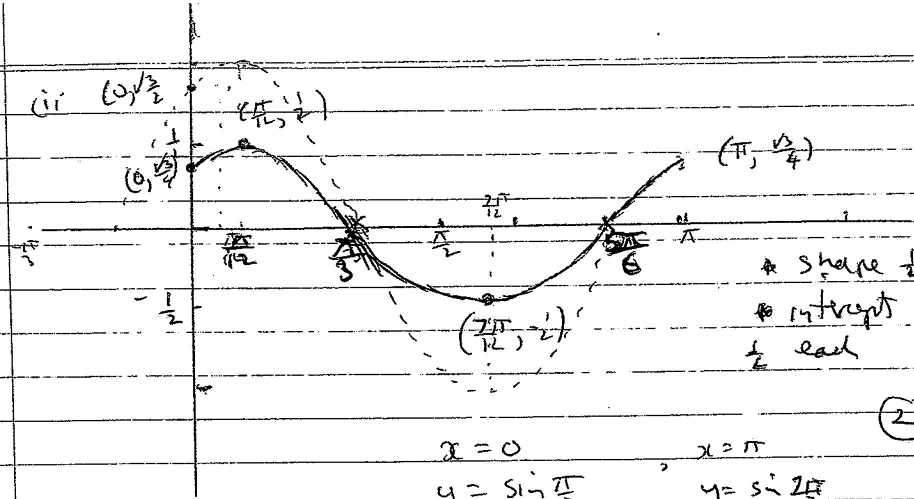
$2x+\frac{\pi}{3} = 0, \pi, 2\pi$ — (1)

$2x = -\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}$

$x = -\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}$

Soln $x = \frac{\pi}{3}, \frac{5\pi}{6}$ — (1)

(ii)



shape $\frac{1}{2}$
intercept $\frac{1}{2}$ each
(2)

$x=0$, $y=\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$x=\pi$, $y=\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

(b) (i) $\int \cos 3x\,dx = \frac{1}{3}\sin 3x + C$ — (1)

(ii) $\int \cos^2 x\,dx = \frac{1}{2}\int [\cos 2x + 1]\,dx$ — (1)

$= \frac{1}{2}\left(\frac{1}{2}\sin 2x + x\right) + C$ — (1)

$= \frac{1}{4}\sin 2x + \frac{x}{2} + C$

(iii) $\int \sin x \cos^2 x\,dx = \int f'(x).f(x)\,dx$

$= \frac{1}{2}\int 2\sin x . \cos^2 x\,dx$ — (1)

$= \frac{1}{2}\left[\frac{1}{3}\cos^3 x\right] + C$ — (1)

$= \frac{1}{6}\cos^3 x + C$

(iv) $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx$ — (1)

$= \tan x - x + C$

(c) $\frac{dy}{dx} = 3\cos 3x$

$\frac{d^2y}{dx^2} = -9\sin 3x$ — (1)

$\frac{d^2y}{dx^2} + 9y$ — (1)

$= -9\sin 3x + 9\sin 3x$

$= 0$.

Q4

(a) $\frac{d}{dx}\left[\frac{1}{2}\log_e\left(\frac{x^2-1}{2x+1}\right)\right] = \frac{1}{2}\left[\frac{d}{dx}\left(\log_e(x^2-1) - \log_e(2x+1)\right)\right]$ — (1)

$= \frac{1}{2}\left(\frac{2x}{x^2-1} - \frac{2}{2x+1}\right)$

$= \frac{1}{x^2-1} - \frac{1}{2x+1}$ — (1)

* $\frac{(2x+1) - (x^2-1)}{(x^2-1)(2x+1)}$

$= \frac{2+2x-x^2}{(x^2-1)(2x+1)}$

(b) L.H.S. $\frac{1}{x-2} - \frac{1}{x-1} = \frac{(x-1)-(x-2)}{(x-2)(x-1)}$ — (1)

$= \frac{1}{(x-2)(x-1)}$

then $\int_3^5 \frac{dx}{x^2-3x+2} = \int_3^5 \frac{dx}{(x-2)(x-1)}$

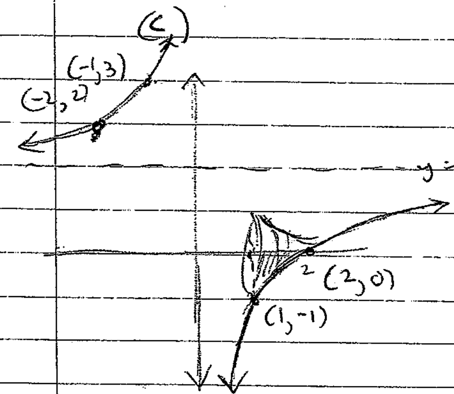
$= \int_3^5 \left[\frac{1}{x-2} - \frac{1}{x-1}\right] dx$ — (1)

$= \left[\log_e(x-2) - \log_e(x-1)\right]_3^5$

$= (\log_e 3 - \log_e 4) - (\overset{=0}{\log_e 1} - \log_e 2)$

$= \log_e\left(\frac{3}{2}\right)$ — (1)

(c)



$V = \pi\int_1^2 \left(1 - \frac{2}{x}\right)^2 dx$ — (1)

$= \pi\int_1^2 \left[1 - \frac{4}{x} + \frac{4}{x^2}\right] dx$

$= \pi\left[x - 4\log_e x - 4x^{-1}\right]_1^2$ — (1)

$= \pi\left[(2 - 4\log_e 2 - 2) - (1 - 0 - 4)\right]$ — (1)

$= \pi\left[3 - 4\log_e 2\right]$ cubic units — (1)

$\doteqdot 0.71$ cubic units.

(d) (i) $\frac{d}{dx}\left[x^3 \ln x\right] = 3x^2 \times \ln x + \frac{1}{x}\cdot x^3$

$= 3x^2 \ln x + x^2$ — (1)

(ii) from (i) $3x^2 \ln x = \frac{d}{dx}\left[x^3 \ln x\right] - x^2$

$x^2 \ln x = \frac{1}{3}\left[\frac{d}{dx}\left[x^3 \ln x\right] - x^2\right]$

$\therefore \int_1^2 x^2 \ln x \, dx = \frac{1}{3}\int_1^2 \frac{d}{dx}\left[x^3 \ln x\right] - x^2 \, dx$ — (1)

$= \frac{1}{3}\left[x^3 \ln x - \frac{1}{3}x^3\right]_1^2$

$= \frac{1}{3}\left[\left(8\ln 2 - \frac{8}{3}\right) - \left(0 - \frac{1}{3}\right)\right]$

$= \frac{1}{3}\left(8\ln 2 - \frac{7}{3}\right)$ — (1)

$\doteqdot 1.07$ (two dp)

Q5

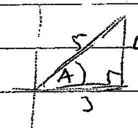
(a) (i) $P = 2\times 8 + 8\times 1.2$
$= 25.6\text{ cm}$ — (1)

(ii) $A = \frac{1}{2}r^2\theta$
$= 32\times 1.2$
$= 38.4\text{ cm}^2$ — (1)

(b) $0° < A < 90°$ and $0 < B < 90°$

(i) $\cos A = \frac{3}{5}$ then

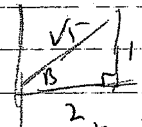
$\therefore \sin A = \frac{4}{5}$



by Pythagoras. — (1)

and $\sin B = \frac{1}{\sqrt5}$ then

$\cos B = \frac{2}{\sqrt5}$



by Pythagoras.

if $A = 2B$

then $\cos A = \cos 2B$ — (1)

$= \cos^2 B - \sin^2 B$

$= \left(\frac{2}{\sqrt5}\right)^2 - \left(\frac{1}{\sqrt5}\right)^2$

$= \frac{4}{5} - \frac{1}{5}$

$\cos A = \frac{3}{5}$ which is required result

(ii) $\sin 3B = \sin(2B + B)$ — (1)

$= \sin 2B\cos B + \cos 2B\sin B$

$= 2\sin B\cos B\cos B + \cos 2B\sin B$

$= 2\times\frac{1}{\sqrt5}\times\left(\frac{2}{\sqrt5}\right)^2 + \frac{3}{5}\times\frac{1}{\sqrt5}$ — (1)

$= \frac{8}{5\sqrt5} + \frac{3}{5\sqrt5}$

$= \frac{11}{\ }$

(c) $\log_a c = \frac{\log_b c}{\log_b a}$; $\log_c b = \frac{\log_b b}{\log_b c}$

$= \frac{1}{\log_b c}$

Given $\frac{1}{\log_b c} = 2$ — (1)

$\log_b c = \frac{1}{2}$

$\therefore \log_a c = \frac{\frac{1}{2}}{5}$ — (1)

$= \frac{1}{10}$

(d)

x	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
$\ln 2x$	$\ln 4$	$\ln 5$	$\ln 6$	$\ln 7$	$\ln 8$

— (1)

two applications of Simpson's rule required, 5 values (4 strips)

$\therefore \frac{4-2}{4} = \frac{1}{2}$ — (1)

$\therefore$ Area $\doteq \frac{\frac{1}{2}}{3}\left[\ln 4 + 2\ln 6 + 4(\ln 5 + \ln 7) + \ln 8\right]$ — (1)

$\doteq \frac{1}{6}(21\ldots$

$\doteq 3.55$ (two dec pl.) — (1)

Q6.

(a) $-4(x^2-x)-5$

$= -4(x^2-x+\frac{1}{4})-5+1$

$= -4(x-\frac{1}{2})^2-4$ — (1)

(i) Expression has max value since $a<0$ giving quadratic as concave down. — (1)

(ii) $x=\frac{1}{2}$, $y=-4$ — (1)

Max value is -4 — (1)

(b) let $m=2^x \Rightarrow$ $m^2-6m+8=0$

$(m-4)(m-2)=0$ — (1)

$\therefore m=4$ and $m=2$

$2^x=4$ $\quad 2^x=2$

$\underline{x=2}$ $\quad \underline{x=1}$ — (1)

(c) Primitive of $\frac{x^2+1}{x^3+3x} \rightarrow \frac{1}{3}\int \frac{3x^2+3}{x^3+3x}\,dx$ — (1)

$= \frac{1}{3}\ln(x^3+3x)+C$

$(1,0)$ lies on curve $\rightarrow 0=\frac{1}{3}(\ln 4)+C$

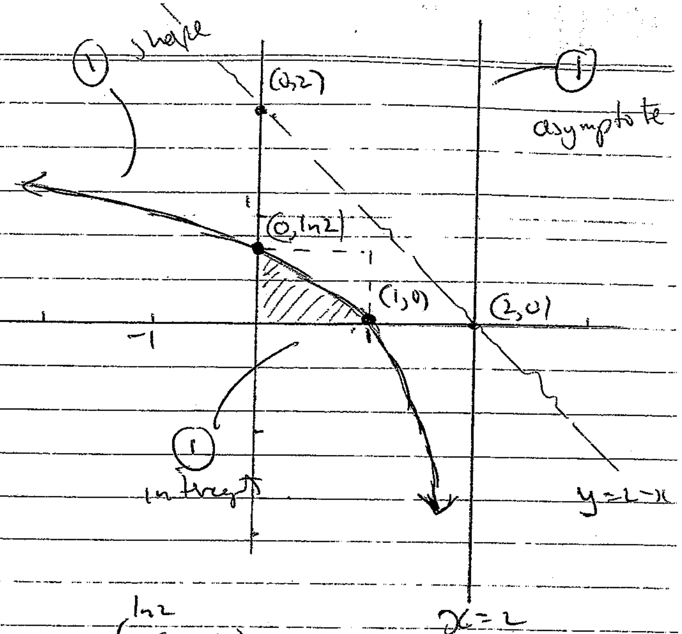
$C=-\frac{1}{3}\ln 4$

$\therefore y=\frac{1}{3}\left[\ln(x^3+3x)-\ln 4\right]$

$= \frac{1}{3}\ln\left(\frac{x^3+3x}{4}\right)$ — (1)

(d) $y=\ln(2-x)$

(i)



(ii) x-intercept $(1,0)$

y-intercept $(0,\ln 2)$

(iii) Area $= 1\times\ln 2 - \int_0^{\ln 2}(2-e^y)\,dy$ — (1)

$= \ln 2 - \left[2y-e^y\right]_0^{\ln 2}$

$y=\ln(2-x)$

$e^y=2-x$

$x=2-e^y$

$= \ln 2 - \left[(2\ln 2-2)-(0-1)\right]$

$= \ln 2 - \left[2\ln 2-1\right]$ — (1)

$= \underline{1-\ln 2}$