St George Girls High School

Year 12

Mid-HSC Course Examination

2008

Mathematics

General Instructions

- Working time: 1½ hours
- Reading time: 5 minutes
- Write using blue or black pen.
- Board-approved calculators may be used.

Total marks – 70

- Attempt Questions 1 – 5
- All questions are of equal value
- Begin each question in a new booklet
- All necessary working should be shown in every question.

Question 1 – **(14 marks)** – (Start a new booklet) **Marks**

a) By sketching the graph, or otherwise, write down the equation of the locus of the point $P(x, y)$ which moves so that it is.

(i) equidistant from the points $A(3, 0)$ and $B(-5,0)$ 2

(ii) 3 units from the x axis 2

b) The point $P(x, y)$ is joined to the points $D(1, 1)$ and $E(3, 5)$

(i) Write down the gradients of PD and PE. 1

(ii) If $D\hat{P}E = 90°$ show that the equation of the locus of P is 3

$$x^2 - 4x + y^2 - 6y + 8 = 0$$

(iii) Hence, or otherwise, give a full geometric description of the locus of P. 2

c) The point $P(x, y)$ moves so that it is equidistant from the point $S(2, 3)$ and the line $y = -1$. 4

Derive the equation of the locus of P.

Question 2 – (14 marks) – (Start a new booklet) **Marks**

a) For the parabola $x^2 - 6x - 4y + 1 = 0$ find:

(i) the coordinates of the vertex 2

(ii) the focal length 1

(iii) the coordinates of the focus 1

(iv) the equation of the directrix 1

b) Differentiate with respect to x

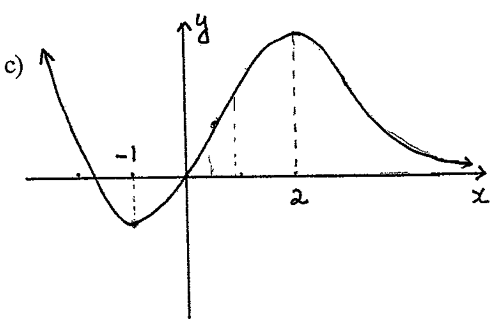
(i) $f(x) = \frac{1}{2x}$ 1

(ii) $h(x) = (x^2 + 2)^{10}$ 2

(iii) $F(x) = \frac{2x+1}{3x+2}$ 2

(iv) $y = x(3x+2)^3$ 2

c)



The graph of $y = f(x)$ is shown on the diagram. 2

Draw a neat sketch of $y = f'(x)$

Question 3 – (14 marks) – (Start a new booklet) **Marks**

a) Find the values of x for which the curve $y = x^3 + 5x^2 + 3x - 7$ is concave down. 3

b) Find the equation of the tangent to the curve $y = x + \frac{2}{x}$ at the point $(1, 3)$ 3

c) Find the coordinates of the point on the curve $y = x^2 - 5x + 2$ where the normal is parallel to the line $x + 3y - 7 = 0$ 4

d) Find the values of x for which $f(x) = 3 + 3x^2 - x^3$ is a decreasing function. 4

Question 4 – **(14 marks)** – (Start a new booklet) **Marks**

a) Find the second derivative of $y = \frac{x}{2x+1}$ 3

b) In a calculus test Philippa correctly showed that $\frac{d^2y}{dx^2} = (x-5)^2$. She then concluded that the curve has a point of inflexion at $x = 5$.

Explain why Philippa did not score full marks for her answer. 2

c) Find primitive functions of:

(i) $6x + 3$ 1

(ii) $(5x-2)^3$ 1

(iii) $x\sqrt{x}$ 2

(iv) $\frac{1}{\sqrt{3x+2}}$ 2

d) The gradient of a curve is given by $\frac{dy}{dx} = 2x + \frac{1}{x^2}$

If the curve passes through the point $(1, 4)$ find its equation. 3

$x^2\sqrt{x}$
$= x^2 \times x^{\frac{1}{2}}$

Question 5 – **(14 marks)** – (Start a new booklet) **Marks**

a) A piece of wire 36 metres long is cut into 2 pieces. The first piece is bent to form a square of side length x metres. The second is bent to form a rectangle with sides of lengths x metres and y metres.

(i) Show that $y = 18 - 3x$ 1

(ii) Hence show that the sum of the areas of the square and rectangle is given by $A = 18x - 2x^2$ 2

(iii) Using calculus, show that the maximum value of A is $\frac{81}{2}$ 3

b) For the curve $y = 6x^2 - x^3$

(i) Find any stationary points and determine their nature. 4

(ii) Sketch the curve for the domain $-1 \le x \le 7$ 2

(iii) For what values of k does the equation $6x^2 - x^3 = k$ have 3 distinct solutions. 2

End of Paper

Karen Q4

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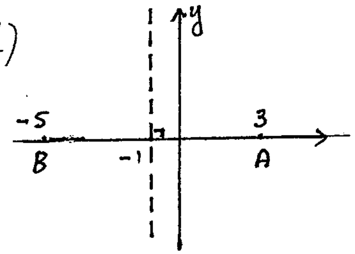
Question 1

(a)

(i)



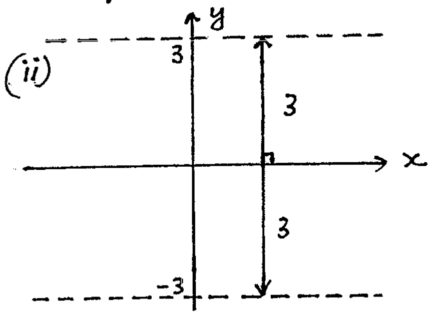
OR $PA^2 = PB^2$

$$(x-3)^2+(y-0)^2 = (x--5)^2+(y-0)^2$$
$$x^2-6x+9+y^2 = x^2+10x+25+y^2$$
$$-16x = 16$$
$$x = -1$$

Locus of P is the perpendicular bisector of interval AB

$\therefore$ Equation is $x = -1$

(ii)



Equation of locus of P is $y = 3$ or -3

OR $|y| = 3$

(b) (i) Gradient of $PD = \frac{y-1}{x-1}$

Gradient of $PE = \frac{y-5}{x-3}$

(ii) If $D\hat{P}E = 90°$ then

$$\frac{y-1}{x-1} \times \frac{y-5}{x-3} = -1$$

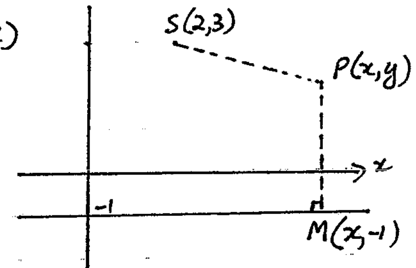
$$(y-1)(y-5) = -(x-1)(x-3)$$
$$y^2-6y+5 = -(x^2-4x+3)$$
$$x^2-4x+3+y^2-6y+5 = 0$$
$$x^2-4x+y^2-6y+8 = 0$$

(iii) $x^2-4x+4+y^2-6y+9 = -8+4+9$

$$(x-2)^2+(y-3)^2 = 5$$

$\therefore$ P moves on a circle with centre $(2,3)$ and radius of $\sqrt{5}$.

(c)



Let M be the foot of the perpendicular from P to line $y = -1$

$\therefore$ Coordinates of M are $(x, -1)$

$$PS = PM$$
$$\therefore PS^2 = PM^2$$
$$(x-2)^2+(y-3)^2 = (x-x)^2+(y--1)^2$$
$$(x-2)^2+y^2-6y+9 = 0+y^2+2y+1$$
$$(x-2)^2 = 8y-8$$

ie $(x-2)^2 = 8(y-1)$ is the equation of the locus of P

Question 2

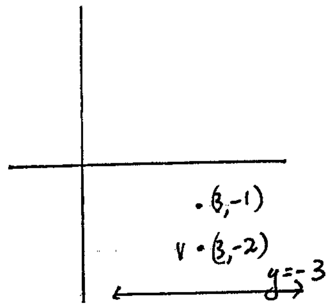
(a) $x^2 - 6x - 4y + 1 = 0$

(i) $x^2 - 6x + 9 = 4y - 1 + 9$

$(x-3)^2 = 4y + 8$

$(x-3)^2 = 4(y+2)$

$\therefore$ Vertex is $(3,-2)$



(ii) Focal length $= 1$

(iii) Focus is $(3,-1)$

(iv) Directrix is $y = -3$

(b) (i) $f(x) = \frac{1}{2}x^{-1}$

$f'(x) = -\frac{1}{2}x^{-2}$

$\left(= \frac{-1}{2x^2}\right)$

(ii) $h(x) = (x^2+2)^{10}$

$h'(x) = 10(x^2+2)^9 . 2x$

$= 20x(x^2+2)^9$

(iii) $F(x) = \dfrac{2x+1}{3x+2}$

$F'(x) = \dfrac{2(3x+2) - 3(2x+1)}{(3x+2)^2}$

$= \dfrac{6x+4-6x-3}{(3x+2)^2}$

$= \dfrac{1}{(3x+2)^2}$

(iv) $y = x(3x+2)^3$

$\frac{dy}{dx} = 1.(3x+2)^3 + x.3(3x+2)^2.3$

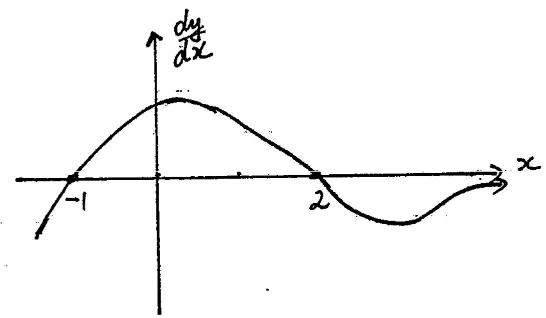
$= (3x+2)^3 + 9x(3x+2)^2$

$[= (3x+2)^2(3x+2+9x)$

$= (3x+2)^2(12x+2)$

$= 2(3x+2)^2(6x+1)\ ]$

(c)



Question 3

(a) $y = x^3 + 5x^2 + 3x - 7$

$\frac{dy}{dx} = 3x^2 + 10x + 3$

$\frac{d^2y}{dx^2} = 6x + 10$

Curve is concave down when $\frac{d^2y}{dx^2} < 0$

$6x + 10 < 0$

$x < -\frac{5}{3}$

(b) $y = x + 2x^{-1}$

$\frac{dy}{dx} = 1 - 2x^{-2}$

When $x = 1$ $\frac{dy}{dx} = 1 - 2 \times 1 = -1$

$\therefore$ Eq^n of tangent is

$y - 3 = -1(x - 1)$

$= -x + 1$

$x + y - 4 = 0$ (or $y = -x + 4$)

(c) $x + 3y - 7 = 0$

$y = -\frac{1}{3}x + \frac{7}{3}$

$\therefore$ Grad of normal $= -\frac{1}{3}$

Grad of tangent $= 3$

$y = x^2 - 5x + 2$

$\frac{dy}{dx} = 2x - 5$

$2x - 5 = 3$

$x = 4$

When $x = 4$

$y = 16 - 20 + 2$

$= -2$

$\therefore$ Point is $(4, -2)$

(d) $f(x) = 3 + 3x^2 - x^3$

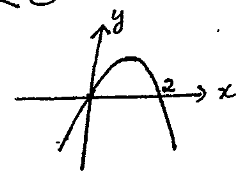
$f'(x) = 6x - 3x^2$

$f(x)$ is decreasing when $f'(x) < 0$

ie $6x - 3x^2 < 0$

$3x(2 - x) < 0$

ie $x < 0$ or $x > 2$



Question 4

(a) $y = \dfrac{x}{2x+1}$

$\dfrac{dy}{dx} = \dfrac{1(2x+1) - x.2}{(2x+1)^2}$ 1

$= \dfrac{2x+1-2x}{(2x+1)^2}$

$= \dfrac{1}{(2x+1)^2}$ 1

$= (2x+1)^{-2}$

$\dfrac{d^2y}{dx^2} = -2(2x+1)^{-3} \times 2$

$= \dfrac{-4}{(2x+1)^3}$ 1

$\dfrac{-8x-4}{(2x+1)^4}$ $2\frac{1}{2}$

$y = x(2x+1)^{-1}$

$y' = x.-1(2x+1)^{-2}.2 + (2x+1)^{-1}.1$

$= \dfrac{-2x}{(2x+1)^2} + \dfrac{1}{2x+1}$

$= \dfrac{-2x+2x+1}{(2x+1)^2} = \dfrac{1}{(2x+1)^2}$

$y'' \quad \dfrac{(2x+1)^2.0 - 1.2(2x+1).2}{(2x+1)^4}$

$= \dfrac{-4(2x+1)}{(2x+1)^4} = \dfrac{-4}{(2x+1)^3}$

1

(b) There must be a change of concavity at a point of inflexion

When $x=4$ $\dfrac{d^2y}{dx^2} = (4-5)^2 = 1 > 0$ $\therefore$ concave up

$x=6$ $\dfrac{d^2y}{dx^2} = (6-5)^2 = 1 > 0$ $\therefore$ concave up

1

$\therefore$ There is no change of concavity at $x=5$

Thus the curve does not have an inflexion point at $x=5$.

(c) (i) $\int 6x+3\,dx = 3x^2+3x+c$ 1

no c $-\frac{1}{2}$

(ii) $\int (5x-2)^3 dx = \dfrac{(5x-2)^4}{5\times 4} + c$

$= \dfrac{(5x-2)^4}{20} + c$ 1

$-\frac{1}{2}$ if no 5

(iii) $\int x\sqrt{x}\,dx = \int x^{3/2} dx$ $\frac{1}{2}$

$= \dfrac{x^{5/2}}{5/2} + c$ $\frac{1}{2}$

$= \dfrac{2}{5}x^{5/2} + c$ 1 $\quad \frac{2}{5}\sqrt{x^5} + c$

$\left[= \dfrac{2}{5}x^2\sqrt{x} + c \right]$

(iv) $\int \dfrac{1}{\sqrt{3x+2}}\,dx = \int (3x+2)^{-1/2} dx$ $\frac{1}{2}$

$= \dfrac{(3x+2)^{1/2}}{3\times\frac{1}{2}} + c$ 1

$= \dfrac{2}{3}\sqrt{3x+2} + c$ $\frac{1}{2}$

leave out $-\frac{1}{2}$

(d) $\dfrac{dy}{dx} = 2x + x^{-2}$

$y = x^2 + \dfrac{x^{-1}}{-1} + c$ 1

$y = x^2 - \dfrac{1}{x} + c$

When $x=1$ $y=4$

$4 = 1 - \dfrac{1}{1} + c$ 1

$c = 4$

$\therefore$ Eq^n of curve is 1

$y = x^2 - \frac{1}{x} + 4$

2

if for incorrect value of c

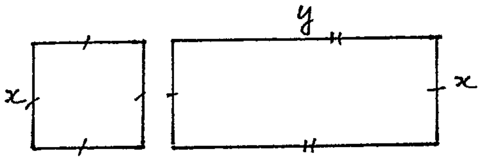
$125x^3 + 3\times 25x^2\times -2 + 3\times 5x\times 4 + 1\times(-2)^3$

$= 125x^3 - 150x^2 + 60x - 8$

$\int \quad = \dfrac{125x^4}{4} - \dfrac{150x^3}{3} + \dfrac{60x^2}{2} - 8x + c$

Question 5

(a)



(i) $4x + 2x + 2y = 36$

$6x + 2y = 36$

$2y = 36 - 6x$

$y = 18 - 3x$

(ii) $A = x^2 + xy$

$= x^2 + x(18 - 3x)$

$= x^2 + 18x - 3x^2$

$= 18x - 2x^2$

(iii) $\frac{dA}{dx} = 18 - 4x$

$\frac{d^2A}{dx^2} = -4$

Stationary points occur when $\frac{dA}{dx} = 0$

$18 - 4x = 0$

$x = \frac{9}{2}$

Since $\frac{d^2A}{dx^2} < 0$ curve is concave down and so there is a max tp at $x = \frac{9}{2}$

$A = 18 \times \frac{9}{2} - 2 \times \left(\frac{9}{2}\right)^2$

$= \frac{81}{2}$

ie Maximum value of A is $\frac{81}{2}$

(b) $y = 6x^2 - x^3$

(i) $\frac{dy}{dx} = 12x - 3x^2$

$\frac{d^2y}{dx^2} = 12 - 6x$

Stat pts occur when $\frac{dy}{dx} = 0$

$12x - 3x^2 = 0$

$3x(4 - x) = 0$

$x = 0, 4$

When $x = 0$ $y = 0$

$\frac{d^2y}{dx^2} = 12$

$\therefore$ Min tp at $(0,0)$

When $x = 4$ $y = 6 \times 16 - 64$

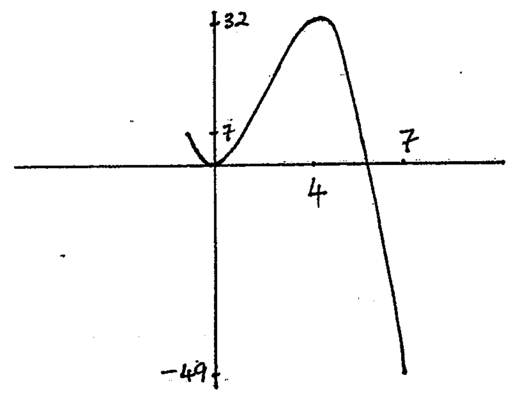
$= 32$

$\frac{d^2y}{dx^2} = -12$

$\therefore$ Max tp at $(4, 32)$

(ii) When $x = -1$ $y = 6(-1)^2 - (-1)^3 = 7$

$x = 7$ $y = 6 \times 7^2 - 7^3 = -49$



(iii) $6x^2 - x^3 = k$ will have 3 distinct roots if the line $y = k$ intersects the above graph 3 times

ie $0 < k \leq 7$