St George Girls High School

Year 12

Common Test 3

2007

Mathematics

General Instructions

1. Working time – 75 minutes.
2. Use only blue or black pen.
3. Board approval calculators may be used.
4. A table of standard integrals is provided.
5. All necessary working should be shown in every question.
6. Start each question on a new page.

Total Marks

1. Attempt Question 1 – 6.
2. All questions are of equal value.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a>0, \quad -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x>0$

Question 1 – (10 marks) – Start a New Page **Marks**

a) The nth term of a series is given by $T_n = 3 - 5n$

(i) What type of series is it? Give justification for your answer. 2

(ii) Find T_{10} 1

(iii) Find S_{10}, the sum of the first 10 terms. 1

b) Differentiate the following functions with respect to x:

(i) e^{7x^2+5} 2

(ii) $x \log_e(2x^3+1)$ 2

(iii) $\dfrac{e^x}{\log_e x}$ 2

Question 2 – (10 marks) – Start a New Page **Marks**

a) Find the primitive functions of these:

(i) e^{3x+1} 1

(ii) $\dfrac{x^2}{x^3-1}$ 2

b) If the sum to infinity of the series: $3 + 3r + 3r^2 + \ldots$ is 5, find the value of r. 2

c) It is given that the nth term of the series $23 + 25 + 27 + \ldots$ is equal to the nth term of the series $1 + 4 + 7 \ldots$. Find the value of n. 2

d) (i) Differentiate the function $f(x) = x \ln x - x$ 1

(ii) and, hence, evaluate $\int_1^3 \ln x \, dx$ 2

Question 3 – (10 marks) – Start a New Page **Marks**

a) For $F(x)=\dfrac{e^x+1}{e^x-1}$,

(i) Find the value of x for which $F(x)$ is undefined. 1

(ii) Show that $F'(x)=\dfrac{-2e^x}{\left(e^x-1\right)^2}$ 1

(iii) Show that $F(x)$ is always decreasing in its domain. 1

(iv) Find the minimum value of $F(x)$ in the domain $0.5\le x\le 2$ 1

b) Classify the following series as arithmetic progressions, geometric progressions or neither

(i) $\sqrt{8}+\sqrt{18}+\sqrt{32}+\ldots$ 1

(ii) $\log 3+\log 6+\log 9+\ldots$ 1

c) Find the volume of the solid of revolution generated when the area under the curve $y=e^{2x}$, between $x=1$ and $x=3$, is rotated about the x-axis (give exact value) 4

Question 4 – (10 marks) – Start a New Page **Marks**

a) The second term of an arithmetic series is $\log 6$ and the fifth term is $\log 162$. Find the common difference and the first term. 3

b) Find the sum of 50 consecutive even integers, the first of which is 12. 2

c) What is the equation of the normal to the curve $y=\ln\left(x^2+x\right)$ at the point $(1,\ln 2)$? 3

d) In a geometric series, the third term is 54 and the sixth term is -1458. Find the common ratio and the second term. 2

Question 5 – (10 marks) – Start a New Page **Marks**

a) Find $f(x)$, and then find $f(3)$, given that $f'(x) = 2x^2 + \dfrac{3}{2x-1}$ and $f(1) = 1$ 3

b) Find $\displaystyle\int \frac{5x^2 - 2x}{x^3}\,dx$ 2

c) (i) The sum of n terms of a certain series is given by $S_n = 4^{n+1} - 3$, $n > 1$.
Show that the nth term, T_n, is given by $T_n = 3 \times 4^n$. 2

(ii) Hence, find T_5 1

d) Find the domain of the function $y = x\ln(x^2 - 1)$ 2

Question 6 – (10 marks) – Start a New Page **Marks**

a) For what values of x does: $(3-x)+(3-x)^2+(3-x)^3+\ldots$ have a limiting sum? 2

b) For the curve $y = \dfrac{e^x}{x}$,

(i) Find the equations of any vertical asymptotes. 1

(ii) Find the coordinates of any stationary points, AND determine their nature. 4

(iii) Determine the y-value when $x = -1$ 1

(iv) Sketch the curve on a number plane, using at least $\frac{1}{3}$ page.
Show all important features of the curve. Show all necessary calculations.
You may use the fact that $\dfrac{e^x}{x} \to \infty$ as $x \to \infty$ 2

End of Paper

YR 12 Mathematics (1) Berdukia

CT 3 — SOLUTIONS.

Q1 (a) (i)

n	1	2	3	4
T_n	-2	-7	-12	-17

... $T_2 - T_1 = -5 = T_3 - T_2$

Series is arithmetic with common difference of -5 ✓

or $T_{n+1} - T_n = (3-5n) - (3-5(n+1))$

$= -5$

$T_{n+2} - T_{n+1} = (3-5(n+1)) - (3-5(n+2))$

$= -5$

Common difference, so arithmetic. 2

(ii) $T_{10} = 3 - 5\times 10$

$= -47$ 1

(iii) $S_{10} = \frac{10}{2}(-2 + -47)$ 1

$= 5 \times -49$

$= -245$

(b) (i) $\frac{d}{dx}\left(e^{7x^2+5}\right) = e^{(7x^2+5)} \times 14x$

$= 14x \cdot e^{(7x^2+5)}$ 2

(ii) $\frac{d}{dx}\left[x\log_e(2x^3+1)\right] = 1\times\log_e(2x^3+1) + x\times\frac{6x^2}{2x^3+1}$ 2

$= \log_e(2x^3+1) + \frac{6x^3}{2x^3+1}$

(iii) $\frac{d}{dx}\left[\frac{e^x}{\log_e x}\right] = \frac{e^x \times \log_e x - \frac{1}{x}\times e^x}{(\log_e x)^2}$ $1\frac{1}{2}$ 2

$= \frac{e^x(x\log_e x - 1)}{x(\log_e x)^2}$

Q2 (a) (i) $\int e^{3x+1}\,dx = \frac{1}{3}e^{3x+1} + C$

(ii) $\int \frac{x^2}{x^3-1}\,dx = \frac{1}{3}\int\frac{3x^2}{x^3-1}\,dx$

$= \frac{1}{3}\log_e(x^3-1) + C$

(b) $S_\infty = \frac{a}{1-r}$, $\left.\begin{array}{l} a=3 \\ r=r\end{array}\right\}$ then $5 = \frac{3}{1-r}$

$5 - 5r = 3$

$5r = 2$

$r = \frac{2}{5}$

(c) Series 1: $T_n = 23 + (n-1)2$

$= 2n + 21$

Series 2: $T_n = 1 + (n-1)3$

$= 3n - 2$

Let terms be equal $3n - 2 = 2n + 21$

$n = 23$

(d) (i) $f'(x) = 1\times\ln x + x\times\frac{1}{x} - 1$

$= \ln x$

ie $\frac{d}{dx}\left[x\ln x - x\right] = \ln x$

(ii) $\int_1^3 \ln x\,dx = \int_1^3\left[\frac{d}{dx}(x\ln x - x)\right]dx$

$= \left[x\ln x - x\right]_1^3$

$= (3\ln 3 - 3) - (1\times\ln 1 - 1)$

$= 3\ln 3 - 2$

Q3

(a) (i) Undefined when $e^x - 1 = 0$

$e^x = 1$

$x = 0$

$\therefore F(0)$ is undefined.

(ii) $F'(x) = \dfrac{e^x(e^x-1) - e^x(e^x+1)}{(e^x-1)^2}$

$= \dfrac{e^{2x} - e^x - e^{2x} - e^x}{(e^x-1)^2}$

$= \dfrac{-2e^x}{(e^x-1)^2}$

(iii) Since $(e^x-1)^2 > 0$ for all x, $x \neq 0$

and $e^x > 0$ for all x

then $-2 \times \left(\dfrac{e^x}{(e^x-1)^2}\right) = -2 \times$ positive for all x, $x \neq 0$

= negative value.

So $F'(x) < 0$ for all x $(x \neq 0)$

ie decreasing function.

(iv) let $\dfrac{-2e^x}{(e^x-1)^2} = 0$

ie $e^x = 0$ no turning pt.

Gives $F(2)$ as min. value.

$F(2) = \cancel{\dfrac{-2e^2}{(e^2-1)^2}}$ $\dfrac{e^2+1}{e^2-1}$

$=$ ______

(b) (i) $\sqrt{8} + \sqrt{18} + \sqrt{32} + \dots = 2\sqrt{2} + 3\sqrt{2} + 4\sqrt{2} \dots$

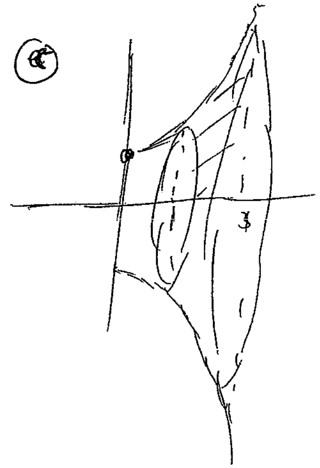
~~Geometric~~ Arithmetic with $d = \sqrt{2}$

(ii) $\log 3 + \log(2\times 3) + \log(3\times 3) + \dots$

$\log 3 + \log 2 + \log 3 + \log 3 + \log 3 \dots$

Series ~~is Geometric with ... log 3~~ is neither!

(c)



$V = \pi \int_1^3 \left[e^{2x}\right]^2 dx$

$= \pi \int_1^3 e^{4x}\, dx$

$= \pi \left[\frac{1}{4} e^{4x}\right]_1^3$

$= \frac{\pi}{4}\left[e^{12} - e^4\right)$ cubic units.

Q4 (a) $T_2 = \log 6$ On that. $a + d = \log 6$
$T_5 = \log 162$ $\quad a + 4d = \log 162$

$\therefore 3d = \log 162 - \log 6$
$= \log\left(\frac{162}{6}\right)$
$= \log 27$
$3d = 3\log 3$
$d = \log 3$

So $a = \log 6 - \log 3$
$= \log 2$

First term $\log 2$ and common diff. $\log 3$.

(b) $S_{50} = \frac{50}{2}\left(12 + [12 + (49) \times 2]\right)$
$= \frac{50}{2}(24 + 98)$
$= 3050$

(c) $\frac{dy}{dx} = \frac{2x+1}{x^2+x}$ at $(1, \ln 2)$
$m = \frac{2\times 1+1}{1^2+1}$
$m = \frac{3}{2}$

Equation of Normal
$y - \ln 2 = -\frac{2}{3}(x - 1)$
$y - \ln 2 = \frac{2}{3} - \frac{2x}{3}$
or $y = \ln 2 + \frac{2}{3} - \frac{2x}{3}$

(d) $T_3 = 54$, $T_6 = -1458$ $\Big\}$ $ar^2 = 54$, $ar^5 = -1458$
$\therefore r^3 = -27$
$r = -3$

So $9a = 54$
$a = 6$
Hence $T_2 = 6 \times -3$
$= -18$

Q5

(a) $f(x) = \frac{2x^3}{3} + \frac{3}{2}\ln(2x-1) + c$

at $x = 1$, $f(1) = \frac{2\times 1^3}{3} + 0 + C = 1$
$C = -\frac{2}{3}$

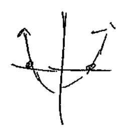
$f(x) = \frac{2x^3}{3} + \frac{3}{2}\ln(2x-1) - \frac{2}{3}$
$\therefore f(3) = 18 + \frac{3}{2}\ln 5 - \frac{2}{3}$ (11.4)

(b) $\int \frac{5}{x} - \frac{2}{x^2}\, dx = 5\ln x + 2x^{-1} + C$
$= 5\ln x + \frac{2}{x} + C$

(c) (i) $T_n = S_n - S_{n-1}$
$= 4^{n+1} - 3 - 4^{(n-1)+1} + 3$
$= 4^{n+1} - 4^n$
$= 4^n(4-1)$
$= 3 \times 4^n$

(ii) $T_5 = 3 \times 4^5$
$= 3072$

(d) $y = x\ln(x^2-1)$
Domain $x^2 - 1 > 0$



$x : x < -1$ & $x > 1$

Q6

(a) $r = \frac{(3-x)^2}{(3-x)}$ Limiting sum iff $|r| < 1$

$r = 3-x$ i.e. $-1 < 3-x < 1$

$-4 < -x < -2$

$4 > x > 2$

Domain : $\underline{2 < x < 4}$

(b) (i) Denominator $x \neq 0$

Vertical asymptote $\underline{x = 0}$

(ii) $\frac{dy}{dx} = \frac{e^x \times x - 1 \times e^x}{x^2}$

$= \frac{e^x(x-1)}{x^2}$

Stationary at $x - 1 = 0$, $e^x > 0$

$x = 1$

Nature

x	$\frac{1}{2}$	1	$\frac{3}{2}$
$\frac{dy}{dx}$	$-\frac{1}{2}e^{\frac{1}{2}}$	0	$\frac{1}{2}e^{\frac{3}{2}}$
	<0 ↘	⟺	↗

Local Min. at

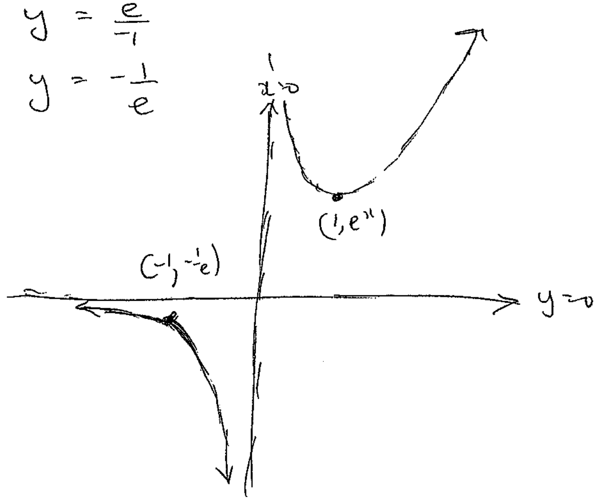
$x = 1$

$y = e^x$

(iii) $x = -1$, $y = \frac{e^{-1}}{-1}$

$y = -\frac{1}{e}$

(iv) ($\frac{1}{2}$ page)



as $x \to 0_+$ $e^x \to 1$

$x \to 0^+$

$\therefore \frac{e^x}{x} \to \infty$

as $x \to 0-$, $e^x \to 1$

$x \to 0^-$