St George Girls High School

Year 12

Common Test 3

May 2008

Mathematics

General Instructions

1. Working time – 70 minutes.
2. Use only blue or black pen.
3. Board approval calculators may be used.
4. A table of standard integrals is provided.
5. All necessary working should be shown in every question.
6. Start each question on a new page.

Total Marks

1. Attempt Question 1 – 5.
2. All questions are of equal value.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

2602-3

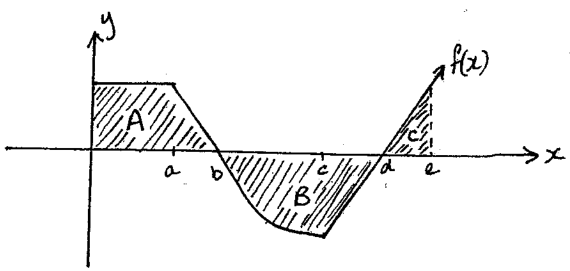
Question 1 – (13 marks) – Start a New Page **Marks**

a) Differentiate with respect to x 4

(i) e^{5x^2+3} (ii) $5x^2 + 3e^{-7x}$

b) Sketch the graph of $\int_{-4}^{4}\sqrt{16-x^2}$ and then use an area formula to find the value of the definite integral. 4

c)



where A, B and C are the shaded areas

The above is a diagram of $f(x)$ for $0 \le x \le e$. State whether the following statement is true giving reasons. 2

$$\int_0^e f(x) = A + B + C$$

d) If $y = \left(x^2 + e^{2x}\right)^4$ find $\dfrac{dy}{dx}$ 3

Question 2 – (13 marks) – Start a New Page **Marks**

a) Write down the derivative of 4

(i) $2e^{3x} + e^{-x}$ (ii) $e^{2x}\left(e^{x} - e^{-x}\right)$

b) If $y = e^{2x} + e^{8x}$

(i) find $\dfrac{d^2y}{dx^2}$ 2

(ii) show that $\dfrac{d^2y}{dx^2} - 10\dfrac{dy}{dx} + 16y = 0$ 2

c) Find the gradient of the tangent to the curve $y = 4e^x$ at point where $x = 1$. 2

d) (i) Explain why every tangent to the curve $y = e^{3x-6}$ has a positive gradient.

(ii) Find the point on the curve $y = e^{3x-6}$ where the gradient is 3. $\cancel{1}$ 2

Question 3 – (10 marks) – Start a New Page **Marks**

a) Differentiate

(i) x^3e^{3x} (ii) $\dfrac{e^x+1}{e^x-1}$ 4

b) For the curve $y = xe^x$

(i) Show that $\dfrac{dy}{dx} = (1+x)e^x$ and $\dfrac{d^2y}{dx^2} = (2+x)e^x$ 2

(ii) Find the stationary point and determine its nature. 2

(iii) Find the coordinates of the point of inflexion. 2

(iv) Given that $y \to 0$ as $x \to -\infty$ sketch the curve and write down its range. 3

Question 4 – (13 marks) – Start a New Page **Marks**

a) Find the following indefinite integrals: 6

(i) $\int e^{-x}\,dx$

(ii) $\int xe^{x^2+7}\,dx$

(iii) $\int 7x^6 + 6e^{7x}\,dx$

b) Find: 2

(i) $\int_{-3}^{0} 3x^2\,dx$

(ii) $\int_{-1}^{3} e^{-\frac{1}{2}x}\,dx$ 4

c) Find the volume of the solid formed when the area enclosed by the ~~curves $y=x^2$~~ and $y=(x-2)^2$ and the x-axis is rotated about the x-axis. 3

Question 5 – (13 marks) – Start a New Page **Marks**

a) Find the area enclosed by the curve $y = x^3$, and the lines $x = -1$ and $x = 3$ 3

b) Use the trapezoidal rule with 3 function values to approximate $\int_1^3 2^x \, dx$ 3

c) Find the area enclosed by the curve $y = \sqrt{3x-5}$, the y-axis and the lines $y = 2$ and $y = 3$. 3

d) Use Simpson's rule with 5 function values to evaluate $\int_1^2 \frac{1}{x} \, dx$ 4

End of Paper

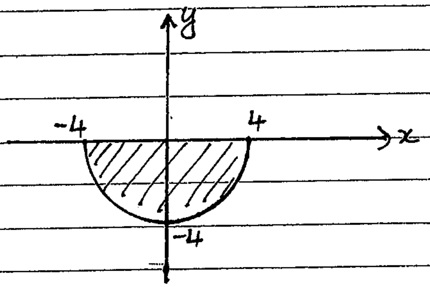
JS Q4

Question 1

(a) (i) $\frac{d}{dx} e^{5x^2+3} = 10x\, e^{5x^2+3}$

(ii) $\frac{d}{dx}\left(5x^2 + 3e^{-7x}\right) = 10x - 21e^{-7x}$

(b)



$A = \frac{\pi r^2}{2}$

$= \frac{1}{2} \times \pi \times 4^2$

$= 8\pi$

$\therefore \int_{-4}^{4} -\sqrt{16-x^2}\, dx = -8\pi$

(c) $\int_0^e f(x)\, dx \neq A+B+C$ since $\int_b^d f(x) dx = -B$

$\therefore \int_0^e f(x)\, dx = A - B + C$

(d) $y = \left(x^2 + e^{2x}\right)^4$

$\frac{dy}{dx} = 4\left(x^2+e^{2x}\right)^3 \times \left(2x + 2e^{2x}\right)$

$= 8\left(x^2+e^{2x}\right)^3\left(x+e^{2x}\right)$

Question 2

(a) (i) $\frac{d}{dx}\left(2e^{3x} + e^{-x}\right)$

$= 6e^{3x} - e^{-x}$

(ii) $e^{2x}\left(e^{x} - e^{-x}\right)$

$= e^{3x} - e^{x}$

$\frac{d\left(e^{3x} - e^{x}\right)}{dx} = 3e^{3x} - e^{x}$

(b) (i) $y = e^{2x} + e^{8x}$

$\frac{dy}{dx} = 2e^{2x} + 8e^{8x}$

$\frac{d^2y}{dx^2} = 4e^{2x} + 64e^{8x}$

(ii) $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 16y$

$= 4e^{2x} + 64e^{8x} - 10\left(2e^{2x} + 8e^{8x}\right) + 16\left(e^{2x} + e^{8x}\right)$

$= e^{2x}(4 - 20 + 16) + e^{8x}(64 - 80 + 16)$

$= 0$

(c) $y = 4e^{x}$

$\frac{dy}{dx} = 4e^{x}$

When $x = 1$ $\frac{dy}{dx} = 4e$ = gradient of tangent where $x = 1$

(d) (i) $y = e^{3x-6}$

$\frac{dy}{dx} = 3e^{3x-6}$

$\frac{dy}{dx} > 0$ for all values of x since $e^k > 0$ for all k

$\therefore$ Every tangent has positive gradient since $\frac{dy}{dx}$ is the gradient of the tangent

(iii) $3e^{3x-6} = 3$

$e^{3x-6} = 1$

$3x-6 = 0$

$x = 2$

$y = 1$

Question 3

(a) (i) $y = x^3 e^{3x}$

$\frac{dy}{dx} = 3x^2 e^{3x} + x^3 . 3e^{3x}$

$= 3x^2 e^{3x}(1+x)$

(ii) $y = \frac{e^x+1}{e^x-1}$

$\frac{dy}{dx} = \frac{e^x(e^x-1) - e^x(e^x+1)}{(e^x-1)^2}$

$= \frac{e^{2x} - e^x - e^{2x} - e^x}{(e^x-1)^2}$

$= \frac{-2e^x}{(e^x-1)^2}$

(b) $y = xe^x$

$u = x \quad v = e^x$

$\frac{du}{dx} = 1 \quad \frac{dv}{dx} = e^x$

(i) $\frac{dy}{dx} = 1.e^x + x.e^x$

$= (1+x)e^x$

$\frac{d^2y}{dx^2} = 1.e^x + (1+x).e^x$

$= (2+x)e^x$

(ii) Stationary point occurs when $\frac{dy}{dx} = 0$

$(1+x)e^x = 0$

$x = -1 \quad (e^x > 0)$

$y = -1.e^{-1}$

$= -\frac{1}{e}$

~~When x=~~ $\frac{d^2y}{dx^2} = (2-1)e^{-1} = e^{-1} > 0$

$\therefore (-1, \frac{-1}{e})$ is a minimum turning point

2 ✓ 5 ✓ 8 ✓
3 ✓ 6 ✓

(iii) Possible inflexion points occur when $\frac{d^2y}{dx^2} = 0$

ie $(2+x)e^{x} = 0$

$x = -2$

$y = -2e^{-2}$

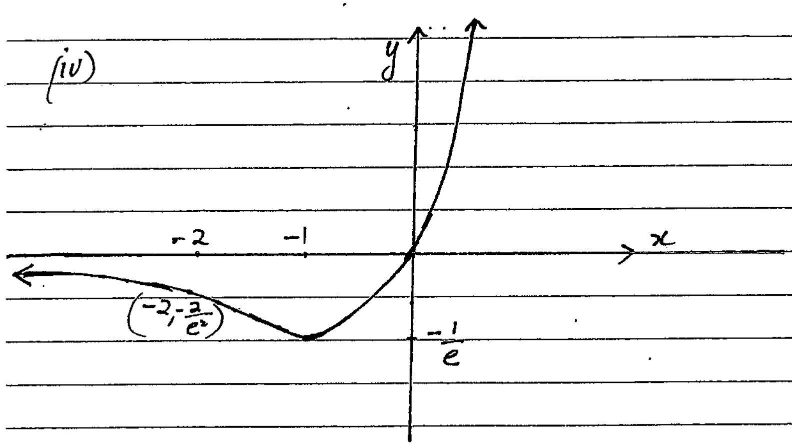
$= -\frac{2}{e^2}$

x	-3	-2	0
$\frac{d^2y}{dx^2}$	$-\frac{1}{e^3}$	0	2
	$\cap$		$\cup$

$\therefore$ Change of concavity

$\therefore \left(-2, -\frac{2}{e^2}\right)$ is an inflexion pt.

(iv)



When $x = 0$

$y = 0$

Range is $y \geq -\frac{1}{e}$

Question 4

(a) (i) $\int e^{-x}\,dx = -e^{-x} + c$ -1 1

(ii) $\int xe^{x^2+7}\,dx = \frac{1}{2}\int 2xe^{x^2+7}\,dx$ -2 for x at front.

$= \left(\frac{1}{2}\right)e^{x^2+7} + c$ $-\frac{1}{2}$ 2

(iii) $\int 7x^6 + 6e^{7x}\,dx = x^7 + \frac{6e^{7x}}{7} + c$ 2

-1 for wrong coefficients.

(b) (i) $\int_{-3}^{0} 3x^2\,dx = \left[x^3\right]_{-3}^{0}$

$= 0 - (-3)^3$

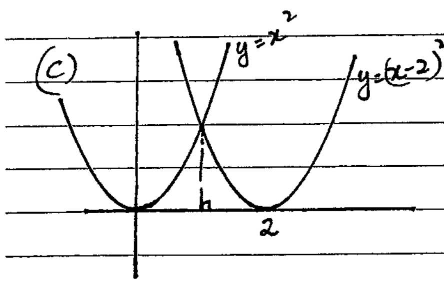
$= (-)27$ $-\frac{1}{2}$ 2

(ii) $\int_{-1}^{3} e^{-\frac{1}{2}x}\,dx = \left[-2e^{-\frac{1}{2}x}\right]_{-1}^{3}$ -2 if $e^{\frac{1}{2}x}$

$= -2\left(e^{-3/2} - e^{\frac{1}{2}}\right)$ $-\frac{1}{2}$

$= 2\left(e^{1/2} - e^{-3/2}\right)$ 2 ~~4~~

(c)



$x^2 = (x-2)^2$ $\quad y = x^2$

$x = 1$ $\quad Y = (x-2)^2$

-2 for wrong region with π

$V = \pi\int_0^1 y^2\,dx + \pi\int_1^2 Y^2\,dx$

$= \pi\int_0^1 x^4\,dx + \pi\int_1^2 (x-2)^4\,dx$

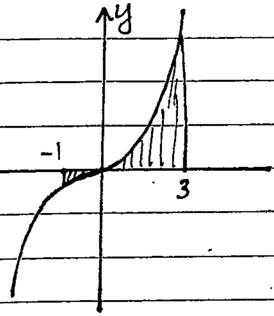
$= \pi\left[\frac{x^5}{5}\right]_0^1 + \pi\left[\frac{(x-2)^5}{5}\right]_1^2$ 3

$\pi\left(\frac{1}{5} - 0\right) + \pi\left(0 - -\frac{1}{5}\right)$

$= \frac{2\pi}{5}$ units^3

Question 5

(a)

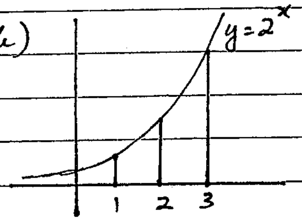


$$\int_{-1}^{0} x^3\,dx = \left[\frac{x^4}{4}\right]_{-1}^{0}$$
$$= 0 - \frac{1}{4}$$
$$= -\frac{1}{4}$$

$$\int_{0}^{3} x^3\,dx = \left[\frac{x^4}{4}\right]_{0}^{3}$$
$$= \frac{81}{4} - 0$$
$$= \frac{81}{4}$$

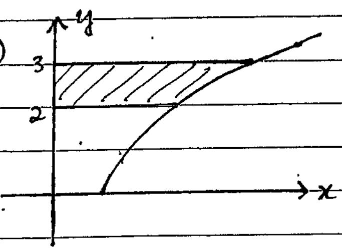
$$\therefore \text{ Area } = \left|-\frac{1}{4}\right| + \left|\frac{81}{4}\right|$$
$$= \frac{82}{4}$$
$$= 20.5$$

(b)



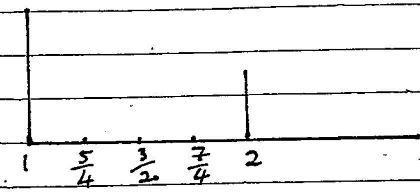
$$\int_{1}^{3} 2^x\,dx \doteqdot \frac{h}{2}(y_1+y_2) + \frac{h}{2}(y_2+y_3) \qquad h=1$$
$$y_1 = 2^1$$
$$= \frac{1}{2}(2+4) + \frac{1}{2}(4+8) \qquad y_2 = 2^2$$
$$= 9 \qquad y_3 = 2^3$$

(c)



$$y = \sqrt{3x-5}$$
$$y^2 = 3x-5$$
$$x = \frac{y^2+5}{3}$$

$$A = \int_{2}^{3} \frac{y^2+5}{3}\,dy$$
$$= \frac{1}{3}\left[\frac{y^3}{3} + 5y\right]_{2}^{3}$$
$$= \frac{1}{3}\left\{\left(\frac{27}{3}+15\right) - \left(\frac{8}{3}+10\right)\right\}$$
$$= \frac{34}{9}$$



$$h = \frac{1}{4}$$
$$y_1 = 1 \quad y_2 = \frac{4}{5} \quad y_3 = \frac{2}{3}$$
$$y_4 = \frac{4}{7} \quad y_5 = \frac{1}{2}$$

$$\int_{1}^{2} \frac{1}{x}\,dx \doteqdot \frac{h}{3}(y_1+4y_2+y_3) + \frac{h}{3}(y_3+4y_4+y_5)$$
$$= \frac{\frac{1}{4}}{3}\left(1+4\times\frac{4}{5}+\frac{2}{3}\right) + \frac{1}{12}\left(\frac{2}{3}+4\times\frac{4}{7}+\frac{1}{2}\right)$$
$$= \frac{1}{12}\left(1+\frac{16}{5}+\frac{2}{3}+\frac{2}{3}+\frac{16}{7}+\frac{1}{2}\right)$$
$$= \frac{1747}{2520}$$
$$= 0.69325\ldots$$