SYDNEY GIRLS HIGH SCHOOL

YEAR 12 MATHEMATICS

ASSESSMENT TASK 3

June 2008

Time allowed: 90 minutes

Topics: Locus and Parabola, Integration, Quadratic functions and Applications of sequences and series

Instructions:

- There are Five (5) questions. Questions **are** of equal value.
- Attempt all questions.
- Show all necessary working. Marks may be deducted for badly arranged work.
- Start each question on a new page. Write on one side of the paper only.

Name: .....

Total : 100 marks

QUESTION 1 (20 marks)

a) The roots of $4x^2-12x-1=0$ are α and β. Find the values of

i) $\alpha+\beta$ 1

ii) $\alpha\beta$ 1

iii) $3\alpha\beta^2+3\alpha^2\beta$ 2

iv) $\beta^2+\alpha^2$ 2

v) $\frac{1}{2\alpha}+\frac{1}{2\beta}$ 2

b) Solve $5-4x-x^2\geq 0$ and graph the answer on a number line. 3

c) i) Find the equation of the directrix of the Parabola $x^2=-12y$ 1

ii) Find the equation of the tangent to this parabola at the point $(-6,-3)$ 2

iii) This tangent meets the directix at T. Find the coordinates of T. 2

d) Given that point A has coordinates $(4,0)$ and O is the origin, find the locus of point $P(x,y)$ which moves so that the line PA is perpendicular to PO. Describe this Locus. 4

QUESTION 2 (20 marks)

a) If $4x^2-5x+6 \equiv a(x-1)^2+b(x-1)+c$ for all values of x, find the values of a, b and c. 3

b) Solve $x^6+7x^3=8$ 3

c) Find the following indefinite integrals

i) $\int(3-2x)^6\,dx$ 2

ii) $\int\frac{t^3-1}{t^2}\,dt$ 2

iii) $\int 2x\sqrt{x}\,dx$ 2

d) For the quadratic equation $(k-4)x^2-11x+10-2k=0$ find the value of k if

i) 4 is a root of this equation 2
ii) The roots are reciprocal 2
iii) The sum of the roots equals their product 2

e) Use the trapezoidal Rule to approximate $\int_0^4\sqrt{16-x^2}\,dx$ using 5 function values. (correct to 2 decimal places) 2

QUESTION 3 (20 marks)

a) i) Sketch the curve $y=x^2-6x+8$ from $x=1$ to $x=6$ 4

ii) Find the area bounded by the parabola, the x- axis and ordinates $x=1$ and $x=6$ 4

b) Solve $4^x-3.2^x+2=0$ 3

c) Evaluate

i) $\int_1^4\frac{dx}{2\sqrt{x}}$ 3

ii) $\int_1^3 3x(4-x)\,dx$ 3

d) The line $y=6x+b$ is a tangent to the parabola $y=x^2+12x-3$.

Find the value of b. 3

QUESTION 4 **(20 marks)**

a) The roots of $x^2-(k+5)x+4k=0$ are such that their product is 2 more than twice their sum. Find the value of k. 3

b) For the Parabola $12y=x^2-48$ write down

i) The focal length 2
ii) The coordinates of the vertex 1
iii) The coordinates of the focus 1
iv) The equation of the directrix 1

c) Lisa Simpson saves \$1200 per year, and at the end of the year invests the years savings at 8% p.a compound interest. How much do her savings amount to at the end of 20 years? 4

d) i) Shade the region which is common to $2x-y+3\geq 0$ and $y\geq x^2$ 3

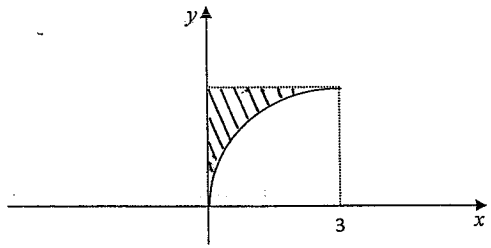
ii) Find the area of this shaded region 3

e) The expression $x^2+kx+16$ is positive definite. Find the value of k. 2

QUESTION 5 **(20 marks)**

a) The region enclosed between the curve $y=-x^2$ and $x=y^2$ is rotated about the x-axis. Find the volume of the solid of revolution. 3

b) i) Given $f(x)=6x-x^2$, find the shaded area. 3



ii) The area between the curve $f(x)=6x-x^2$ and the $y-$axis is rotated about the x-axis. Find the volume of the solid of revolution. 3

c) The volume $V=\pi\int_1^2 \frac{dx}{x^3}$ was formed by rotating an area about the x-axis. Find the size of this area. 3

Question 5 continued overleaf.....

d) Tom Cruze buys a golden Porsche for $400,000 and agrees to pay it off at the same amount each month over 10 years. The interest rate is 18% p.a reducible monthly.

i) If monthly repayments are $M and A_n is the amount owing after n repayments, show that the amount owing after the second payment is given by

$$A_2 = 400000 \times (1.015)^2 - M(1.015+1)$$ 2

ii) Hence find the amount of each monthly repayment 3

iii) If Tom had difficulties on his repayments and made no payments in the last 3 years, how much does he then owe at the end of 10 year?

(assume there is no bank charges or penalty for no payment) 3

THE END

Yr 12 2008 Assessment Task 3

Q1

a) i) $\alpha + \beta = \frac{-b}{a}$

$= \frac{12}{4}$

$= 3$

ii) $\alpha\beta = \frac{c}{a}$

$= \frac{-1}{4}$

iii) $3\alpha\beta(\beta+\alpha)$

$= 3 \times -\frac{1}{4}(3)$

$= -\frac{9}{4}$

iv) $\beta^2 + \alpha^2 = (\beta+\alpha)^2 - 2\alpha\beta$

$= 3^2 - 2\times(-\frac{1}{4})$

$= 9 + \frac{1}{2}$

$= 9\frac{1}{2}$

v) $\frac{1}{2\alpha} + \frac{1}{2\beta} = \frac{\beta+\alpha}{2\alpha\beta}$

$= \frac{3}{2\times(-\frac{1}{4})}$

$= \frac{3}{-\frac{1}{2}}$

$= -6$

b) $x^2 + 4x - 5 \leq 0$

$(x+5)(x-1) \leq 0$

$-5 \leq x \leq 1$

c) i) $4a = 12$

$a = 3$

$y = 3$

ii) $y = \frac{-x^2}{12}$

$y' = \frac{-2x}{12}$

$= \frac{-x}{6}$

at $x = -6$

$m = 1$

$y + 3 = 1(x+6)$

$y = x + 6 - 3$

$\boxed{y = x + 3}$

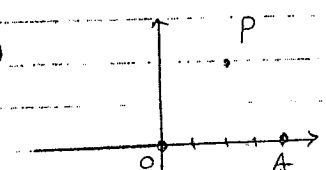
iii) at $y = 3$

$3 = x + 3$

$x = 0$

$T(0, 3)$

d)



$m_{PA} = \frac{y-0}{x-4}$ $\quad m_1 \times m_2 = -1$

$m_{PO} = \frac{y-0}{x-0}$ $\quad \frac{y}{x-4} \times \frac{y}{x} = -1$

PA ⊥ PO $\therefore$ $\quad y^2 = -x^2 + 4x$

$y^2 - 4x + 4x^2 = 0$

2) a)

$a(x-1)^2 + b(x-1) + c$

$= a(x^2 - 2x + 1) + bx - b + c$

$= ax^2 - 2ax + a + bx - b + c$

$= ax^2 - x(2a - b) + a - b + c$

$\boxed{a = 4}$, $\quad 2a - b = 5$

$8 - b = 5$

$\boxed{b = 3}$

$a - b + c = 6$

$4 - 3 + c = 6$

$\boxed{c = 5}$

b) Let $m = x^3$

$m^2 + 7m - 8 = 0$

$(m+8)(m-1) = 0$

$m = -8$ or 1

$x^3 = -8 \rightarrow \boxed{x = -2}$

$x^3 = 1 \rightarrow \boxed{x = 1}$

c) i) $\int (3-2x)^6 \, dx$

$= \frac{(3-2x)^7}{7\times -2} + C$

$= \frac{-(3-2x)^7}{14} + C$

ii) $\int \frac{t^3 - 1}{t^2} \, dt$

$= \int t - t^{-2} \, dt$

$= \frac{t^2}{2} - \frac{t^{-1}}{-1} + C$

$= \frac{t^2}{2} + \frac{1}{t} + C$

iii) $\int 2x\sqrt{x} \, dx$

$= \int 2x \cdot x^{\frac{1}{2}} \, dx$

$= \int 2x^{\frac{3}{2}} \, dx$

$= \frac{2x^{\frac{5}{2}}}{\frac{5}{2}} + C$

$= \frac{4x^{\frac{5}{2}}}{5} + C$

$= \frac{4\sqrt{x^5}}{5} + C$

d) i) $(k-4)\times 16 - 44 + 10 - 2k = 0$

$16k - 64 - 44 + 10 - 2k = 0$

$14k = 98$

$\boxed{k = 7}$

ii) $\alpha, \frac{1}{\alpha}$

$\alpha \times \beta = \frac{c}{a}$

$\alpha \times \frac{1}{\alpha} = \frac{10 - 2k}{k-4}$

$K - 4 = 10 - 2k$

$3k = 14$

$k = \frac{14}{3}$

$\boxed{k = 4\frac{2}{3}}$

iii) $\alpha+\beta = \alpha\beta$

$$\frac{11}{k-4} = \frac{10-2k}{k-4}$$

$10-2k = 11$

$-2k = +1$

$\boxed{k = -\frac{1}{2}}$

e)

x	f(x)	w	wf(x)
0	4	1	4
1	$\sqrt{15}$	2	$2\sqrt{15}$
2	$\sqrt{12}$	2	$4\sqrt{3}$
3	$\sqrt{7}$	2	$2\sqrt{7}$
4	0	1	0

$A \doteq \frac{1}{2}\sum wf(x)$

$= \frac{1}{2} \times (4+2\sqrt{15}+4\sqrt{3}+2\sqrt{7})$

$\doteq 11.98$

3) a)

$(x-4)(x-2)=0$

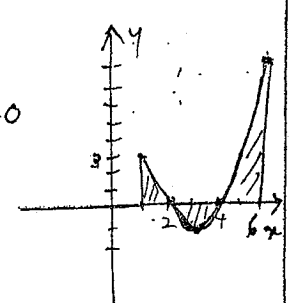
$x=4,\ x=2$

at $x=1$

$y=3$

at $x=6$

$y=8$



ii) $A = \int_0^1 x^2-6x+8\,dx$

$+ \left| \int_2^4 x^2-6x+8\,dx \right|$

$+ \int_4^6 x^2-6x+8\,dx$

$= \frac{x^3}{3} - \frac{6x^2}{2} + 8x \Big]_0^2 +$

$\left| \left[\frac{x^3}{3} - 3x^2 + 8x\right]_2^4 \right| +$

$\left[\frac{x^3}{3} - 3x^2 + 8x\right]_4^6$

$= \left[\frac{8}{3} - 12 + 16 - (0)\right] + \left|\left(\frac{64}{3} - 3(16) + 32\right)\right.$

$\left. - \left(\frac{8}{3} - 12 + 16\right)\right| + \left[\frac{216}{3} - 108\right.$

$\left. + 48 - \left(\frac{64}{3} - 48 + 32\right)\right]$

$= \left[1\frac{1}{3}\right] + \left|-5\frac{1}{3} - 6\frac{2}{3}\right| + \left(12 - 5\frac{1}{3}\right)$

$= 1\frac{1}{3} + 1\frac{1}{3} + 6\frac{2}{3}$

$= 9\frac{1}{3}\ u^2$

b) Let $m = 2^x$

$m^2 - 3m + 2 = 0$

$(m-2)(m-1) = 0$

$m = 2$ or 1 $\therefore 2^x = 2$

$\boxed{x=1}$ or $2^x = 1$ $\therefore \boxed{x=0}$

c) i) $\frac{1}{2}\int_1^4 x^{-\frac{1}{2}}\,dx$

$= \frac{1}{2}\left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4$

$= \frac{1}{2}\left[2\sqrt{x}\right]_1^4$

$= \frac{1}{2}\left[4-2\right]$

$= 1$

ii) $\int_1^3 3x(4-x)\,dx$

$= \int_1^3 12x - 3x^2\,dx$

$= \left[\frac{12x^2}{2} - \frac{3x^3}{3}\right]_1^3$

$= \left[6x^2 - x^3\right]_1^3$

$= (54-27) - (6-1)$

$= 22$

d) $x^2 + 12x - 3 = 6x + b$

$x^2 + 12x - 6x - 3 - b = 0$

$x^2 + 6x - (3+b) = 0$

For tangent $\Delta = 0$

$\Delta = 36 + 4 \times 1 \times (3+b)$

$0 = 36 + 12 + 4b$

$48 + 4b = 0$

$4b = -48$

$\boxed{b = -12}$

4) a)

$\alpha \quad \beta$

$\alpha\beta = 2 + 2(\alpha+\beta)$

$4k = 2 + 2(k+5)$

$4k = 2 + 2k + 10$

$2k = 12$

$\boxed{k=6}$

b) i) $x^2 = 12y + 48$

$x^2 = 12(y+4)$

$4a = 12 \quad \therefore \boxed{a=3}$

ii) $V(0, -4)$

iii) $F(0, -1)$

iv) $y = -7$

c) $A_1 = 1200(1+0.08)^{19}$

$A_2 = 1200(1+0.08)^{18}$

$\vdots$

$A_{20} = 1200$

Total $= A_1 + A_2 + \dots + A_{20}$

$= 1200(1.08)^{19} + 1200(1.08)^{18} + \dots + 1200$

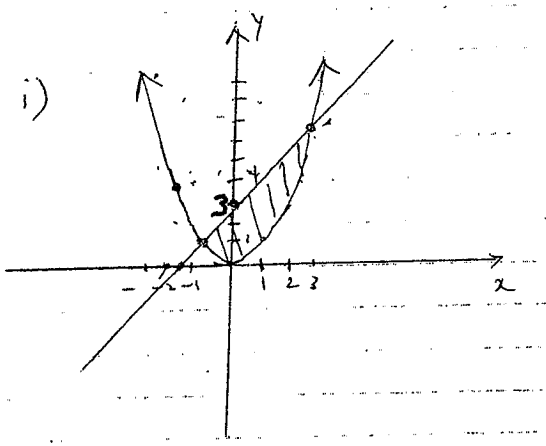
$= 1200\underbrace{(1 + 1.08 + \dots + 1.08^{19})}_{\text{G.P}}$

$a = 1 \quad r = 1.08$

$$\text{Total} = 1200 \times \frac{1(1.08^{20} - 1)}{0.08}$$

$= 54914.36$

d) i)



$2x - y + 3 = 0$

$y = x^2$

$2x - x^2 + 3 = 0$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3$ or -1

ii) $A = \int_{-1}^{3} 2x + 3 - x^2 \, dx$

$A = \left[\frac{2x^2}{2} + 3x - \frac{x^3}{3}\right]_{-1}^{3}$

$= \left[x^2 + 3x - \frac{x^3}{3}\right]_{-1}^{3}$

$= \left(9 + 9 - \frac{27}{3}\right) - \left(1 - 3 + \frac{1}{3}\right)$

$= 9 + \frac{5}{3}$

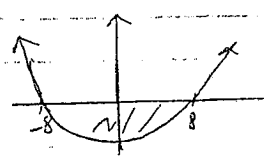
$= 10\frac{2}{3}$

e) $a > 0 \quad \Delta < 0$

$\Delta = k^2 - 64$

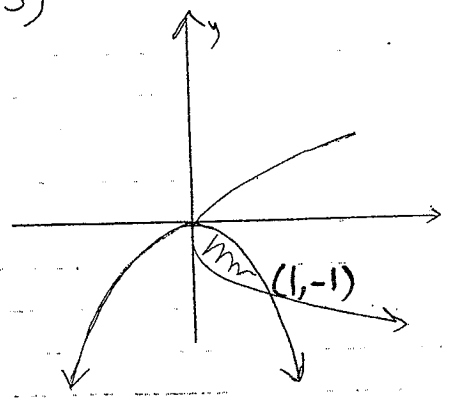
$k^2 - 64 < 0$

$(k-8)(k+8) < 0$



$-8 < k < 8$

5)



$y = -x^2$

$x = y^2$

$x = (-x^2)^2$

$x = x^4$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

$x = 0$ or 1

$V = \pi \int_0^1 x - x^4 \, dx$

$= \pi \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$

$= \pi\left[\frac{1}{2} - \frac{1}{5}\right)$

b) i) $x = 3 \rightarrow y = 9$

Shaded area $= 3 \times 9 - \int_0^3 6x - x^2 \, dx$

$\vdots$

$= 27 - \left[\frac{6x^2}{2} - \frac{x^3}{3}\right]_0^3$

$= 27 - \left[3x^2 - \frac{x^3}{3}\right]_0^3$

$= 27 - [27 - 9]$

$= 9 \ u^2$

ii) Volume $= \pi \times 9^2 \times 3$

$\pi \int_0^3 (6x - x^2)^2 \, dx$

$= 243\pi - \pi \int_0^3 36x^2 - 12x^3 + x^4$

$= 243\pi - \pi\left[\frac{36x^3}{3} - \frac{12x^4}{4} + \frac{x^5}{5}\right]_0^3$

$= 243\pi - \pi\left[12x^3 - 3x^4 + \frac{x^5}{5}\right]_0^3$

$= 243\pi - \pi\left[324 - 243 + \frac{243}{5}\right]$

$= 243\pi - 129\frac{3}{5}\pi$

$= 113\frac{2}{5}\pi$

c)

$$V = \pi \int_1^2 \frac{1}{x^3}\,dx$$

$$\text{Area} = \int_1^2 \sqrt{\frac{1}{x^3}}\,dx$$

$$= \int_1^2 \frac{1}{(x^3)^{\frac{1}{2}}}\,dx$$

$$= \int_1^2 x^{-\frac{3}{2}}\,dx$$

$$= \left[\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}}\right]_1^2$$

$$= \left[\frac{-2}{\sqrt{x}}\right]_1^2$$

$$= \frac{-2}{\sqrt{2}} + 2$$

$$= 2 - \sqrt{2}\ u^2$$

ii)

$$A_{120} = 400000(1.015)^{120} - M(1 + 1.015 + \ldots 1.015^{119})$$

$A_{120} = 0$

$$\therefore M\underbrace{(1 + 1.015 + \ldots 1.015^{119})}_{\text{G.P a=1 r=1.015}} = 400000(1.015)^{120}$$

$$M = \frac{400000(1.015)^{120} \times 0.015}{1(1.015^{120} - 1)}$$

$M = \$7207.41$

iii)

$$A_{84} = 4000000(1.015)^{84} - 7207.41 \times \frac{(1.015^{84} - 1)}{0.015}$$

$= \$199361.50$

Amount owing after 3 more years

$$A = 199361.50(1.015)^{36}$$

$= 340736.622$

d) i) $A_1 = 400000(1.015)^1 - M$

$A_2 = (400000(1.015) - M)1.015 - M$

$= 400000(1.015)^2 - 1.015M - M$