St George Girls High School

Year 11 – Higher School Certificate Course

Assessment Task 1

2009

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 60 minutes.
- Write using black or blue pen.
- Attempt all questions.
- Start each question on a new page.
- Show ALL working.
- Marks for each question are shown in right column
- Complete cover sheet clearly showing
  - your name
  - mathematics class and teacher

Question 1 – (10 Marks) – Start a New Page Marks

a) Given that $y = x^2 - x$, prove that $\frac{dy}{dx} - \frac{d^2y}{dx^2} = \frac{2y-x}{x}$ 2

b) (i) Given that $f(x) = 2^x$ copy and complete the following table 1

x	0	1	2	3
$f(x)$				

(ii) With these 4 function values estimate $\int_0^3 2^x \, dx$ using the trapezoidal rule. 2

c) (i) Sketch the graph of $y = x^2 - 2x - 3$ clearly showing x and y intercepts. 2

(ii) Calculate the area bounded by the graph, the x axis and the ordinates at $x = 0,\ x = 4$ 3

Question 2 – (10 Marks) – Start a New Page

a) Differentiate $(x^2 + 8x)^5$ and hence find $\int (x+4)(x^2+8x)^4 \, dx$ 3

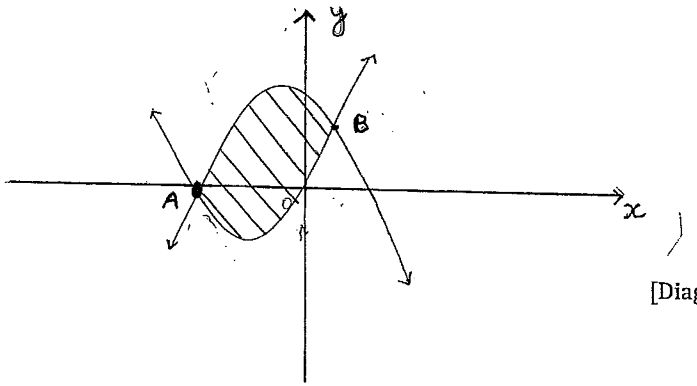
b) For the curve $y = f(x)$, $f''(x) = 3x$ and there is a stationary point at $(2, 2)$. Find the equation of the curve. 4

c) (i) Prove that the function $y = x^3 - 3x^2 + 5x - 3$ is increasing for all values of x 3

(ii) Hence deduce the number of solutions to the equation $x^3 - 3x^2 - 5x \mp 3 = 0$

Question 3 – (10 Marks) – Start a New Page Marks

a) The diagram shows the curves $y = x^2 + x$ and $y = 2 + x - x^2$. A and B are points of intersection of the curves. 6



[Diagram not to scale]

(i) Find the coordinates of A and B

(ii) Calculate the shaded area.

b) A football has a volume approximately the same as the volume generated by rotating the ellipse $9x^2 + 16y^2 = 144$ about the x-axis. Find the volume of the football. 4

Question 4 – (10 Marks) – Start a New Page Marks

For the function $y = \frac{x-1}{x^2}$, $x \neq 0$

a) Give the equation of the vertical asymptote. 1

b) Find the $\lim_{x \to +\infty} \frac{x-1}{x^2}$ and $\lim_{x \to -\infty} \frac{x-1}{x^2}$ 2

c) Find any stationary points. 3

d) Determine the nature of any stationary points. 2

e) Sketch the curve $y = \frac{x-1}{x^2}$ showing all of the above information on your graph. 2

Question 5 – (10 Marks) – Start a New Page

a) (i) Evaluate $\int_1^4 5x\sqrt{x}\ dx$ 2

(ii) Find the indefinite integral of $(x^2 + 1)^2$ 2

b) The hourly cost of running a boat is $\$\left(\frac{v^2}{20} + 10\right)$, where v is the speed in knots.

(i) For a trip of D nautical miles show that the cost is $\$\frac{D}{v}\left(\frac{v^2}{20} + 10\right)$ 2

(ii) What is the cost of the most economical trip. 4

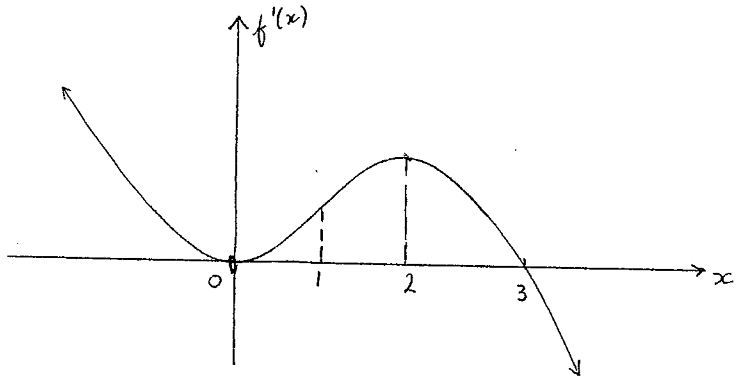
Question 6 – (10 Marks) – Start a New Page

Marks

a) Show that $\int x^2(x-1)\,dx \not\equiv (\int x^2\,dx) \times (\int (x-1)\,dx)$ 2

b) Find the volume of the solid formed when the region bounded by the parabola $y = 4 - x^2$ and the x-axis is rotated about the y-axis. 4

c) 4



Given the graph of $y = f'(x)$, state values of x where $y = f(x)$:

(i) is decreasing

(ii) has a maximum turning point

(iii) has an inflexion

(iv) is concave up

End of Paper

EXT I 2009 HSC ASSESSMENT #1

Question 1

(a) $y = x^2 - x$

$\frac{dy}{dx} = 2x - 1$

$\frac{d^2y}{dx^2} = 2$

LHS $= \frac{dy}{dx} - \frac{d^2y}{dx^2}$

$= 2x - 1 - 2$

$= 2x - 3$

RHS $= \frac{2y - x}{x}$

$= \frac{2(x^2 - x) - x}{x}$

$= \frac{2x^2 - 2x - x}{x}$

$= \frac{2x^2 - 3x}{x}$

$= 2x - 3$

$=$ LHS

$\therefore \frac{dy}{dx} - \frac{d^2y}{dx^2} = \frac{2y - x}{x}$

(b)

(i)

x	0	1	2	3
$f(x)$	1	2	4	8

(ii) $\int_0^3 2^x dx = \frac{1-0}{2}\left[1 + 2(2+4) + 8\right]$

$= \frac{1}{2}\left[1 + 12 + 8\right]$

$= \frac{13}{2}$

$= 6.5$

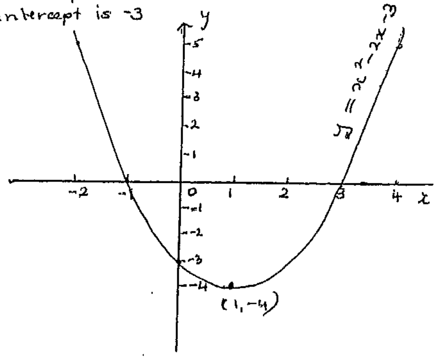
(c) $y = x^2 - 2x - 3$

$= (x-3)(x+1)$

if $x = 1$ $y = -4$

x intercepts are 3 and -1

y intercept is -3



(ii) Area $= \left|\int_0^3 (x^2 - 2x - 3)dx\right| + \int_3^4 (x^2 - 2x - 3)dx$

$= \left|\left[\frac{x^3}{3} - \frac{2x^2}{2} - 3x\right]_0^3\right| + \left[\frac{x^3}{3} - x^2 - 3x\right]_3^4$

$= \left|\left(\frac{27}{3} - 9 - 9\right) - 0\right| + \left[\left(\frac{64}{3} - 16 - 12\right) + 9\right]$

$= 9 + 2\frac{1}{3}$

$= 11\frac{1}{3}$

$\therefore$ Area is $11\frac{1}{3}$ units^2

Question 2

(a) Let $y = (x^2 + 8x)^5$

$\frac{dy}{dx} = 5(x^2 + 8x)^4 \cdot (2x + 8)$

$= 5(x^2 + 8x)^4 \cdot 2(x + 4)$

$= 10(x^2 + 8x)^4(x + 4)$

$\therefore \int (x+4)(x^2 + 8x)^4 dx = \frac{1}{10}(x^2 + 8x)^5 + C$

(b) $f''(x) = 3x$

$f'(x) = \frac{3x^2}{2} + c_1$

when $x = 2$ $f'(x) = 0$

$0 = \frac{3 \cdot 4}{2} + c_1$

$-6 = c_1$

$f'(x) = \frac{3x^2}{2} - 6$

$f(x) = \frac{3}{2}\frac{x^3}{3} - 6x + C_2$

when $x = 2$ $f(2) = 2$

$2 = \frac{8}{2} - 12 + C_2$

$C_2 = 10$

$\therefore f(x) = \frac{x^3}{2} - 6x + 10$

(c)(i) $y = x^3 - 3x^2 + 5x - 3$

Function is increasing if $\frac{dy}{dx} > 0$

$\frac{dy}{dx} = 3x^2 - 6x + 5$

$3x^2 - 6x + 5 > 0$ if

$\Delta < 0$ and $a > 0$

$a = 3$ $\therefore a > 0$

$\Delta = b^2 - 4ac$

$= 36 - 4 \times 3 \times 5$

$= -24$

$\therefore 3x^2 - 6x + 5 > 0$ for all x

Hence $y = x^3 - 3x^2 + 5x - 3$ is increasing for all values of x

(ii) As $y = x^3 - 3x^2 + 5x - 3$ is increasing for all values of x there is only one solution to $x^3 - 3x^2 - 5x + 3 = 0$

Question 3

(a) (i) $y = x^2 + x$ —— (1)

$y = 2 + x - x^2$ —— (2)

Sub (1) into (2)

$x^2 + x = 2 + x - x^2$

$2x^2 - 2 = 0$

$2(x^2 - 1) = 0$

$2(x-1)(x+1) = 0$

$\therefore x = 1$ or -1

if $x = 1$ $y = 1^2 + 1 = 2$

if $x = -1$ $y = (-1)^2 + (-1) = 0$

$\therefore$ A has co-ordinates $(-1, 0)$

B has co-ordinates $(1, 2)$

(ii) Shaded area $= \int_{-1}^{1} [(2 + x - x^2) - (x^2 + x)]\,dx$

$= \int_{-1}^{1} (2 - 2x^2)\,dx$

$= \left[2x - \frac{2x^3}{3} \right]_{-1}^{1}$

$= \left(2 - \frac{2}{3}\right) - \left(-2 + \frac{2}{3}\right)$

$= 2\frac{2}{3}$

$\therefore$ Shaded area is $2\frac{2}{3}$ units^2

(b) $V_x = \pi \int_a^b y^2\,dx$

when $y = 0$ $9x^2 = 144$

$x^2 = \frac{144}{9}$

$x = \pm\sqrt{16}$

$= \pm 4$

$9x^2 + 16y^2 = 144$

$y^2 = \frac{144 - 9x^2}{16}$

$V_x = \pi \int_{-4}^{4} \frac{144 - 9x^2}{16}\,dx$

$= \frac{\pi}{16} \left[144x - \frac{9x^3}{3} \right]_{-4}^{4}$

$= \frac{\pi}{16} \left[(576 - 192) - (-576 + 192) \right]$

$= \frac{\pi}{16} \cdot 768$

$= 48\pi$

$\therefore$ Exact volume is 48π units^3

Question 4

$y = \frac{x-1}{x^2}$, $x \neq 0$

(a) Vertical asymptote is $x = 0$

(b) $\lim_{x \to \infty} \frac{x-1}{x^2} = \lim_{x \to \infty} \frac{1}{x} - \frac{1}{x^2}$

$= 0^+$

$\lim_{x \to -\infty} \frac{1}{x} - \frac{1}{x^2}$

$= 0^-$

(c) Stationary points when $\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{x^2 \cdot 1 - (x-1) \cdot 2x}{x^4}$

$= \frac{x^2 - 2x^2 + 2x}{x^4}$

$= \frac{-x^2 + 2x}{x^4}$

$= \frac{-x + 2}{x^3}$

$0 = \frac{-x + 2}{x^3}$

$\therefore x = 2$ $y = \frac{2-1}{4} = \frac{1}{4}$

Stationary point at $(2, \frac{1}{4})$

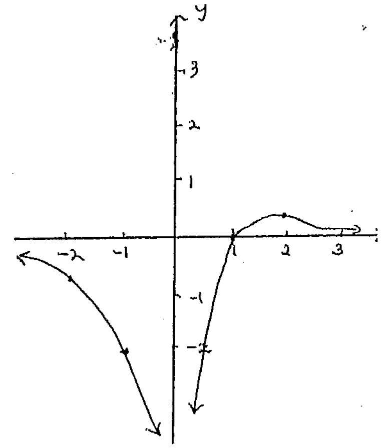
(d) Test nature of stationary point

x	1	2	3
y''	2	0	$-\frac{1}{3}$

/ — \

$\therefore$ stationary point is a maximum

(e)



x intercept at $x = 1$

Question 5

(a) (i) $\int_1^4 5x\sqrt{x}\,dx$

$= 5\int_1^4 x^{3/2}\,dx$

$= 5\left[\frac{x^{5/2}}{5/2}\right]_1^4$

$= \left[2x^{5/2}\right]_1^4$

$= 2.4^{5/2} - 2$

$= 64 - 2$

$= 62$

(ii) $\int (x^2+1)^2\,dx$

$= \int (x^4+2x^2+1)\,dx$

$= \frac{x^5}{5} + \frac{2x^3}{3} + x + C$

(b) (i) $S = \frac{D}{T}$

$\therefore T = \frac{D}{S}$

$= \frac{D}{V}$

Time of journey is $\frac{D}{V}$ hours

$\therefore$ Cost $= \$\left(\frac{V^2}{20}+10\right)\times\frac{D}{V}$

$= \$\,\frac{D}{V}\left(\frac{V^2}{20}+10\right)$

(ii) Stationary points $\frac{dC}{dV} = 0$

$C = \frac{DV}{20} + \frac{100D}{V}$

$= \frac{D}{20}.V + 100D.V^{-1}$

$\frac{dC}{dV} = \frac{D}{20} - 100DV^{-2}$

$= \frac{D}{20} - \frac{100D}{V^2}$

$0 = \frac{D}{20} - \frac{100D}{V^2}$

$\frac{100D}{V^2} = \frac{D}{20}$

$200 = V^2$

$V = \pm\sqrt{200}$

$= \pm 10\sqrt{2}$ $\quad V > 0$

$\therefore V = 10\sqrt{2}$ knots

check for nature of stationary point

$\frac{d^2C}{dV^2} = 200DV^{-3}$

$= \frac{200D}{V^3}$

when $V = 10\sqrt{2}$ $\quad \frac{d^2C}{dV^2} = \frac{20D}{1000\times2\sqrt{2}}$

$= \frac{D}{100\sqrt{2}}$

$\frac{d^2C}{dV^2} > 0$

$\therefore$ curve is concave up hence a minimum turning point when $V = 10\sqrt{2}$

$\therefore$ Most economical cost of trip is

$\$\,\frac{D}{10\sqrt{2}}\left(\frac{200}{20}+10\right) = \$\,\frac{2D}{\sqrt{2}}$

$= \$\,\sqrt{2}D$

Question 6

(a) $\int x^2(x-1)\,dx = \int(x^3-x^2)\,dx$

$= \frac{x^4}{4} - \frac{x^3}{3} + C_1$

$\int x^2\,dx \int (x-1)\,dx$

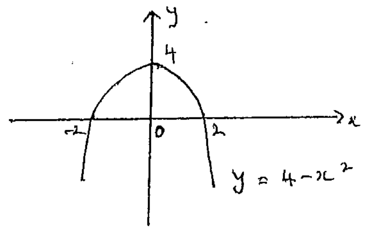
$= \frac{x^3}{3}.\left(\frac{x^2}{2}-x\right) + C_2$

$= \frac{x^5}{6} - \frac{x^4}{3} + C_2$

$\neq \frac{x^4}{4} - \frac{x^3}{3} + C_1$

$\therefore \int x^2(x-1)\,dx \neq \int x^2\,dx \int (x-1)\,dx$

(b)



$V_y = \pi\int_a^b x^2\,dy$

$x^2 = 4-y$ $\quad$ when $y = 0$ $x = \pm 2$

$\therefore V_y = \pi\int_{-2}^{2}(4-y)\,dy$

$= \pi\left[4y - \frac{y^2}{2}\right]_{-2}^{2}$

$= \pi\left[(8-2)-(-8-2)\right]$

$= \pi.16$

$= 16\pi$

$\therefore$ Volume is 16π units^3

(a) (i) decreasing if $f'(x) < 0$

decreasing for $x > 3$

(ii) maximum turning point when $x = 3$

(iii) Inflexion at $x = 0$

(iv) Concave up $0 < x < 2$