St George Girls High School

Year 12

Mid-HSC Course Examination

2012

Mathematics

General Instructions

- Working time: 90 minutes
- Reading time: 5 minutes
- Write using blue or black pen.
- Board-approved calculators may be used.
- Begin each question in a new booklet
- All necessary working should be shown in every question.
- A table of standard integrals is provided.

Total marks – 65

- Attempt Questions 1 – 5
- All questions are of equal value
- Multiple Choice answers should be written in your booklet ie. A, B, C or D

Question 1 – (13 marks) – (Start a new booklet) **Marks**

Parts a) and b) are multiple choice questions.

Write down the letter matching the correct answer in your booklet

a) $\int (x^3 - 3x)\ dx =$ 1

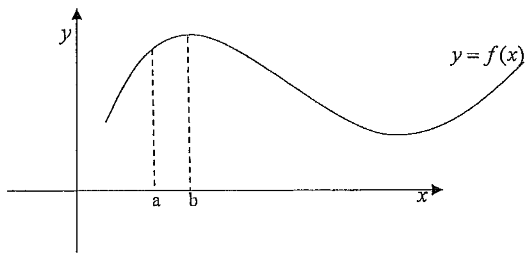
A. $3x^2 - 3 + C$

B. $4x^4 - 6x^2 + C$

C. $\dfrac{x^4}{3} - 3x^2 + C$

D. $\dfrac{x^4}{4} - \dfrac{3x^2}{2} + C$

b) The graph of the function $y = f(x)$ is graphed below 1



For x values such that $a < x < b$,

A. $f'(x) > 0$ and $f''(x) > 0$

B. $f'(x) > 0$ and $f''(x) < 0$

C. $f'(x) < 0$ and $f''(x) > 0$

D. $f'(x) < 0$ and $f''(x) < 0$

Question 1 – (cont'd) Marks

c) Find any points of inflexion of the curve $y = \frac{2x^3}{3} - 4x^2$. Justify your answer. 2

d) For what values of x is the curve $y = 18x - 4x^2$ increasing? 2

e) A function $f(x)$ is defined by $f(x) = x(x^2 - 12)$

(i) Find all solutions to $f(x) = 0$ 2

(ii) Find the coordinates of the turning points of the graph $y = f(x)$ and determine their nature. 3

(iii) Hence, sketch the graph of $y = f(x)$ in the domain $\{x: -3 \leq x \leq 4\}$, showing all intercepts with the axes and all turning points. 2

Question 2 – (13 marks) – (Start a new booklet) Marks

Parts a) and b) are multiple choice questions.

Write down the letter matching the correct answer in your booklet

a) Note: C is a real constant. $\int (2x+5)^{\frac{3}{2}}\, dx$ is equal to: 1

A. $\frac{2}{5}(2x+5)^{\frac{5}{2}} + C$

B. $\frac{1}{2}(2x+5)^{\frac{1}{2}} + C$

C. $\frac{1}{5}(2x+5)^{\frac{5}{2}} + C$

D. $\frac{4}{5}(2x+5)^{\frac{5}{2}} + C$

b) $\int_1^4 2(f(x)+1)\, dx$ can be written as: 1

A. $2\int_1^4 (f(x)+1)\, dx$

B. $2\int_1^4 f(x)\, dx + 3$

C. $2\int_1^4 f(x)\, dx$

D. $2\int_1^4 f(x)\, dx + x$

c) Find the primitives of:

(i) $3x^2 - 3x + 7$ 1

(ii) $\frac{x^4 - 3x}{x^3}$ 2

(iii) $\frac{6}{\sqrt[3]{x}} + \frac{12}{x^4}$ 2

Question 2 – (cont'd) Marks

d) Evaluate

(i) $\int_1^2 6x^4\, dx$ 1

(ii) $\int_0^4 (2x-3)\, dx$ 1

(iii) $\int_1^9 \left(2x+9\sqrt{x}\right)\, dx$ 2

(iv) $\int_1^2 (x+1)(3x-1)\, dx$ 2

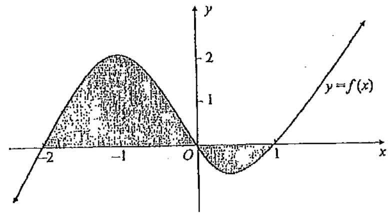
The examination continues on the next page

Question 3 – (13 marks) – (Start a new booklet) Marks

Parts a) and b) are multiple choice questions.

Write down the letter matching the correct answer in your booklet

a) 1



The total area of the shaded region shown is given by

A. $\int_{-2}^{1} f(x)\, dx$

B. $\int_{-2}^{0} f(x)\, dx + \int_0^1 f(x)\, dx$

C. $\int_{-2}^{0} f(x)\, dx - \int_0^1 f(x)\, dx$

D. $\int_{-1}^{2} f(x)\, dx$

b) The area of the region enclosed by the curve $y = x^2$ and the line $y = 4$ is: 1

A. $\frac{8}{3}$

B. $\frac{16}{3}$

C. $\frac{32}{3}$

D. $\frac{26}{3}$

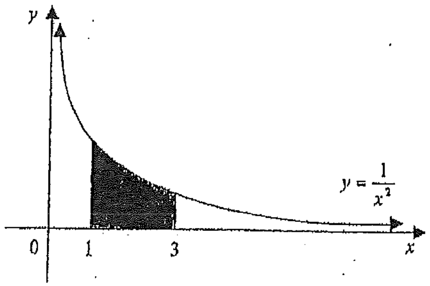
Question 3 – (cont'd) **Marks**

c) If $\frac{dy}{dx} = 6x - 1$ and the function passes through the point (1, 5), find an expression for y. 2

d) The curve $y = x^3 + bx^2 + 5x - 6$ has a stationary point at when $x = 1$. Find the value of the constant, b. 2

e) Find the value of k if $\int_4^k (x-4)\ dx = 8$ 2

f)



NOT TO SCALE

The diagram above shows the area bounded by the graph $= \frac{1}{x^2}$, (for $x > 0$), the x-axis and the lines $x = 1$ and $x = 3$.

(i) Find the shaded area. Leave your answer as a fraction. 2

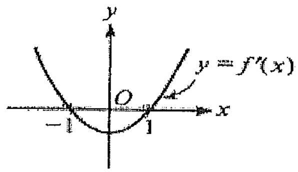
(ii) Find the volume of the solid formed when the shaded area is rotated about x-axis. Leave your answer in exact form. 3

<u>Question 4</u> – (13 marks) – (Start a new booklet) **Marks**

<u>Parts a) and b) are multiple choice questions.</u>

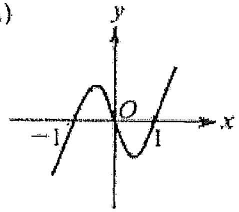
Write down the letter matching the correct answer in your booklet

a) 1

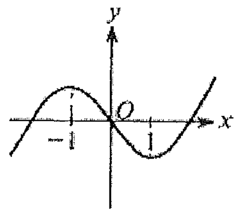


The graph of the <u>derivative</u> of f is shown in the figure above. Which of the following could be the graph of f ?

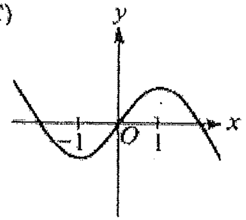
(A)



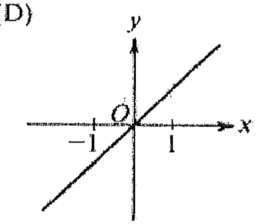
(B)



(C)



(D)



b) The region enclosed by the graph of $y = x^2$, the line $x = 2$, and the x-axis is revolved about the y-axis. The volume of the solid generated is: 1

A. 8π

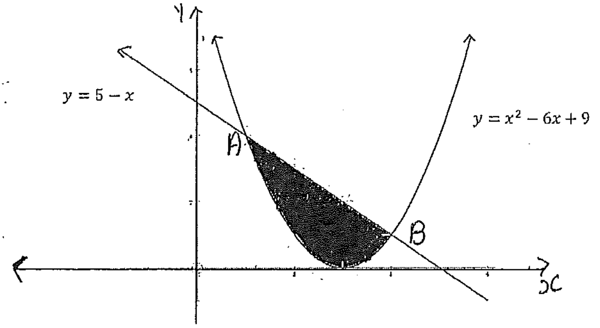
B. $\frac{32}{5}\pi$

C. $\frac{16}{3}\pi$

D. 4π

Question 4 – (cont'd) **Marks**

c)



The diagram shows the graphs of $y = x^2 - 6x + 9$ and $y = 5 - x$.

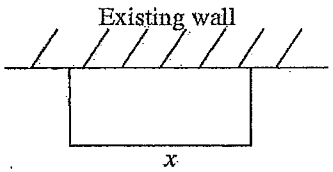
The graphs intersect at the points A and B as shown.

(i) Find the x coordinates of the points A and B. 2

(ii) Find the area of the shaded region between $y = x^2 - 6x + 9$ and $y = 5 - x$. 3

d) A landscaper is constructing a rectangular garden bed. The area needs to be fenced to keep out the deer. Three of the sides are to be enclosed using 40 m of fencing, while an existing wall will form the fourth side of a rectangle.

(i) If x is the side opposite the wall show that the area is given by $A = 20x - \frac{1}{2}x^2$ 2



(ii) Find the dimension of the rectangle that makes the area a maximum. Justify your answer. 3

(iii) Find the maximum area. 1

Question 5 – (13 marks) – (Start a new booklet) **Marks**

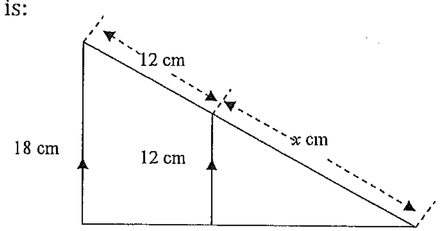
Parts a), b), c) and d) are multiple choice questions.

Write down the letter matching the correct answer in your booklet

a) Which of the following is NOT a property of a rectangle? 1

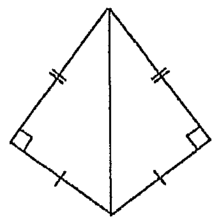
A. adjacent sides are equal
B. all angles are equal
C. diagonals bisect each other
D. opposite sides are parallel

b) The value of x is: 1



A. 6 cm
B. 8 cm
C. 18 cm
D. 24 cm

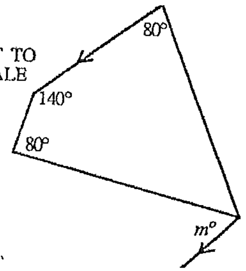
c) Using the information in the diagram, which of the following tests for congruence is not suitable?



A. AAS (two angles and a side of one triangle are respectively equal to two angles and the corresponding side of the other triangle)
B. RHS (the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and one side of the other right-angled triangle)
C. SAS (two sides and the included angle of one triangle are respectively equal to two sides and the included angle of the other triangle)
D. SSS (the three sides of one triangle are respectively equal to the three sides of the other triangle)

Question 5 – (cont'd) Marks

d)

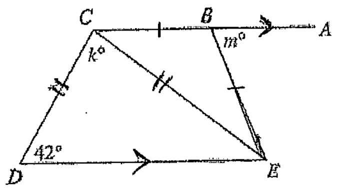


Find the value of m. 1

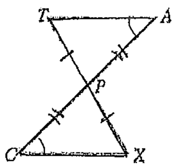
A. 20
B. 40
C. 60
D. 80

e) In the diagram below, AC is parallel to DE and $CD = CE$. 4

Find the values of k and m, giving reasons for your answers.



f) In the diagram below, $TP = XP$ and $AP = CP$



(i) Prove $\Delta TAP \equiv \Delta XCP$ 3

(ii) Hence prove that $TA \| XC$ 2

End of Paper

a) $\int x^3 - 3x \, dx$

$= \frac{x^4}{4} - \frac{3x^2}{2} + c$

$\therefore$ D 1

13/13

b) B 1

c) $y = \frac{2x^3}{3} - 4x^2$

$y' = \frac{2 \times 3x^2}{3} - 4 \times 2x$

$= 2x^2 - 8x$

$y'' = 4x - 8$

$4x - 8 = 0$

$4x = 8$

$x = 2$

x	1	2	3
y''	$\cap$	$-$	$\cup$

$\therefore x = 2$

$y = \frac{2(2)^3}{3} - 4(2)^2$

$= -10\frac{2}{3}$

$\therefore (2, -10\frac{2}{3})$ is a point of inflexion, as points before and after it change concavity.

2

d) $y = 18x - 4x^2$

$y' = 18 - 8x$

$18 - 8x > 0$

$18 > 8x$

$2\frac{1}{4} > x$

$\therefore$ The curve is increasing when $x < 2\frac{1}{4}$

2

e) (i) $f(x) = x(x^2 - 12)$

$x(x^2 - 12) = 0$

$x = 0$ or $x^2 - 12 = 0$

$x^2 = 12$

$x = \pm\sqrt{12}$

$\therefore x = 0, \sqrt{12}$ or $-\sqrt{12}$

2

(ii) $u = x$ $\quad v = x^2 - 12$

$u' = 1$ $\quad v' = 2x$

$f'(x) = vu' + uv'$

$= x^2 - 12 + 2x^2$

$= 3x^2 - 12$

$3x^2 - 12 = 0$

$3x^2 = 12$

$x^2 = 4$

$x = \pm 2$

$y = 2(2^2 - 12)$

$= -16$

Show values

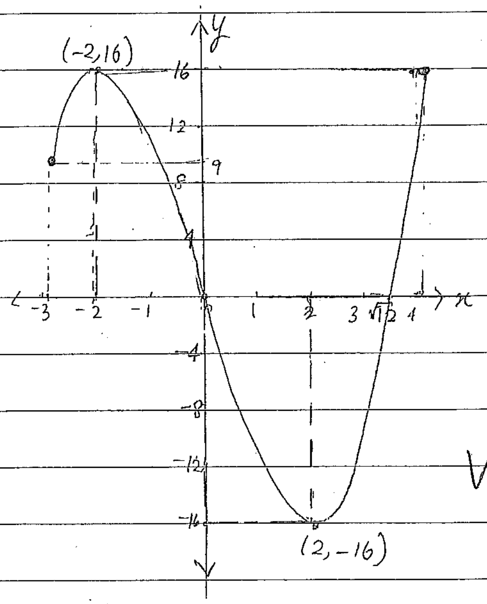
x	-3	-2	-1
$f'(x)$	$+$	0	$-$

x	1	2	3
$f'(x)$	$-$	0	$+$

maximum turning point at $(-2, 16)$

$\therefore$ minimum turning point at $(2, -16)$

3



✓ 2

$y' = 3x^2 - 12$

$y'' = 6x$

a) $\int (2x+5)^{\frac{3}{2}}\, dx$

$= \dfrac{(2x+5)^{\frac{5}{2}}}{\frac{5}{\cancel{2}} \times \cancel{2}} + c$

$= \dfrac{1}{5} (2x+5)^{\frac{5}{2}} + C$

$\therefore$ C ✓ 1

b) A ✓ 1

c) (i) $\int (3x^2 - 3x + 7)\, dx$

$= \dfrac{3x^3}{3} - \dfrac{3x^2}{2} + 7x + c$ ✓ 1

$= x^3 - \dfrac{3}{2}x^2 + 7x + C$ ✗

(ii) $\dfrac{x^4 - 3x}{x^3} = \dfrac{x^4}{x^3} - \dfrac{3x}{x^3} = x - 3x^{-2}$

$\int (x - 3x^{-2})\, dx$

$= \dfrac{x^2}{2} - \dfrac{3x^{-1}}{-1} + c$

$= \dfrac{x^2}{2} + \dfrac{3}{x} + c$ ✓ 2

(iii) $\frac{6}{3\sqrt{x}} + \frac{12}{x^4} = 6x^{-\frac{1}{3}} + 12x^{-4}$

$\int (6x^{-\frac{1}{3}} + 12x^{-4})\, dx$

$= \frac{3 \times 6x^{\frac{2}{3}}}{2} + \frac{12x^{-3}}{-3} + C$

$= 9x^{\frac{2}{3}} - 4x^{-3} + C$

$= 9\sqrt[3]{x^2} - \frac{4}{x^3} + C = 9x^{\frac{2}{3}} - \frac{4}{x^3} + C$ ✓ 2

a) (i) $\int_1^2 6x^4\, dx$

$= \left[\frac{6x^5}{5}\right]_1^2$

$= \frac{6(2)^5}{5} - \frac{6(1)^5}{5}$

$= \frac{192}{5} - \frac{6}{5} = \frac{186}{5} = 37\frac{1}{5}$ ✓ 1

(ii) $\int_0^4 (2x-3)\, dx = \left[2x^2/2 - 3x\right]_0^4$

$= \left[x^2 - 3x\right]_0^4$

$= (4^2 - 3(4)) - 0$ ✓ 1

$= 4$

(iii) $\int_1^9 (2x + 9\sqrt{x})\, dx$

$= \int_1^9 (2x + 9x^{\frac{1}{2}})\, dx$

$= \left[2x^2/2 + \frac{2 \times 9x^{3/2}}{3}\right]_1^9 = \left[x^2 + 6x^{\frac{3}{2}}\right]_1^9$

$= (9^2 + 6(9)^{\frac{3}{2}}) - (1^2 + 6(1)^{3/2})$

$= 243 - 7$ ✓ 2

$= 236$

(iv) $\int_1^2 (x+1)(3x-1)\, dx = \int_1^2 (3x^2 - x + 3x - 1)\, dx$

$= \int_1^2 (3x^2 + 2x - 1)\, dx$

$= \left[\frac{3x^3}{3} + \frac{2x^2}{2} - x\right]_1^2$

$= \left[x^3 + x^2 - x\right]_1^2$

$= (2^3 + 2^2 - 2) - (1^3 + 1^2 - 1)$

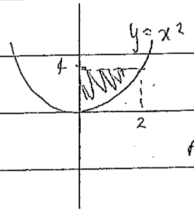
$= 10 - 1$

$= 9$ ✓ 2

13

a) C ✓

b)



$y = x^2$

$x = \sqrt{y}$

$A = \int_0^4 y^{\frac{1}{2}}\, dy$

$= \left[\frac{2y^{3/2}}{3}\right]_0^4$

$= \frac{16}{3}$

$\therefore$ B ✗

c) $\frac{dy}{dx} = 6x - 1$

$y = \frac{6x^2}{2} - x + C$

$= 3x^2 - x + C$ ✓ (1, 5)

$5 = 3(1)^2 - 1 + C$

$= 3 - 1 + C$

$= 2 + C$ $\quad \therefore y = 3x^2 - x + 3$ ✓

$\therefore C = 5 - 2$

$= 3$

d) $y = x^3 + bx^2 + 5x - 6$

$y' = 3x^2 + 2bx + 5$

$f'(1) = 3(1)^2 + 2b(1) + 5 = 0$ ✓

$3 + 2b + 5 = 0$

$2b + 8 = 0$ $\quad \cancel{2b = -8}$

$b = -4$ ✓

e) $\int_4^k (x-4)\, dx = \left[\frac{x^2}{2} - 4x\right]_4^k$

$= \left(\frac{k^2}{2} - 4k\right) - \left(\frac{4^2}{2} - 4(4)\right)$

$= \frac{k^2}{2} - 4k + 8 = 8$

$\frac{k^2}{2} - 4k = 0$ ✓

$k^2 - 8k = 0$

$k(\cancel{8}k - 8) = 0$ ✓

$\cancel{k = 0}$ or $k = 8$

$k = 8$ (k is the larger integrand, $k > 4$)

f) $A = \int_1^3 \frac{1}{x^2}\, dx$ $\quad$ As $\int_a^b f(x)\,dx = -\int_b^a f(x)\, dx$ So $k=0$, is O.K.

(i) $= \int_1^3 x^{-2}\, dx$

$= \left[\frac{x^{-1}}{-1}\right]_1^3$ ✓

$= \left[-\frac{1}{x}\right]_1^3$

$= \frac{-1}{3} - \frac{-1}{1}$ ✓

$= \frac{2}{3}\ u^2$ ✓

$y = \frac{1}{x^2}$

(ii) $V_x = \pi \int_1^3 y^2\, dx$

$= \pi \int_1^3 x^{-4}\, dx$ $\quad y^2 = \frac{1}{x^4}$

$= \pi \left[\frac{x^{-5}}{-5}\right]_1^3$

$= \pi \left[\frac{-1}{5x^5}\right]_1^3 = \pi\left(-\frac{1}{1215} + \frac{1}{5}\right) = \frac{242}{1215}\pi\ u^3$

(ii) $V_x = \pi \int_1^3 y^2 \, dx$

$= \pi \int_1^3 \frac{1}{x^4} \, dx$

$= \pi \int_1^3 x^{-4} \, dx$

$= \pi \left[\frac{x^{-3}}{-3} \right]_1^3$

$= \pi \left(-\frac{3^{-3}}{3} + \frac{1^{-3}}{3} \right)$

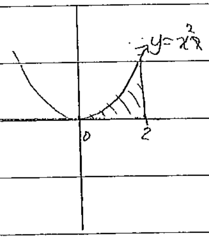
$= \pi \left(-\frac{1}{81} + \frac{1}{3} \right)$

$= \frac{26}{81} \pi$ units^3

a) B

1

b)



$V_x = \pi \int_0^2 y^2 \, dx$

$= \pi \int_0^2 x^4 \, dx$

$= \pi \left[\frac{x^5}{5} \right]_0^2$

$= \frac{32}{5} \pi$

$\therefore$ B

c) (i) $x^2 - 6x + 9 = 5 - x$

S P N
-5 4 $-4, -1$

$x^2 - 5x + 4 = 0$

$(x-4)(x-1) = 0$

$x = 4$ or $x = 1$

$y = 5 - 4$ $= 1$ $\qquad y = 5 - 1$ $= 4$

$\therefore$ $(4, 1)$ and $(1, 4)$ points of intersection.

$A(1, 4)$ and $B(4, 1)$

2

(ii) $A = \int_1^4 5 - x - (x^2 - 6x + 9) \, dx$

$= \int_1^4 (5 - x - x^2 + 6x - 9) \, dx$

$= \int_1^4 (5x - x^2 - 4) \, dx$

$= \left[\frac{5x^2}{2} - \frac{x^3}{3} - 4x \right]_1^4$

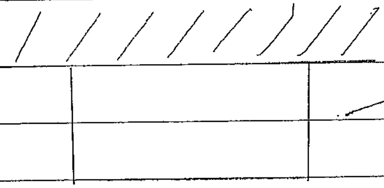
$= \left(\frac{5(4)^2}{2} - \frac{4^3}{3} - 4(4) \right) - \left(\frac{5}{2} - \frac{1}{3} - 4 \right)$

$= \left(40 - \frac{64}{3} - 16 \right) - -\frac{11}{6}$

$= \frac{9}{2} = 4\frac{1}{2}$ units^2

3

d)



(i) $\frac{40-x}{2} = \frac{40}{2} - \frac{x}{2}$

x

$= 20 - \frac{1}{2}x$

$\therefore$ Area $= x \times (20 - \frac{1}{2}x)$

$= 20x - \frac{1}{2}x^2$

2

(ii) $\frac{dA}{dx} = 20 - 2 \times \frac{1}{2}x$

$= 20 - x$

$20 - x = 0$

$x = 20$

x	19	20	21
$\frac{dA}{dx}$	+	0	−

3

$\therefore$ maximum at $x = 20$

(iii) $A = 20x - \frac{1}{2}x^2$

$= 20(20) - \frac{1}{2}(20)^2$

$= 400 - 200$

$= 200\ m^2$

1

a) A

1

b) $\frac{12}{x} = \frac{18}{12+x}$

$12(12+x) = 18x$

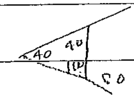
$144 + 12x = 18x$

$6x = 144$

$x = 24$ $\therefore$ D

1

c) A



1

d) B

1

e) $\triangle CDE$ is isosceles (in an isosceles triangle, two sides are equal, CD = CE)

$\therefore \angle CDE = \angle CED$ (equal angles are opposite equal sides)

$= 42°$

$\therefore k° = 180 - (2 \times 42)$

$= 96°$ (angle sum of a triangle is 180°)

Extending BC to G and ED to F

~~$\angle FDC = 180 - 42°$~~ $\angle DCG = 42°$ (alternate angles are equal on parallel lines, CB ∥ DE)

$\angle BCE = 180 - (42 + 96)$

$= 42°$ (angle sum of a straight angle is 180°)

ΔCBE is isosceles (isosceles triangle has 2 sides equal)

$\therefore \angle CEB = \angle BCE$ (equal angles are opposite equal sides)

$= 42°$

$\therefore \angle BED = 42° + 42°$

$= 84°$ (by addition, $\angle BEC + \angle CED$)

$\therefore m° = 84°$ (alternate angles on parallel lines are equal, $BC \parallel DE$)

f) (i) In ΔTAP and ΔXCP

$TP = XP$ (given)

$CP = AP$ (given)

$\angle TPA = \angle CPX$ (vertically opposite angles are equal)

$\therefore \Delta TAP \equiv \Delta XCP$ (SAS - two sides and the included angle of one triangle ΔTAP are respectively equal to two sides and the included angle of the other triangle ΔXCP)

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(ii) $\angle TAP = \angle XCP$ (corresponding angles of congruent triangles, $\Delta TAP \equiv \Delta XCP$)

$\therefore TA \parallel XC$ (alternate angles are equal on parallel lines)

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e) ΔDEC is an isosceles triangle (an isosceles triangle has 2 sides equal, $DC = EC$)

$\therefore \angle CDE = \angle DEC = 42°$ (equal angles are opposite equal sides in a triangle)

$\therefore K = 180 - (2 \times 42) = 96°$ (angle sum of a triangle is 180°)

$\angle BCE = \angle DEC = 42°$ (alternate angles on parallel lines are equal, BC is parallel to DE)

ΔBCE is isosceles (an isosceles triangle has two sides equal, $BC = BE$)

$\therefore \angle BEC = \angle BCE = 42°$ (equal angles are opposite equal sides in a triangle)

$\angle BED = \angle BEC + \angle CED$

$= 42° + 42°$

$= 84°$ (by addition)

$\therefore$ ~~$m° = 84°$~~

$\angle ABE = \angle BED$

$= 84°$ (alternate angles on parallel lines are equal, $BC \parallel ED$)

$\therefore m = 84°$

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