St George Girls High School

Year 12

Mid-HSC Course Examination

2011

Mathematics

General Instructions

- Working time: 90 hours
- Write using blue or black pen.
- Board-approved calculators may be used.
- Begin each question in a new booklet
- All necessary working should be shown in every question.
- A table of standard integrals is provided.

Total marks – 72

- Attempt Questions 1 – 6
- All questions are of equal value

TABLE OF STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\quad x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Note $\ln x = \log_e x,\quad x > 0$

Question 1 – (12 marks) – (Start a new booklet) Marks

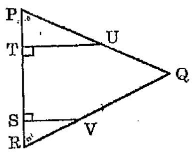
a) Given $y = \frac{5x-2}{3x+1}$, find y' and y'' 2

b) $f(x) = x^3 - 3x^2 - 9x + 5$. Find the values of x for which

(i) the function $y = f(x)$ is increasing 2

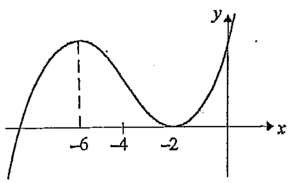
(ii) the curve $y = f(x)$ is concave up 2

c) In ΔPQR, UT and VS are both perpendicular to PR, $\angle PUT = \angle RVS$. Prove that ΔPQR is isosceles. 2



d) Find y as a function of x if $\frac{dy}{dx} = 9x^2 + 4$, and $y = 1$ when $x = 0$ 2

e) Below is the sketch of a curve $y = f(x)$. On the sheet provided, sketch $y = f'(x)$ 2



Question 2 – (12 marks) – (Start a new booklet) Marks

a) If $f(x) = ax^3 + 6x^2 + 4x + 1$ and $f''(1) = 0$ find the value of a 2

b) If $y = 3x^2 - x^3$ is defined for $-2 \le x \le 4$.

(i) Find the co-ordinates of any stationary points and determine the nature of these stationary points. 4

(ii) Find the coordinates of any points of inflexion. 2

(iii) Sketch the curve on the sheet provided, showing all essential features. 2

(iv) Find the absolute (global) maximum and minimum values of the function on the defined domain. 2

Question 3 – (12 marks) – (Start a new booklet) **Marks**

a) Evaluate without using Integration **2**

$$\int_{-4}^{4} \sqrt{16 - x^2}\ dx$$

Leave answer in exact form.

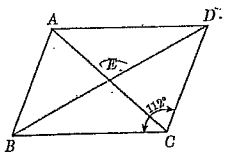
b) Find the primitive functions of the following

(i) $(2x + 3)^4$ **1**

(ii) $\sqrt{x}$ **1**

c) The gradient function of a curve is given by $f'(x) = x(2 - 3x)$. If $f(2) = 3$, find $f(x)$. **2**

d) In the figure, $ABCD$ is a rhombus whose diagonals intersect at E.



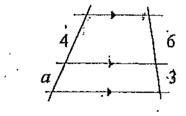
If $\angle BCD = 112°$, giving reasons, find the size of:

(i) $\angle AED$ **2**

(ii) $\angle ABD$ **4**

Question 4 – (12 marks) – (Start a new booklet) **Marks**

a) Find the value of the pronumeral, give a reason for your answer. **2**



b) Find

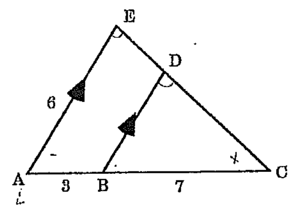
(i) $\int 2x + 11\ dx$ **1**

(ii) $\int \frac{dx}{\sqrt{9-2x}}$ **1**

c) Evaluate the following definite integral **3**

$$\int_{0}^{5} x(x+1)(x-1)\ dx$$

d) In the diagram, $AE \parallel BD$, $AB = 3$ cm, $BC = 7$ cm, $EA = 6$ cm



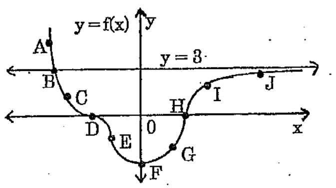
(i) Prove that the triangles ACE and BCD are similar. **3**

(ii) Find the length of BD **2**

Marks

Question 5 – (12 marks) – (Start a new booklet)

a) $y = f(x)$ is the function shown in the diagram.



Write down the points for which

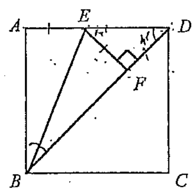
(i) $y > 0$ 1

(ii) $y > 0$ and $y'' > 0$ 1

(iii) $y' > 0$ and $y'' < 0$ 1

(iv) $y'' = 0$ 1

b) $ABCD$ is a square. EB bisects $\angle ABD$. EF is perpendicular to BD. Prove that



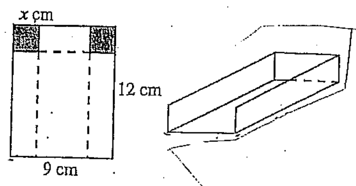
(i) ΔBFE and ΔBAE are congruent. 3

(ii) $FD = AE$ 3

c) Find the size of an interior angle of a regular octagon. 2

Question 6 – (12 marks) – (Start a new booklet) Marks

a)



A rectangle of cardboard measures 12 cm by 9 cm. From two corners, squares of side x cm are removed, as shown above.

The remainder is folded along the dotted lines to form a tray.

(i) Show that the volume, $V\text{cm}^3$, of the tray is given by 2

$$V = 2x^3 - 33x^2 + 108x$$

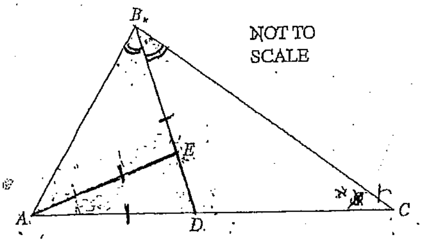
(ii) Find the maximum possible volume of the tray. 4

b) A continuous curve $y = f(x)$ has the following properties over the interval $a \le x \le b$ where a and b are positive

$$f(x) > 0,\ f'(x) > 0,\ f''(x) > 0$$

Sketch a curve satisfying these properties. 1

c)



In the diagram, ABC is a triangle where BD bisects $\angle ABC$ and $AD = AE = BD$.

(i) Prove $\angle DCB = \angle EAB$. Hint: Let $\angle ABD = x$ 3

(ii) Given ΔABE is similar to ΔCBD, prove $AE^2 = BE \times CD$ 2

End of Paper

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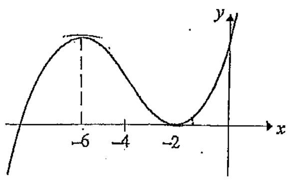
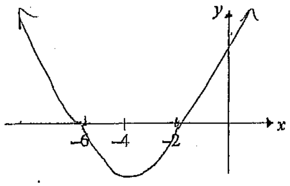
Question 1 e)

$y = f(x)$



$y = f'(x)$



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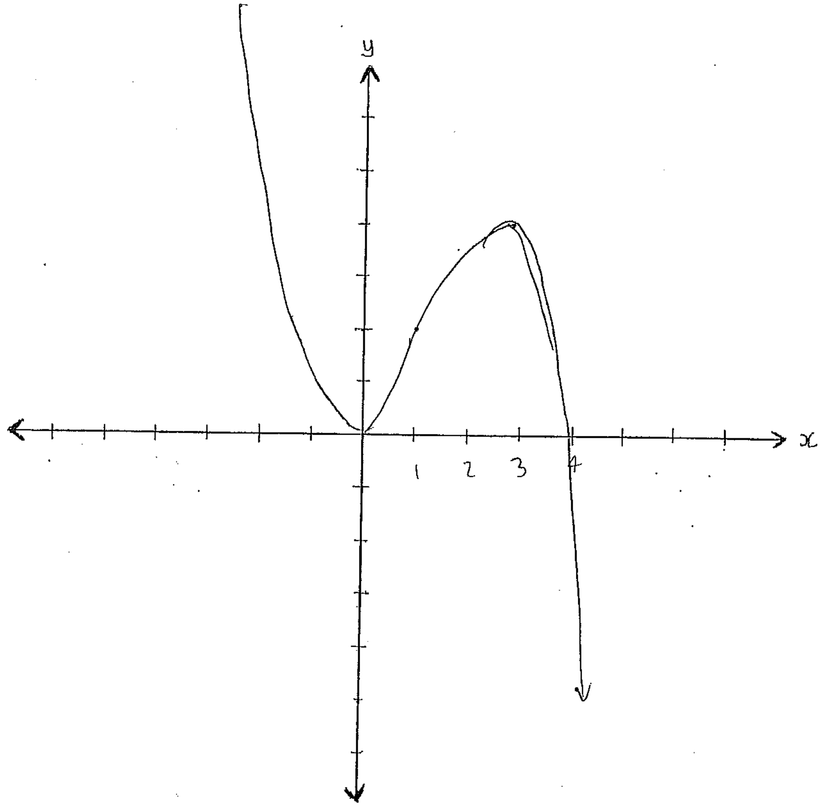
Question 2 b) (iii)



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Q1

a) $y = \dfrac{5x-2}{3x+1}$

$y' = \dfrac{5(3x+1) - 3(5x-2)}{(3x+1)^2}$

$= \dfrac{15x + 5 - 15x + 6}{(3x+1)^2}$

$= \dfrac{11}{(3x+1)^2}$

$= 11(3x+1)^{-2}$

$y'' = -22(3x+1)^{-3}(3)$

$= -\dfrac{66}{(3x+1)^3}$

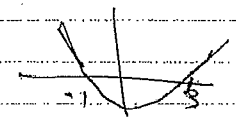
b) $f(x) = x^3 - 3x^2 - 9x + 5$

a) increasing $f'(x) > 0$

$f'(x) = 3x^2 - 6x - 9$

$= 3(x^2 - 2x - 3)$

$= 3(x-3)(x+1)$



$\therefore$ increasing $x < -1$ or $x > 3$

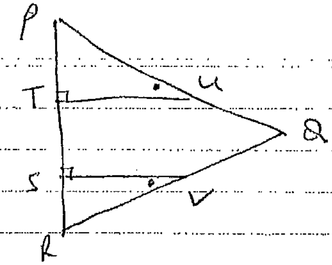
b) concave up $f''(x) > 0$

$f''(x) = 6x - 6$

$6x - 6 > 0$

$x > 1$

c)



$\angle PUT = \angle RVS$ (given)

$\angle PTU = \angle RSV$ (both 90°)

$\therefore \Delta PTU ||| \Delta RSV$ (equiangular)

$\therefore \angle P = \angle R$ (corresponding angles in similar triangles)

So ΔPQR is isosceles (base angles equal)

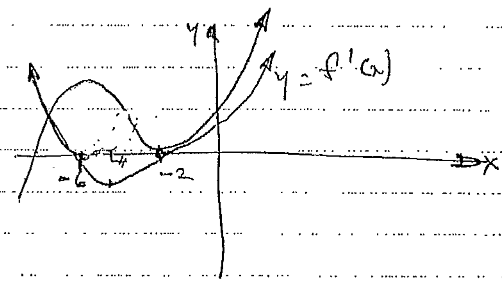
d) $\dfrac{dy}{dx} = 9x^2 + 4$

$\therefore y = 3x^3 + 4x + c$

when $x = 0$ $y = 1$ $\therefore c = 1$

$y = 3x^3 + 4x + 1$

e) ON SHEET



Q2

a) $f(x) = ax^3 + 6x^2 + 4x + 1$

$f'(x) = 3ax^2 + 12x + 4$

$f''(x) = 6ax + 12$

$f''(1) = 0$ $6a + 12 = 0$ $a = -2$

g) $y = 3x^2 - x^3 \qquad -2 \le x \le 4$

$y' = 6x - 3x^2 \qquad y' = 0 \qquad 6x - 3x^2 = 0$

$3x(2 - x) = 0$

$x = 0$ or 2

$y'' = 6 - 6x$

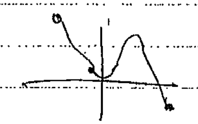
(i) at $x = 0 \quad y'' = 6 > 0 \quad \cup \quad \therefore$ min t.p $(0, 0)$

at $x = 2 \quad y'' = -6 < 0 \quad \cap \quad \therefore$ max t.p $(2, 4)$

(ii) also $y'' = 0$ at $x = 1$ pt of inflexion $(1, 2)$

(iii) at $x = -2 \qquad x = 4$

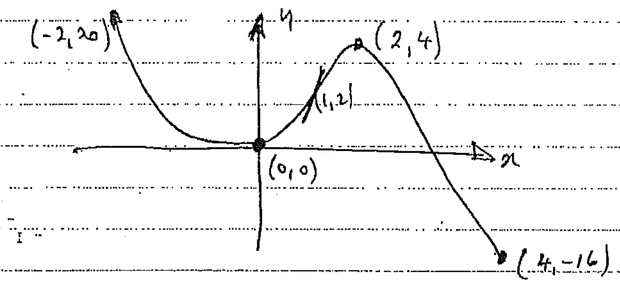
$y = 20 \qquad y = -16$



Global max $(-2, 20)$

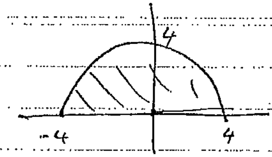
min $(4, -16)$

(iv)



Q3

a) $\int_{-4}^{4} \sqrt{16 - x^2}\, dx$



$= \frac{1}{2}\pi(4)^2$

$= 8\pi$

b) (i) $(2x+3)^4 \qquad$ Primitive $= \frac{(2x+3)^5}{5 \cdot 2} + c$

$= \frac{1}{10}(2x+3)^5 + c$

(ii) $\sqrt{x} = x^{1/2} \qquad$ Primitive $= \frac{2}{3} x^{3/2} + c$

$= \frac{2}{3}\sqrt{x^3} + c$

c) $f'(x) = x(2 - 3x)$

$= 2x - 3x^2$

$f(x) = x^2 - x^3 + c$

$f(2) = 3 \qquad 3 = (2)^2 - (2)^3 + c$

$3 = 4 - 8 + c$

$\therefore c = 7$

$f(x) = x^2 - x^3 + 7$

d) (i) $\angle AED = 90°$ (diagonals of rhombus perpendicular)

(ii) $\angle ABC = 68°$ (co-interior angles on parallel lines)

$\angle ABD = 34°$ (diagonals of rhombus bisect angles of rhombus.

Q4

a) $\frac{a}{4} = \frac{3}{6}$ (ratio of intercepts on parallel lines)

$a = 2$

b) i) $\int (2x+11)\,dx = x^2 + 11x + c$

ii) $\int \frac{dx}{\sqrt{9-2x}} = \int (9-2x)^{-1/2}\,dx$

$= \frac{(9-2x)^{\frac{1}{2}}}{\frac{1}{2} \cdot (-2)} + c$

$= -\sqrt{9-2x} + c$

c) $\int_0^5 x(x+1)(x-1)\,dx$

$= \int_0^5 x(x^2-1)\,dx$

$= \int_0^5 (x^3 - x)\,dx$

$= \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_0^5$

$= \left[\frac{1}{4}(5)^4 - \frac{1}{2}(5)^2\right] - \left[0 - 0\right]$

$= \frac{625}{4} - \frac{25}{2}$

$= \frac{625 - 50}{4}$

$= \frac{575}{4}$

d) (i) $\angle A = \angle B$ (corresponding angles on parallel lines)

$\angle C$ is common

$\therefore \triangle AEC \parallel\!| \triangle BDC$ (equiangular)

$\frac{BD}{6} = \frac{7}{10}$

$10\,BD = 42$

$BD = 4\cdot2$

Question 5

a) i) A, B, C, I, J

ii) A, B, C

iii) I, J

iv) E, D, H.

b) i) $\angle ABE = \angle FBE$ (EB bisects $\angle ABF$)

$\angle BAE = \angle BFE$ (both $90°$)

EB is common

$\therefore \triangle BAE \equiv \triangle BFE$ (A.A.S)

ii) $AE = EF$ (corresponding angles in congruent triangles)

$\angle BDE = 45°$ (diagonals on a square bisect angles of square)

$\therefore \angle FED = 45°$ (angle sum of $\triangle EFD$)

So $\triangle FED$ is isosceles

$EF = FD$ (sides of isosceles triangle)

$\therefore AE = FD$

c) Exterior Angle $= \frac{360}{8} = 45^\circ$ or $\frac{(8-2)\times 180^\circ}{8} = 135^\circ$

$\therefore$ Interior Angle $= 135^\circ$

Q6

a) i) $V = Ah$ $0 < x < \frac{9}{2}$

$= (9-2x)(12-x)\cdot x$

$= (108 - 33x + 2x^2)\cdot x$

$= 2x^3 - 33x^2 + 108x$

ii) $\frac{dV}{dx} = 6x^2 - 66x + 108$

$= 6(x^2 - 11x + 18)$

$= 6(x-9)(x-2)$

$\frac{dV}{dx} = 0$ at $x = 9$ or 2

as $0 < x < 9/2$ $\therefore x = 2$.

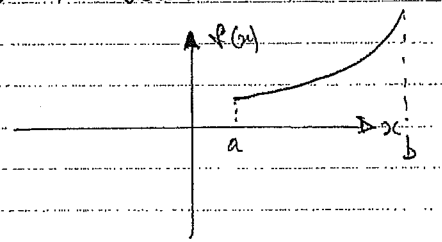
$\frac{d^2V}{dx^2} = 12x - 66$

at $x = 2$ $\frac{d^2V}{dx^2} = -42$

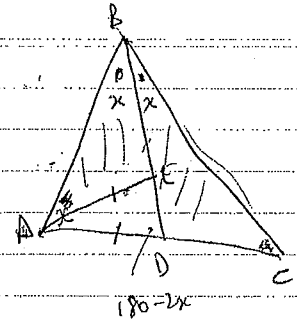
< 0 $\therefore$ max t.p $x = 2$

Max. volume $100\ cm^3$

b)



c)



(i) $\angle AED = \angle ADE$
(base angles of isosceles ΔAED)

$\angle ADE = \angle DCB + \angle DBC$
(exterior angle of ΔDBC)

$\angle AED = \angle EAB + \angle EBA$
(exterior angle of ΔEBA)

as $\angle EBA = \angle DBC$ (BD bisects $\angle ABC$) ~~given~~

$\therefore \angle DCB = \angle EAB$

(ii) $\frac{AE}{BE} = \frac{CD}{BD}$ (corresponding sides in similar triangles)

as $AE = BD$ (given)

$AE^2 = BE \times CD$.