



# Mathematics Extension 1

### General Instructions

- Time allowed – 75 minutes
- Write using blue or black pen
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.
- Write on one side of the page only.
- Start each question on a new page.

Total marks – 66

- Attempt Questions 1 – 6
- All questions are of equal value

Question	Mark
Question 1	/11
Question 2	/11
Question 3	/11
Question 4	/11
Question 5	/11
Question 6	/11
<b>Total</b>	<b>/66</b>

### Question 1 (11 marks)

Marks

- a) Evaluate  $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$  (2)
- b) The polynomial  $P(x) = 2x^3 - 5x^2 - 3x + 1$  has zeroes  $\alpha, \beta, \gamma$ . Evaluate:
- $\alpha + \beta + \gamma, \alpha\beta + \alpha\gamma + \beta\gamma, \alpha\beta\gamma$  (3)
  - $3\alpha + 3\beta + 3\gamma - 4\alpha\beta\gamma$  (2)
  - $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$  (1)
- c) Find the equation of the tangent to the curve  $y = \tan^{-1}(ax + b)$  at the point where the curve crosses the  $x$ -axis. (3)

### Question 2 (11 marks)

- a) Differentiate  $x \sin^{-1} 2x$  (2)
- b) The cubic polynomial equation  $x^3 = ax^2 + bx + c$  has three real roots, two of which are opposites. Prove that:
- one of the roots is  $a$  (1)
  - the other roots are  $\sqrt{b}, -\sqrt{b}$  (2)
  - $ab + c = 0$  (2)
- c) Sketch  $y = 3 \cos^{-1} \frac{x}{2}$ . Give the domain and range. (4)

**Question 3 (11 marks)**

**Marks**

- a) i. Express  $\sin x - \sqrt{3} \cos x$  in the form  $A \sin(x - \alpha)$  with  $A > 0$  and  $0 < \alpha < \frac{\pi}{2}$ . (4)
- ii. Find the general solution to  $\sin x - \sqrt{3} \cos x = \frac{2}{\sqrt{2}}$  (2)
- b) Solve the equation  $5 \cos^2 x + \sin^2 x = 3 \sin 2x$  for  $0^\circ \leq x \leq 360^\circ$  (3)
- c) Explain why the graph of a cubic polynomial with three distinct zeroes must have two turning points. (2)

**Question 4 (11 marks)**

- a) Find the value of  $\sin^{-1}(\cos \frac{2\pi}{3})$  (2)
- b) i. Show that the equation  $3 \sin x - 2 \cos x = 2$  can be written as  $3t - 2 = 0$  where  $t = \tan \frac{x}{2}$ . (3)
- ii. Hence solve the equation for  $0^\circ \leq x \leq 360^\circ$  giving solutions correct to the nearest minute. (3)
- c) The polynomial  $y = 2x^3 - x^2 + ax + b$  has a remainder of 16 when divided by  $x - 1$  and a remainder of -17 when divided by  $x + 2$ . Find  $a$  and  $b$ . (3)

**Question 5 (11 marks)**

**Marks**

- a) Show that if  $x - 2$  is a factor of  $x^3 + ax^2 + bx + c$  it is also a factor of  $ax^3 + bx^2 + cx + 16$  (4)
- b) i. Factorise the polynomial  $P(x) = x^3 - 6x^2 - 9x + 14$  completely. (4)
- ii. Without using calculus, sketch the graph of  $P(x)$ , showing all intercepts with the axes. (3)
- c) Evaluate  $\sin[(2 \tan^{-1}(1))]$  (2)

**Question 6 (11 marks)**

- a) If  $y = \tan^{-1} x$
- i. Prove that  $\sec^2 y = 1 + x^2$  (3)
- ii. Hence or otherwise prove that  $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$  (2)
- b) i. If  $\theta = \tan^{-1} A + \tan^{-1} B$  show that  $\tan \theta = \frac{A + B}{1 - AB}$  (3)
- ii. Hence solve  $\tan^{-1} 3x + \tan^{-1} 2x = \frac{\pi}{4}$  (3)

QUESTION 1

$$\begin{aligned} a) \int_0^3 \frac{dx}{\sqrt{3^2-x^2}} &= \left[ \sin^{-1} \frac{x}{3} \right]_0^3 \\ &= \sin^{-1} 1 - \sin^{-1} 0 \\ &= \frac{\pi}{2} - 0 \\ &= \frac{\pi}{2} \end{aligned}$$

b) (i)  $a = 2, b = -5, c = -3, d = 1$

$$\begin{aligned} \alpha + \beta + \gamma &= \frac{-b}{a} & \alpha\beta + \alpha\gamma + \beta\gamma &= \frac{c}{a} & \alpha\beta\gamma &= \frac{-d}{a} \\ &= \frac{5}{2} & &= \frac{-3}{2} & &= \frac{-1}{2} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad 3(\alpha + \beta + \gamma) - 4\alpha\beta\gamma & \\ &= 3 \times \frac{5}{2} - 4 \times \left(\frac{-1}{2}\right) \\ &= \frac{15}{2} + 2 \\ &= \frac{19}{2} \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} & \\ &= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} \\ &= \frac{-\frac{3}{2} \div \frac{-1}{2}}{1} \\ &= 3 \end{aligned}$$

c) the curve crosses the x axis when  $y = 0$

$$\begin{aligned} \tan^{-1}(ax+b) &= 0 \\ (ax+b) &= 0 \\ ax &= -b \\ x &= \frac{-b}{a} \end{aligned}$$

$$y' = \frac{a}{1+(ax+b)^2}$$

when  $x = \frac{-b}{a}$

$$\begin{aligned} y' &= \frac{a}{1+(a(\frac{-b}{a})+b)^2} \\ &= a \end{aligned}$$

Equation of the tangent

$$\begin{aligned} y - 0 &= a(x + \frac{b}{a}) \\ y &= ax + b \end{aligned}$$

Question 2

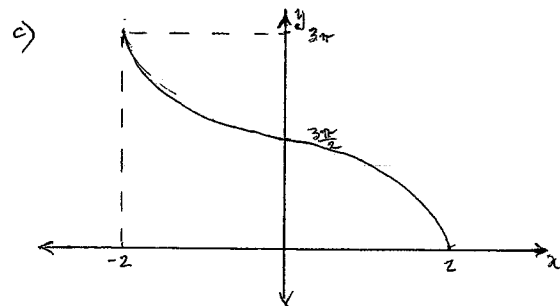
$$\begin{aligned} a) \frac{d}{dx}(x \sin^{-1} 2x) &= \sin^{-1} 2x + \frac{2}{\sqrt{1-(2x)^2}} \\ &= \sin^{-1} 2x + \frac{2}{\sqrt{1-4x^2}} \end{aligned}$$

b) Let the roots be  $\alpha, \beta, -\beta$

$$\begin{aligned} \text{(i)} \quad \alpha + \beta + (-\beta) &= \frac{a}{1} \\ \alpha &= a \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \alpha\beta + \alpha(-\beta) - \beta^2 &= -b \\ \alpha\beta - \alpha\beta - \beta^2 &= -b \\ -\beta^2 &= -b \\ \beta^2 &= b \\ \beta &= \pm\sqrt{b} \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad \alpha\beta(-\beta) &= c \\ -\alpha\beta^2 &= c & \text{since } \alpha = a, \beta^2 = b \\ -ab &= c \\ ab + c &= 0 \end{aligned}$$



Domain  $-2 \leq x \leq 2$   
Range  $0 \leq y \leq 3\pi$

### Question 3

a) (i)  $\sin x - \sqrt{3} \cos x = A \sin(x - \alpha)$

$A = \sqrt{a^2 + b^2}$ , where  $a = 1$ ,  $b = \sqrt{3}$

$= \sqrt{1+3}$

$= \sqrt{4} = 2$

$\tan \alpha = \frac{\sqrt{3}}{1}$

$\alpha = \frac{\pi}{3}$

$\sin x - \sqrt{3} \cos x = 2 \sin(x - \frac{\pi}{3})$

(ii)  $2 \sin(x - \frac{\pi}{3}) = \frac{2}{\sqrt{2}}$

$\sin(x - \frac{\pi}{3}) = \frac{1}{\sqrt{2}}$

$x - \frac{\pi}{3} = \frac{\pi}{4} + 2n\pi, \frac{3\pi}{4} + 2n\pi$

where  $n$  is an integer

$x = \frac{7\pi}{12} + 2n\pi, \frac{13\pi}{12} + 2n\pi$

$n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{3} \Rightarrow \frac{\pi}{3}(3n+1) \pm \dots$

b)  $5 \cos^2 x + \sin^2 x = 3 \times 2 \sin x \cos x$   
 $= 6 \sin x \cos x$

$5 \cos^2 x - 6 \sin x \cos x + \sin^2 x = 0$   $0^\circ \leq x \leq 360^\circ$

$(5 \cos x - \sin x)(\cos x - \sin x) = 0$

either  $5 \cos x - \sin x = 0$

$5 \cos x = \sin x$

$5 = \tan x$

$x = 78^\circ 41', 258^\circ 41'$

or  $\cos x - \sin x = 0$

$\cos x = \sin x$

$1 = \tan x$

$x = 45^\circ, 225^\circ$

c) If a cubic polynomial has 3 distinct zeroes then it must cross the  $x$  axis three times (each zero cannot have a multiplicity greater than 1. For a curve to cross the  $x$  axis 3 times it must change direction twice which means it must have two turning points.

### Question 4

a) Let  $x = \cos \frac{2\pi}{3}$

$= -\frac{1}{2}$

$\sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}$

b) (i)  $\sin x = \frac{2t}{1+t^2}$ ,  $\cos x = \frac{1-t^2}{1+t^2}$ , where  $t = \tan \frac{x}{2}$

$3(\frac{2t}{1+t^2}) - 2(\frac{1-t^2}{1+t^2}) = 2$

$\frac{6t - 2 + 2t^2}{1+t^2} = 2$

$\frac{6t - 2 + 2t^2 - 2(1+t^2)}{1+t^2} = 0$

$\frac{6t - 2 + 2t^2 - 2 - 2t^2}{1+t^2} = 0$

$\frac{6t - 4}{1+t^2} = 0$

$\therefore 6t - 4 = 0$

$3t - 2 = 0$

(ii)  $t = \frac{2}{3}$

$\tan \frac{x}{2} = \frac{2}{3}$

$0^\circ \leq x \leq 360^\circ$

$0^\circ \leq \frac{x}{2} \leq 180^\circ$

$\frac{x}{2} = 33^\circ 41'$

$x = 67^\circ 22'$

NOTE: CHECK  $x = 180^\circ!$

c)  $P(x) = 2x^3 - x^2 + ax + b$

$P(1) = 16$

$2 - 1 + a + b = 16$

$a + b = 15$  — ①

① - ②

$3a = 12$

$a = 4$

$b = 11$

$P(2) = -17$

$-16 - 4 - 2a + b = -17$

$-2a + b = 3$  — ②

### Question 5

a) Let  $P(x) = x^3 + ax^2 + bx + c$   
 if  $x-2$  is a factor of  $P(x)$  then  $P(2) = 0$

$$P(2) = 8 + 4a + 2b + c$$

$$8 + 4a + 2b + c = 0$$

$$4a + 2b + c = -8$$

$$\text{Let } Q(x) = ax^3 + bx^2 + cx + 16$$

if  $x-2$  is a factor of  $Q(x)$  then  $Q(2) = 0$

$$Q(2) = 8a + 4b + 2c + 16$$

$$= 2(4a + 2b + c) + 16$$

$$= 2 \times (-8) + 16$$

$$= 0$$

$\therefore x-2$  is a factor of  $ax^3 + bx^2 + cx + 16$  if  
 it is a factor of  $x^3 + ax^2 + bx + c$

b) (i)  $P(x) = x^3 - 6x^2 - 9x + 14$

$$P(1) = 1 - 6 - 9 + 14$$

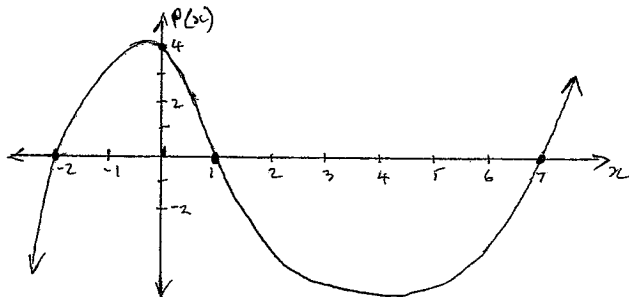
$$= 0$$

$\therefore x-1$  is a factor of  $P(x)$

$$P(x) = (x-1)(x^2 - 5x - 14)$$

$$= (x-1)(x-7)(x+2)$$

(ii)



$$\begin{array}{r} x^2 - 5x - 14 \\ x-1 \overline{) x^3 - 6x^2 - 9x + 14} \\ \underline{x^3 - x^2} \phantom{+ 14} \\ -5x^2 - 9x \phantom{+ 14} \\ \underline{-5x^2 + 5x} \phantom{+ 14} \\ -14x + 14 \\ \underline{-14x + 14} \\ 0 \end{array}$$

c)  $\tan^{-1}(1) = \frac{\pi}{4}$

$$\sin \frac{2\pi}{4}$$

$$= \sin \frac{\pi}{2}$$

$$= 1$$

### Question 6

a) (i)  $\tan y = x$

$$\tan^2 y = x^2$$

$$\tan^2 y + 1 = 1 + x^2$$

$$\sec^2 y = 1 + x^2$$

(ii) Let  $y = \tan^{-1} x$

$$x = \tan y$$

$$\frac{dx}{dy} = \sec^2 y$$

$$= 1 + x^2$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

b) (i)  $\tan \theta = \tan(\tan^{-1} A + \tan^{-1} B)$

$$= \frac{\tan(\tan^{-1} A) + \tan(\tan^{-1} B)}{1 - \tan(\tan^{-1} A) \tan(\tan^{-1} B)}$$

$$= \frac{A + B}{1 - AB}$$

(ii)  $\tan(\tan^{-1} 3x + \tan^{-1} 2x) = \tan \frac{\pi}{4}$

$$\frac{3x + 2x}{1 - 3x \times 2x} = 1$$

$$\frac{5x}{1 - 6x^2} = 1$$

$$5x = 1 - 6x^2$$

$$6x^2 + 5x - 1 = 0$$

$$(6x-1)(x+1) = 0$$

$$x = \frac{1}{6} \text{ or } x = -1$$

NOTE: CHECK both ...