St George Girls High School

Year 12

Common Test 3

May 2012

Mathematics Extension 1

General Instructions

- Working time – 70 minutes
- Reading time – 5 minutes
- Write using blue or black pen
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Total marks – 66

- Attempt Questions 1 – 6
- All questions are of equal value

Question	Mark
Question 1	/11
Question 2	/11
Question 3	/11
Question 4	/11
Question 5	/11
Question 6	/11
Total	/66

Students are advised that this is a School Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

TABLE OF STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n\neq -1;\quad x\neq 0,\ \text{if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\quad a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Note $\ln x = \log_e x,\quad x>0$

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Question 1 – Start a New Booklet – (11 marks) **Marks**

a) Which of the following is <u>not</u> a polynomial? 1

A. $x^2 - 4x + 6$

B. 8

C. $4x + \frac{1}{x} + 3$

D. $\sqrt{2}\,x^3 + \pi\, x$

b) If α and β are the roots of $px^2 + qx + r = 0$ then $\alpha + \beta =$ 1

A. $-\frac{q}{p}$

B. pqr

C. $-\frac{r}{p}$

D. can not be determined from information provided

c) Show $(x + 2)$ is a factor of $P(x) = x^3 + 3x^2 + 7x + 10$ and hence find all the factors of $P(x)$. 3

d) If α, β and γ are the roots of $2x^3 - 6x^2 + x + 2 = 0$ find the values of:

(i) $\alpha + \beta + \gamma$ 1

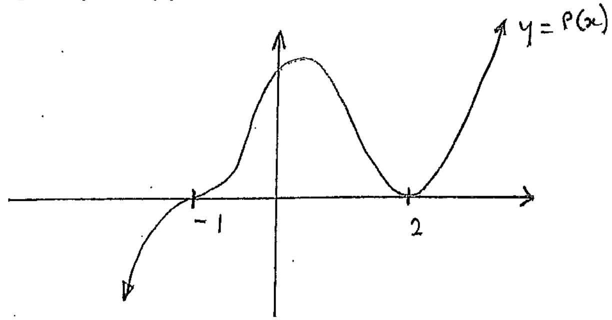
(ii) $\alpha\beta + \beta\gamma + \gamma\alpha$ 1

(iii) $(\alpha - 1)(\beta - 1)(\gamma - 1)$ 2

(iv) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ 2

Question 2 – Start a New Booklet – (11 marks) **Marks**

a) The graph of $y = P(x)$ is shown below. 1



Which of the following is possible?

A. $y = (1 + x)^4\ (x - 2)^3$

B. $y = (1 + x)^3\ (x - 2)^4$

C. $y = (1 - x)^4\ (x + 2)^3$

D. $y = (1 - x)^3\ (x + 2)^4$

b) Two of the roots of $x^3 + qx + r = 0$ are 5 and 1, the other root is: 1

A. $-r$

B. $-\frac{r}{q}$

C. 5

D. -6

c) Solve for $x:$ $2x^3 + 3x^2 - 23x - 12 > 0$ 5

d) Given $x^3 - 5x^2 + 7x + k = 0$ has a double integral root, find the value of k 4

Question 3 – Start a New Booklet – (11 marks) Marks

a) The graph of $y = (x-1)^3(x+3)(2-x)^2$ cuts the y axis at the point where $y =$ 1

A. 6

B. -6

C. 12

D. -12

b) For a given polynomial $P(x)$, $P(-1) = 6$ and $P(2) = 3$. When $P(x)$ is divided by $(x+1)(x-2)$ the remainder is: 1

A. $5 - 2x$

B. $x + 1$

C. $-x + 5$

D. $3x + 2$

c) If $P(x) = 4x^3 - 6x^2 + 3x + 1$ and $Q(x) = 3x^2 - 4$ find: 3

(i) degree $[P(x) + Q(x)]$

(ii) $P(x) - Q(x)$

(iii) $P(x) \,.\, Q(x)$

d) If $4x^2 - 3x + 1 \equiv a(x-1)^2 + b(x-1) + c$ find the values of a, b and c 3

e) Two of the roots of the equation $2x^3 + 3x^2 + \cdots = 0$ are $x = -3$ and $x = 4$ 3

(i) Find the third root.

(ii) Write the complete equation.

Question 4 – Start a New Booklet – (11 marks) Marks

a) If $\sin\theta = -\frac{1}{2}$ and $\tan\theta < 0$ then $\cos\theta =$ 1

A. $\frac{\sqrt{3}}{2}$

B. $-\frac{\sqrt{3}}{2}$

C. $\frac{\sqrt{5}}{2}$

D. $-\frac{\sqrt{5}}{2}$

b) The bearing of B from A is $040°T$, C is due north of A and $\angle ABC$ is $80°$. Then the bearing of C from B is: 1

A. $060°T$

B. $120°T$

C. $300°T$

D $320°T$

c) Express $\sin x + \cos x$ in the form $R\sin(x+\alpha)$ and hence solve $\sin x + \cos x = 1$ for $0 \le x \le 2\pi$ 4

d) Solve for $0 \le \theta \le 2\pi$ $\cos 2\theta = 4\cos^2\theta - 2\sin^2\theta$ 3

e) Find $\int_0^{\frac{\pi}{4}} \cos^2 x \; dx$ 2

Question 5 – Start a New Booklet – (11 marks) Marks

a) If $t = \tan\frac{1}{2}\theta$ then $\sin\theta =$ 1

A. $\frac{1-t^2}{1+t^2}$

B. $\frac{1+t^2}{1-t^2}$

C. $\frac{2t}{1-t^2}$

D. $\frac{2t}{1+t^2}$

b) Given $\sin\left(\frac{1}{2}x + \frac{\pi}{2}\right) = 1$ and $-\pi \leq x \leq \pi$ then $x =$ 1

A. $\frac{\pi}{2}$

B. $-\frac{\pi}{2}$

C. 0

D. π

c) Solve for $0 \leq \theta \leq 2\pi$

(i) $\text{cosec}\ \theta = -2$ 2

(ii) $\tan\left(2\theta - \frac{\pi}{4}\right) + 1 = 0$ 4

d) Simplify 3

(i) $\sin 6x\ \cos 6x$

(ii) $\frac{1-\cos 2\theta}{1+\cos 2\theta}$

Question 6 – Start a New Booklet – (11 marks) Marks

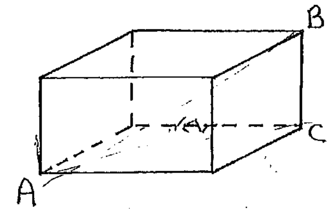
a) If $0 \leq \theta \leq 2\pi$ then the number of solutions to $\tan 2\theta = \sqrt{3}$ is 1

A. 1

B. 2

C. 3

D. 4

b) 

A rectangular prism is shown with base dimension 6 m by 4 m, and height 3 m. 1

The size of $\angle BAC$ to the nearest degree is:

A. 20°

B. 23°

C. 26°

D. 28°

c) If $\cos 2\theta = \cos\theta$ then one possible set of values for θ, for some integer n, is: 1

A. $2n\pi + \frac{2\pi}{3}$

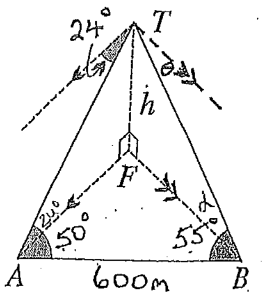
B. $2n\pi + \frac{\pi}{3}$

C. $n\pi + \frac{2\pi}{3}$

D. $n\pi + \frac{\pi}{3}$

Question 6 (cont'd) **Marks**

d)



The points A and B are 600 m apart in a horizontal plane ABF. From the top of a vertical tower TF the angle of depression to point A is 24°.

Also $\angle TAB = 50°$ and $\angle TBA = 55°$

(i) Show $AT = \dfrac{600 \sin 55°}{\sin 75°}$ 1

(ii) Hence find the height (TF) of the tower correct to the nearest metre. 2

(iii) Find correct, to the nearest degree, the angle of depression of B from T (ie. θ) 3

e) Solve for $0 \leq \theta \leq 2\pi$ 2

$$\sin 2\theta - \cos\theta = 0$$

End of Paper

YEAR 12 MATHEMATICS EXT 1 COMMON TEST 3 MAY 2012

SOLUTIONS

Question 1

a) C b) A

c) $P(-2) = (-2)^3 + 3(-2)^2 + 7(-2) + 10$
$= -8 + 12 - 14 + 10$
$= 0$

$\therefore (x+2)$ is a factor of $P(x)$

$$\begin{array}{r} x^2 + x + 5 \\ x+2 \,\overline{)\, x^3 + 3x^2 + 7x + 10} \\ \underline{x^3 + 2x^2} \\ x^2 + 7x \\ \underline{x^2 + 2x} \\ 5x + 10 \\ \underline{5x + 10} \\ 0 \end{array}$$

$\therefore P(x) = (x+2)(x^2+x+5)$

d) $a=2$, $b=-6$, $c=1$, $d=2$

(i) $\alpha+\beta+\gamma = -\frac{b}{a}$
$= 3$

(ii) $\alpha\beta+\beta\gamma+\alpha\gamma = \frac{c}{a}$
$= \frac{1}{2}$

(iii) $(\alpha-1)(\beta-1)(\gamma-1)$
$= \alpha\beta\gamma - (\alpha\beta+\alpha\gamma+\beta\gamma) + (\alpha+\beta+\gamma) - 1$
$= -1 - \frac{1}{2} + 3 - 1$
$= \frac{1}{2}$

$\alpha\beta\gamma = \frac{-d}{a}$
$= -1$

(iv) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$
$= \frac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma}$
$= \frac{1}{2} \div -1$
$= -\frac{1}{2}$

Question 2

a) B b) D

c) $P(3) = 54 + 27 - 69 - 12$
$= 0$

$\therefore x-3$ is a factor of $P(x) = 2x^3+3x^2-23x-12$

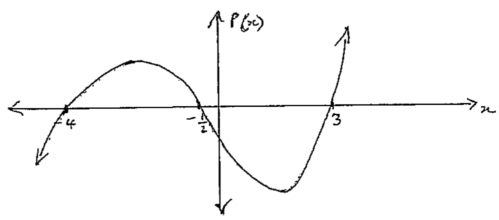
$P(-4) = -128 + 48 + 92 - 12$
$= 0$

$\therefore x+4$ is a factor of $P(x)$

$(x+4)(x-3)$
$= x^2+x-12$

$$\begin{array}{r} 2x + 1 \\ x^2+x-12 \,\overline{)\, 2x^3+3x^2-23x-12} \\ \underline{2x^3+2x^2-24x} \\ x^2 + x - 12 \\ \underline{x^2 + x - 12} \\ 0 \end{array}$$

$P(x) = (x-3)(x+4)(2x+1)$



$\therefore P(x) > 0$ when $-4 < x < -\frac{1}{2}$ or $x > 3$

d) $a=1$, $b=-5$, $c=7$, $d=k$

Let the roots be α, α, β where α is an integer

$\alpha+\alpha+\beta = -\frac{b}{a}$
$2\alpha+\beta = 5 \Rightarrow \beta = 5-2\alpha$ ——— ①

$\alpha^2+\alpha\beta+\alpha\beta = \frac{c}{a}$
$\alpha^2+2\alpha\beta = 7$ ——— ②

$\alpha^2\beta = \frac{d}{a}$
$\alpha^2\beta = -k$ ——— ③

sub ① into ②
$\alpha^2 + 2\alpha(5-2\alpha) = 7$
$-3\alpha^2+10\alpha-7 = 0$
$(\alpha-1)(3\alpha-7) = 0$
$\alpha = 1$ or $\frac{7}{3}$

For $\alpha = 1$ $\beta = 3$
from ③
$-k = 1\times 3$
$k = -3$

Question 3

a) D b) C

c) (i) 3 (ii) $4x^3 - 9x^2 + 3x + 5$

(iii) $P(x).Q(x) = 12x^5 - 16x^3 - 18x^4 + 24x^2 + 9x^3 - 12x + 3x^2 - 4$
$= 12x^5 - 18x^4 - 7x^3 + 27x^2 - 12x - 4$

d) Let $x=1$
$4 - 3 + 1 \equiv a\times 0 + b\times 0 + c$
$c = 2$

Let $x = 0$
$4\times 0 - 3\times 0 + 1 = a - b + c$
$1 = a - b + 2$
$a - b = -1$ ——— ①

Let $x = 2$
$4\times 4 - 3\times 2 + 1 = a + b + c$
$11 = a + b + 2$
$a + b = 9$ ——— ②

① + ②
$2a = 8$
$a = 4$
sub $a=4$ into ②
$b = 5$

$\therefore a = 4$, $b = 5$, $c = 2$

e) $a=2$, $b=3$, Let α be the third root

(i) $\alpha+\beta+\gamma = -\frac{b}{a}$
$\alpha - 3 + 4 = -\frac{3}{2}$
$\alpha = -\frac{5}{2}$

(ii) $(x+3)(x-4)(2x+5) = 0$
$2x^3 - 2x^2 + 5x^2 - 24x - 5x - 60 = 0$
$2x^3 + 3x^2 - 29x - 60 = 0$

Question 4

a) A b) C

c) $R = \sqrt{1^2+1^2}$
$= \sqrt{2}$

$R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$
$= R\cos\alpha\sin x + R\sin\alpha\cos x$

$R\cos\alpha = 1$ $R\sin\alpha = 1$
$\cos\alpha = \frac{1}{\sqrt{2}}$ $\sin\alpha = \frac{1}{\sqrt{2}}$

$\therefore \alpha = \frac{\pi}{4}$

$\sin x + \cos x = 1$ $0 \le x \le 2\pi$
$\sqrt{2}\sin(x+\frac{\pi}{4}) = 1$ $\frac{\pi}{4} \le x+\frac{\pi}{4} \le \frac{9\pi}{4}$
$\sin(x+\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$x+\frac{\pi}{4} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}$
$x = 0, \frac{\pi}{2}, 2\pi$

d) $\cos 2\theta = 4\cos^2\theta - 2\sin^2\theta$
$2\cos^2\theta - 1 = 4\cos^2\theta - 2 + 2\cos^2\theta$
$1 = 4\cos^2\theta$
$\cos^2\theta = \frac{1}{4}$
$\cos\theta = \pm\frac{1}{2}$
$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

e) $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

$\frac{d}{dx}(\sin 2x) = 2\cos 2x$
$\int \cos 2x\,dx = \frac{1}{2}\sin 2x +$

$\int_0^{\frac{\pi}{4}} \cos^2 x\,dx$
$= \int_0^{\frac{\pi}{4}} \frac{1}{2}(\cos 2x + 1)\,dx$
$= \frac{1}{2}\left[\frac{1}{2}\sin 2x + x\right]_0^{\frac{\pi}{4}}$
$= \frac{1}{2}\left(\frac{1}{2}\sin\frac{\pi}{2} + \frac{\pi}{4} - \frac{1}{2}\sin 0 - 0\right)$
$= \frac{1}{2}\left(\frac{1}{2} + \frac{\pi}{4}\right)$
$= \frac{1}{4} + \frac{\pi}{8}$

Question 5

a) D   b) C

c)(i) $\sin\theta = -\frac{1}{2}$

$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

(ii) $\tan\left(2\theta - \frac{\pi}{4}\right) = -1$ $0 \leq \theta \leq 2\pi$

$-\frac{\pi}{4} \leq 2\theta - \frac{\pi}{4} \leq \frac{15\pi}{4}$

$2\theta - \frac{\pi}{4} = -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$

$2\theta = 0, \pi, 2\pi, 3\pi, 4\pi$

$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

d) (i) $\sin 2\theta = 2\sin\theta\cos\theta$

$2\sin 6x\cos 6x = \sin 12x$

$\therefore \sin 6x\cos 6x = \frac{1}{2}\sin 12x$

(ii) $\frac{1-(2\cos^2\theta - 1)}{1+(2\cos^2\theta - 1)}$

$= \frac{1 - 2\cos^2\theta + 1}{2\cos^2\theta}$

$= \frac{2 - 2\cos^2\theta}{2\cos^2\theta}$

$= \frac{1-\cos^2\theta}{\cos^2\theta}$

$= \frac{\sin^2\theta}{\cos^2\theta}$

$= \tan^2\theta$

Question 6

a) D   b) B   c) A

d) (i) In $\triangle ABT$ $\angle ATB = 75°$ (angle sum of a $\triangle$)

$\frac{AT}{\sin 55°} = \frac{600}{\sin 75°}$

$AT = \frac{600\sin 55°}{\sin 75°}$

(ii) In $\triangle AFT$ $\angle TAF = 24°$ (alternate angles on parallel lines)

$\sin 24° = \frac{FT}{AT}$

$TF = AT\sin 24°$

$= \frac{600\sin 55°\sin 24°}{\sin 75°}$

$= 206.959$

$\doteqdot 207$ m to the nearest metre

(iii) In $\triangle ABT$

$\frac{BT}{\sin 50°} = \frac{600}{\sin 75°}$

$BT = \frac{600\sin 50°}{\sin 75°}$

In $\triangle BFT$ $\angle FBT = \theta$ (alternate angles on parallel lines)

$\sin\theta = \frac{TF}{BT}$

$= \frac{600\sin 55°\sin 24°}{\sin 75°} \times \frac{\sin 75°}{600\sin 50°}$

$= \frac{\sin 55°\sin 24°}{\sin 50°}$

$= 25.78°$

$\doteqdot 26°$ to the nearest degree

e) $2\sin\theta\cos\theta - \cos\theta = 0$

$\cos\theta(2\sin\theta - 1) = 0$

$\cos\theta = 0$ $2\sin\theta = 1$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$