St George Girls High School

Year 12

Common Test 3

May 2012

Mathematics Extension 2

General Instructions

- Working time – 75 minutes
- Write using blue or black pen.
- Board-approved calculators may be used.
- All necessary working must be shown in every question.
- Write on one side of the page only.
- Start each question in a new booklet.

Total marks – 48

- Attempt Questions 1, 2 and 3
- Questions are not all of equal value.

Students are advised that this is a School Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Question 1 – (16 marks) – **Start a New Booklet** Marks

Part a) is a multiple choice question.

Write down the letter matching the correct answer in your booklet.

a) The equation $24x^3 - 12x^2 - 6x + 1$ has roots α, β, γ. 1

What is the value of α if $\alpha = \beta + \gamma$

(A) $-\frac{1}{2}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) 1

b) With respect to coordinate axis the line $x = 1$ is a directrix and the point $(2, 0)$ a focus of a conic with eccentricity $e = \sqrt{2}$.

(i) Find the equation of the conic and show that it is a rectangular hyperbola. 3

[Hint: Let $P(x, y)$ be a point on the curve]

(ii) Find the equation of the normal to the curve at $Q(x_0, y_0)$ 2

c) The roots of the equation $x^3 + 7x^2 - 5x - 1 = 0$ are α, β and γ.

Find:

(i) $\alpha^2 + \beta^2 + \gamma^2$ 1

(ii) $\alpha^3 + \beta^3 + \gamma^3$ 2

(iii) the cubic equation whose roots are α^2, β^2 and γ^2 2

d) (i) Solve for z: $z^5 = 1$ 2

(ii) If α is a fifth root of unity $(\alpha \neq 1)$ show that $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$ 1

(iii) Find a quadratic equation whose roots are $(\alpha + \alpha^4)$ and $(\alpha^2 + \alpha^3)$ 2

Question 2 – (16 marks) – **Start a New Booklet** Marks

Part a) is a multiple choice question.

Write down the letter matching the correct answer in your booklet.

a) The points $P\left(p,\frac{1}{p}\right)$ and $Q\left(q,\frac{1}{q}\right)$ lie on $xy=1$ with $0<q<p$. If the chord PQ passes through $A(2,0)$ then 1

(A) $p+q=2p$

(B) $p+q=2$

(C) $p+q=1$

(D) $p+q=\frac{1}{pq}$

b) (i) The rectangular hyperbola $\mathcal{H}$ has Cartesian equation $x^2-y^2=8$.

(α) Write down its eccentricity, the coordinates of its foci S and S', the equation of each directrix and each asymptote. 3

(β) Sketch the curve indicating on your diagram the foci, directrices and asymptotes. 2

c) For what real values of c is $(z-ci)$ a factor of $P(z)=z^4-z^3+9z^2-4z+20$? 4

Hence, solve $P(z)=0$

d) Express $\frac{2x^2+3x-7}{(x^2+4)(x-5)}$ as partial fractions. 2

e) Given $P(x)=x^4-3x^3-6x^2+28x-24$ has a root of multiplicity three, solve $P(x)=0$ 4

Question 3 – (16 marks) – **Start a New Booklet** Marks

Part a) is a multiple choice question.

Write down the letter matching the correct answer in your booklet

a) If $y=mx+c$ is tangential to the hyperbola $xy=4$, then m equals. 1

(A) $\frac{16}{c^2}$

(B) $\frac{4+cx}{x^2}$

(C) $-\frac{c^2}{16}$

(D) $-\frac{c^2}{4}$

b) The tangents at the points $P\left(cp,\frac{c}{p}\right)$ and $Q\left(cq,\frac{c}{q}\right)$ which are in the first quadrant on the rectangular hyperbola $xy=c^2$ intersect at point R.

The chord PQ produced passes through the point $T(0,c)$

(i) Show that the coordinates of R are $\left(\frac{2cpq}{p+q},\frac{2c}{p+q}\right)$ 2

(ii) Show that $pq=p+q$ 2

(iii) Hence, (or otherwise) find the locus of R and describe it geometrically. 2

Question 3 continued next page

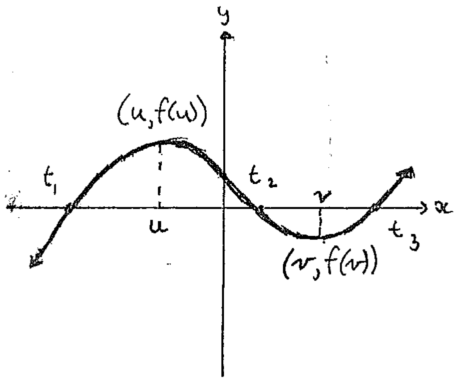
Question 3 (cont'd) **Marks**

c) Let $f(t) = t^3 + ct + d$ where c, d are constant. Suppose that the equation $f(t) = 0$ has three distinct real roots t_1, t_2 and t_3

(i) Find $t_1 + t_2 + t_3$ 1

(ii) Show that $t_1^2 + t_2^2 + t_3^2 = -2c$. Deduce that $c < 0$ 2

(iii) Since the roots are real and distinct the graph of $y = f(t)$ has two turning points at $t = u$ and $t = v$ $[u < v]$, <u>and</u> $f(u).f(v) < 0$ as shown below. 3



Show that $27d^2 + 4c^3 < 0$

d) The line $3y = 5x + 1$ is the equation of the diagonal of a square. One of the squares vertices has coordinates $(3, 11)$.

Find one other vertex. 3

End of Paper

(i) $\alpha+\beta+\gamma = \frac{12}{24} = \frac{1}{2}$

B.

$\alpha+\alpha = \frac{1}{2}$

$2\alpha = \frac{1}{4}$, $\alpha = \frac{1}{4}$

1

b). Let $P(x,y)$.

$(ae, 0)$ $ae = 2$. $\frac{a}{e} = 1$. $e = \sqrt{2}$.

$a\sqrt{2} = 2$ a

$\cancel{a = \frac{1}{\sqrt{2}}}$. $\boxed{a = \sqrt{2}}$

2

~~eqn~~ as $e=\sqrt{2}$, the equation must be a rectangular hyperbola.

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

$\therefore$ equation is. $x^2 - y^2 = 2$.

$b^2 = a^2(e^2-1)$.

$= 2(2-1) = 2.$, $b^2 = 2$.

mention since $a=b$

$y = \pm\frac{b}{a}x \Rightarrow y = \pm x$ asymptote perpendicular condition for rectangular hyperbola

$\frac{x^2}{2} - \frac{y^2}{2} = 1$, $x^2 - y^2 = 2$.

(ii). at $Q(x_0, y_0)$.

$2x - 2y\frac{dy}{dx} = 0$.

$\frac{dy}{dx} = \frac{2y}{2x}$, $\frac{dy}{dx} = \frac{y}{x}$. at $x = x_0$ $y = y_0$

$\frac{1}{2}a^2 = c^2$

$\frac{1}{2}(4) = c^2$

$c^2 = 2$.

$xy = 2$.

$\frac{dy}{dx} = \frac{y_0}{x_0}$. ~~$M_{normal}: -\frac{x_0}{y_0}$.~~

$2x - 2y\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} = 2x$

$\frac{dy}{dx}$ $\frac{2x}{2y} = \frac{x}{y}$

~~$y - y_0 = -\frac{x_0}{y_0}(x - x_0)$~~

~~$yy_0 - y_0^2 = -xx_0 + x_0^2$~~

~~$xx_0 + yy_0 = x_0^2 + y_0^2$.~~

2

at $x = x_0$, $y = y_0$ $\frac{dy}{dx} = \frac{x_0}{y_0}$ $M_{normal}: -\frac{y_0}{x_0}$.

$y - y_0 = -\frac{y_0}{x_0}(x - x_0)$

$x_0y - x_0y_0 = -xy_0 + x_0y_0$

$= x_0y + xy_0 = 2x_0y_0$.

c) $x^3 + 7x^2 - 5x - 1 = 0$. $a=1$, $b=7$, $c=-5$, $d=-1$

(i) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)$

$\alpha+\beta+\gamma = -7$. $= (-7)^2 - 2(-5)$

$\alpha\beta+\beta\gamma+\alpha\gamma = -5$. $= 49 + 10$

$= 59$.

1

(ii) $\alpha^3+\beta^3+\gamma^3$.

Since α is a root,

$\alpha^3 + 7\alpha^2 - 5\alpha - 1 = 0$, ——— ①

Similarly, $\beta^3 + 7\beta^2 - 5\beta - 1 = 0$, ——— ②

$\gamma^3 + 7\gamma^2 - 5\gamma - 1 = 0$. ——— ③

① + ② + ③.

$\alpha^3+\beta^3+\gamma^3 + 7(\alpha^2+\beta^2+\gamma^2) - 5(\alpha+\beta+\gamma) - 3 = 0$.

$\alpha^3+\beta^3+\gamma^3 + 7(59) - 5(-7) - 3 = 0$.

$\alpha^3+\beta^3+\gamma^3 = -445$.

2

(iii). $\alpha^2, \beta^2, \gamma^2$

~~x~~ $x^2 = y$, $y = \sqrt{x}$.

$x\sqrt{x} + 7x - 5\sqrt{x} - 1 = 0$.

$\sqrt{x}(x-5) = 1 - 7x$.

$x(x-5)^2 = (1-7x)^2$.

$x(x^2 - 10x + 225) = 1 - 14x + 49x^2$

$x^3 - 10x^2 + 225x = 1 - 14x + 49x^2$

$x^3 - 59x^2 + 239x - 1 = 0$.

$1\frac{1}{2}$

$\cancel{\alpha^2, \beta^2, \gamma^2}$ $\qquad z^5 = 1.$

$\cancel{x\sqrt{x} + \sqrt{x} - 5\sqrt{x} - 1 = 0}$ $\qquad z^5 - 1 = 0.$

$\cancel{\sqrt{x}(x-5) = 1 - \sqrt{x}}$ $\qquad (z-1)(z^4$

$\cancel{x\,(x^2}$ $\qquad 1 - i - 1 + i + 1$

(iii). $z^4 + z^3 + z^2 + z + 1$

$z^2(z^2+1) + z(z^2+1) + 1$

$(z^2+z)(z^2+1) + 1 = 0.$

$z\,(z^4) = 1.$

$(\alpha + \alpha^4)$, $(\alpha^2 + \alpha^3)$

SUM: $\alpha + \alpha^2 + \alpha^3 + \alpha^4.$ $\qquad \alpha^5 = 1$

PRODUCT: $\alpha^3 + \alpha^4 + \alpha^6 + \alpha^7.$ $\qquad$ 2

$x^2 - \underbrace{(\alpha + \alpha^2 + \alpha^3 + \alpha^4)}_{=-1}x + \underbrace{(\alpha^3 + \alpha^4 + \alpha^6 + \alpha^7)}_{-1}$

$\alpha \qquad \alpha^2$

quadratic: $\boxed{x^2 + x - 1 = 0}$

d) (i) $z^5 = 1$ $\qquad z^5 - 1 = 0$

$z^5 - 1 = 0$ $\qquad z = 1$

$(z-1)(z^4 + z^3 + z^2 + z + 1) = 0.$

$z^4 + z^3 + z^2 + z + 1 = 0.$

$\cos 5\theta + i\sin 5\theta = 1$ $\qquad k=0,\ \theta = 0$ $\qquad k = 4,\ \theta = \frac{8\pi}{5}.$

$\cos 5\theta = 1$ $\qquad k=1,\ \theta = \frac{2\pi}{5}.$

$5\theta = 2k\pi$ $\qquad k=2,\ \theta = \frac{4\pi}{5}.$

$\theta = \frac{2k\pi}{5}.$ $\qquad k=3,\ \theta = \frac{6\pi}{5}$

z: $\cos 0,\ \cos\frac{2\pi}{5},\ \cos\frac{4\pi}{5},\ \cos\frac{6\pi}{5},\ \cos\frac{8\pi}{5}$

$1,\ \cos\frac{2\pi}{5},\ \cos\frac{4\pi}{5},\ \cos\frac{6\pi}{5},\ \cos\frac{8\pi}{5}.$ $\qquad$ 2

(ii). If α is fifth root of unity,

$\alpha^5 - 1 = 0$

$(\alpha - 1)(\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1) = 0.$

$\therefore$ $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$, as required. $\qquad$ 1

$\cancel{(iii)\ (\alpha + \alpha^4)\ (\alpha^2 + \alpha^3).\qquad \alpha^5 = 1.}$

$\cancel{(x - (\alpha + \alpha^4))(x - (\alpha + \alpha^3))}$

$\cancel{x^2 - (\alpha + \alpha + \alpha^4 + \alpha^3)x + (\alpha^2 + \alpha^4 + \alpha^5 + \alpha^7)}$

$\cancel{x^2 - (2\alpha + \alpha^4 + \alpha^3)x + (\alpha^2 + \alpha^4 + \alpha^5 + \alpha^7) = 0.}$

$\cancel{x^2 - (2\alpha + \alpha^4 + \alpha^3)x + (\alpha^2 + \alpha^4 + 1 + \alpha^2) = 0}$

$\cancel{x^2 - (2\alpha + \alpha^4 + \alpha^3)x + (2\alpha^2 + \alpha^4 + 1) = 0.}$

$P(p, \frac{1}{p})$ $Q(q, \frac{1}{q})$ $xy = 1$.

$m_{PQ} = \dfrac{\frac{1}{q} - \frac{1}{p}}{q - p} = \dfrac{p-q}{pq(q-p)} = \dfrac{-1}{pq}$.

$y - \frac{1}{p} = \frac{-1}{pq}(x - p)$ $(2, 0)$

$2 = p+q$ B. ✓ (1)

$pqy - q = -x + p$ $\Rightarrow x + pqy = 2$.

$x + pqy = p + q$.

b) (i) $x^2 - y^2 = 8$. $e = \sqrt{2}$. ✓

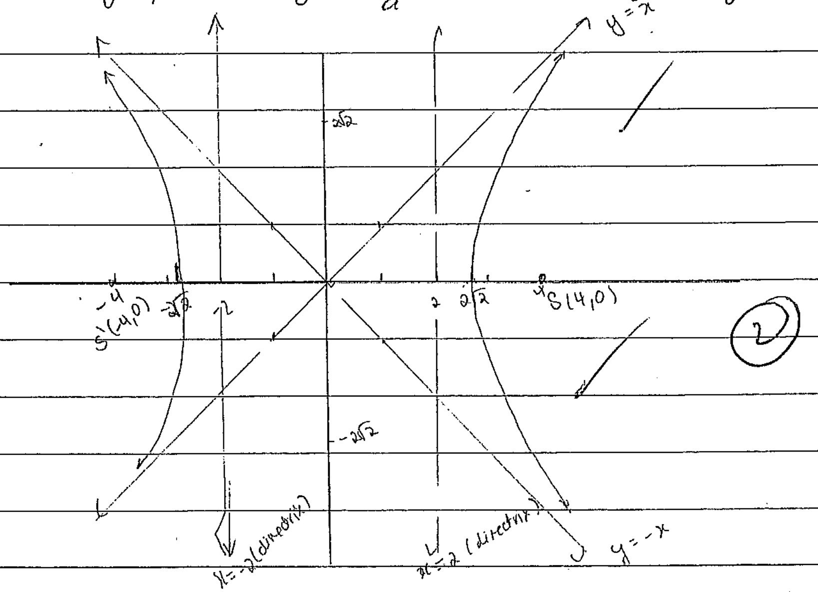
(a) $a^2 = 8$, $a = 2\sqrt{2}$.

$e = \sqrt{2}$, $S(\pm ae, 0) = S(4,0)$ or $S'(-4,0)$

directrix: $x = \pm \frac{a}{e} = \pm 2$

$= x = 2$, $x = -2$.

asymptote: $y = \pm \frac{b}{a} x = \pm x$. $y = x$, $y = -x$. (3)



(2)

c) $(z - ci)$ $P(z) = z^4 - z^3 + 9z^2 - 4z + 20$.

$P(ci) = c^4 + ic^3 - 9c^2 - 4ci + 20 = 0$. ✓

$c^4 - 9c^2 + 20 + i(c^3 - 4c) = 0$.

$\boxed{c^4 - 9c^2 + 20 = 0}$ ✓ (1)

$\cancel{i^4c^4 - (ci)^3 + 9(ci)^2 - 4(ci) + 20}$

$\cancel{c^4 + ic^3 + 9c^2}$

$c^2 = \dfrac{9 \pm \sqrt{81 - (4 \times 20)}}{2}$

$= \dfrac{9 \pm \sqrt{1}}{2} = c^2 = \dfrac{9+1}{2}$ or $c^2 = \dfrac{9-1}{2}$

$c^2 = 10$, $c^2 = 8$.

$c = \pm\sqrt{10}$ $c = \pm 2\sqrt{2}$.

roots of $P(z)$ are $\sqrt{10}i$, $-\sqrt{10}i$, $2\sqrt{2}i$, $-2\sqrt{2}i$ ✓

d) $\dfrac{2x^2 + 3x - 7}{(x^2+4)(x-5)} \equiv \dfrac{Ax+B}{x^2+4} + \dfrac{C}{x-5}$.

$2x^2 + 3x - 7 = (Ax+B)(x-5) + C(x^2+4)$

$x=5$. $58 = 29C$, $C = 2$. $(Ax+3)(x-5) + 2(x^2+4)$.

$x=0$ $-7 = -5B + 8$.

$-15 = -5B$, $B = 3$. ✓ (2)

$x=1$ $-2 = -4(A+3) + 2(1+4)$ $-2 = -4(A+3) + 10$.

$-2 - 10 = -4(A+3)$ $-12 = -4(A+3)$

$\cancel{3 = A+}$ $-12 = -4(A+3)$

$3 = A + 3$ ✓ $A = 0$.

$= \boxed{\dfrac{3}{x^2+4} + \dfrac{2}{x-5}}$

c) $P(x) = x^4 - 3x^3 - 6x^2 + 28x - 24$.

$P'(x) = 4x^3 - 9x^2 - 12x + 28$

$P''(x) = 12x^2 - 18x - 12 = 0$.

$12x^2 - 18x - 12 = 0$.

$2x^2 - 3x - 2 = 0$

$2x \quad +1$

$x \quad -2$

$(2x+1)(x-2) = 0$.

$x = \frac{-1}{2}, \quad x = 2$.

$P'(2) = 4(8) - 9(4) - 24 + 28$

$= 32 - 36 + 4 = 0$.

$P(2) = 16 - 3(8) - 6(4) + 56 - 24$

$= 0$.

$\therefore$ 2 is the root of multiplicity 3.

Let the remaining root be α.

$6 + \alpha = 3$.

$\alpha = -3$.

4

$\therefore$ ~~P(x)~~ $P(x) = 0$ when $x = 2, 2, 2, -3$.

$\frac{13}{16}$

$y = mx + c \qquad xy = 4. \qquad y = \frac{4}{x}$.

$y + x\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-y}{x}$.

$x(mx + c) = 4$

$mx^2 + cx = 4$

$mx^2 + cx - 4 = 0$.

$\Delta = 0 \qquad c^2 - 4(m \times -4) = 0$

$c^2 + 16m = 0$.

$16m = -c^2$

$m = \frac{-c^2}{16}$.

C.

b). $P(cp, \frac{c}{p}) \qquad Q(cq, \frac{c}{q})$. 1st quadrant

$xy = c^2$.

TANGENTS:

at P: $x = cp \quad y = \frac{c}{p} \implies \frac{dy}{dx} = \frac{-1}{p^2}$.

$\frac{dx}{dp} = c \qquad \frac{dy}{dp} = -cp^{-2}$.

$y - \frac{c}{p} = \frac{-1}{p^2}(x - cp)$

$p^2y - cp = -x + cp$

$x + p^2y = 2cp$. — ① $\qquad$ At Q, $x + q^2y = 2cq$ — ②

(i) ① − ②

$y(p^2 - q^2) = 2c(p - q)$.

since $p \neq q$, $\qquad y = \frac{2c}{p+q}$.

sub $y = \frac{2c}{p+q}$ into ①.

$$x + \frac{2cp^2}{p+q} = 2cp$$

$$x = 2cp - \frac{2cp^2}{p+q}$$

$$= \frac{2cp^2 + 2cpq - 2cp^2}{p+q}$$

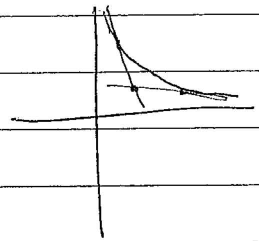
$$= \frac{2cpq}{p+q}.$$ ✓ 2

∴ R has coordinates $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$, as required.

(ii) SHOW: $pq = p+q$.

CHORD: PQ : $m_{PQ} = \frac{c(q-p)}{cpq(p-q)} = \frac{-1}{pq}$. ✓

$$y - \frac{c}{p} = \frac{-1}{pq}(x - cp)$$



$$pqy - cq = -x + cp$$

$$x + pqy = c(p+q).$$

If passes T, $x = 0$, $y = c$.

$$pqc = c(p+q).$$ ✓ 2

since $c \neq 0$, $pq = p+q$, as required.

(iii) $\left(\frac{2cpq}{pq}, \frac{2c}{p+q}\right) = \left(2c, \frac{2c}{p+q}\right)$.

~~The locus is the straight line $x = 2c$,~~

Sub $x = 2c$ into $xy = c^2$

$$2cy = c^2$$

$$y = \frac{c}{2}.$$

∴ the locus is the straight line $x = 2c$, where y is between: $0 < y < \frac{c}{2}$, as $p+q > 0$. ✓ 2

c). $f(t) = t^3 + ct + d$. $f(t) = 0$ t_1, t_2, t_3.

$a = 1, b = 0, c = c, d = d$. ✓ 1

(i) $t_1 + t_2 + t_3 = 0$.

(ii) $t_1^2 + t_2^2 + t_3^2 = -2c$.

LHS $= (t_1 + t_2 + t_3)^2 - 2(t_1 t_2 + t_2 t_3 + t_1 t_3)$.

$\Sigma \alpha\beta = c.$ $= 0 - 2(c)$

$= -2c$ = RHS, as required. ✓

since ~~$t^2 > 0$~~ $t_1^2 > 0$ and $t_2^2 > 0$, $t_3^2 \geq 0$ why?

c must be < 0 as $t_1^2 + t_2^2 + t_3^2$ cannot be negative.

(iii). $27d^2 + 4c^3 < 0$. Stat. pts?? ✓

$f'(t) = 3t^2 + c$ $3t^2 + c = 0$.

~~$3t^2 + c = 0$~~ $3(u^2 + v^2) + 2c = 0$.

$f'(u) = 0$ $3u^2 + c = 0$ $f(u) = u^3 + cu + d$

$f'(v) = 0$. $3v^2 + c = 0$. $f(v) = v^3 + cv + d$.

$f(v) < 0$.

Next booklet.