St George Girls High School

Trial Higher School Certificate Examination

2012

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen.
- Write your student number on each booklet.
- Board-approved calculators may be used.
- The mark allocated for each question is listed at the side of the question.
- Marks may be deducted for careless or poorly presented work.
- Diagrams are not drawn to scale.

Total Marks – 100

Section I – Pages 2 – 4
10 marks
- Attempt Questions 1 – 10.
- Allow about 15 minutes for this section.
- Answer on the sheet provided.

Section II – Pages 5 – 12
60 marks
- Attempt Questions 11 – 14
- Allow about 1 hour 45 minutes for this section.
- Begin each question in a new booklet.
- Show all necessary working in Questions 11 – 14.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a \neq 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Section I – (10 marks) Marks

Answer this section on the answer sheet provided at the back of this paper.
Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

1. The value of $\lim\limits_{x \to 0} \dfrac{\sin 4x}{9x}$ is:

A. $2\frac{1}{4}$

B. 1

C. $\frac{4}{9}$

D. 0

2. For the function $f(x) = 3\sin^{-1}\left(\frac{x}{4}\right)$ the domain and range of $y = f(x)$ are:

A. domain $\left\{x:\ -\frac{3\pi}{2} \leq x \leq \frac{3\pi}{2},\ x \in \mathbb{R}\right\}$
range $\{y:\ -4 \leq y \leq 4,\ y \in \mathbb{R}\}$

B. domain $\{x:\ -1 \leq x \leq 1,\ x \in \mathbb{R}\}$
range $\{y:\ -3 \leq y \leq 3,\ y \in \mathbb{R}\}$

C. domain $\{x:\ -3 \leq x \leq 3,\ x \in \mathbb{R}\}$
range $\left\{y:\ -\frac{\pi}{2} \leq y \leq \frac{\pi}{2},\ y \in \mathbb{R}\right\}$

D. domain $\{x:\ -4 \leq x \leq 4,\ x \in \mathbb{R}\}$
range $\left\{y:\ -\frac{3\pi}{2} \leq y \leq \frac{3\pi}{2},\ y \in \mathbb{R}\right\}$

3. Solve for x, $\dfrac{2x+1}{1-x} \geq 1$

A. $0 \leq x < 1$

B. $x \leq 0$ or $x > 1$

C. $x > 0$ or $x > 1$

D. $0 < x \leq 1$

Section I (cont'd) Marks

4. A particle is oscillating in Simple Harmonic Motion where its position x metres from a fixed point O on the same line as its motion after t seconds is given by $x = 2\cos\left(3t + \frac{\pi}{6}\right)$. What is the maximum speed of the particle?

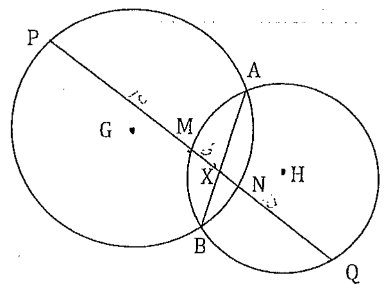
A. 2 m/s

B. 6 m/s

C. 0 m/s

D. $\frac{\pi}{9}$ m/s

5.



AB is a common chord to the circles with centres G and H.

PQ is drawn intersecting circle centre G at P and N, intersecting circle centre H at M and Q and intersecting AB at X as shown in the diagram.

If $PM = 18$, $MX = 6$ and $NQ = 15$ then the length NX is:

A. 5

B. 4

C. 3

D. 2

6. The derivative of $\tan^{-1}\dfrac{2x}{3}$ is:

A. $\dfrac{1}{3+4x^2}$

B. $\dfrac{1}{\frac{9}{4}+x^2}$

C. $\dfrac{6}{9+4x^2}$

D. $\dfrac{3}{4+x^2}$

Section I (cont'd) Marks

7. The exact value of $\sin^{-1}\left(\cos\frac{2\pi}{3}\right)$ is:

A. $\frac{\pi}{6}$

B. $-\frac{\pi}{6}$

C. $\frac{\pi}{3}$

D. $-\frac{\pi}{3}$

8. Consider $(1+2x)^n$. If the ratio of the coefficient of x^4 to the coefficient of x^6 is $5:8$ then the value of n is:

A. 5

B. 6

C. 7

D. 8

9. The polynomial $P(x) = x^4 - 2x^3 - 7x^2 + 20x - 12$ has a zero of multiplicity 2 at $x =$:

A. 1

B. -3

C. 2

D. -2

10. A particle moves in a straight line. At time t seconds, where $t \geq 0$, its displacement x metres from the origin and its velocity v metres per second are such that $v = 25 + x^2$.

If $x = 5$ initially, then t is equal to:

A. $25x + \frac{x^3}{3}$

B. $25x + \frac{x^3}{3} + \frac{500}{3}$

C. $\tan^{-1}\left(\frac{x}{5}\right) - \frac{\pi}{4}$

D. $\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) - \frac{\pi}{20}$

Section II – Show all working

Question 11 – Start A New Booklet – (15 marks) Marks

a) (i) Find the derivative of $\log_e(\cos^2 x)$ 1

(ii) $\int_0^1 \frac{x^2}{x^3+1}\,dx$ 1

b) In the expression of $\left(x^2 + \frac{2}{x}\right)^{10}$ find the coefficient of x^2 1

c) The quadratic polynomial $ax^2 + bx + 14$ leaves a remainder of -12 when divided by $(x-1)$, and has $(x+2)$ as a factor. Find the values of a and b. 2

d) Find the acute angle between the lines $y = 5 - x$ and $\sqrt{3}y = x + 1$ 1

e) (i) Show that the area of an equilateral triangle of side length x is given by 1

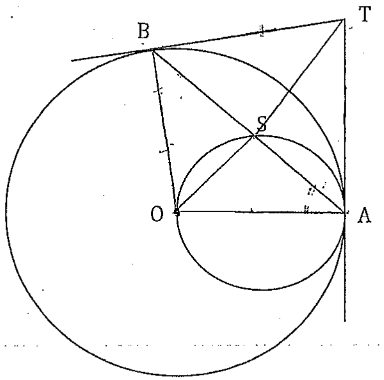
$$A = \frac{\sqrt{3}}{4}x^2$$

(ii) The sides of an equilateral triangle are increasing at the rate of 5 mm/s. At what rate is the area of the triangle increasing at the instant the sides are 10 cm long. 2

Question 11 continued on next page

Question 11 (cont'd) Marks

f)



Two circles touch internally at a point A and the smaller of the two circles passes through O, the centre of the larger circle.

AB is any chord of the larger circle through S, a point on the smaller circle. The tangents to the larger circle at A and B meet at the point T

Prove:

(i) AB is bisected at S. 4

(ii) O, S and T are collinear. 2

Question 12 – Start A New Booklet – (15 marks) Marks

a) Use the principle of Mathematical Induction to prove that $7^n + 2$ is divisible by 3 for all positive integers n 2

b) (i) Show that $\sin(A + B) + \sin(A - B) = 2\sin A\cos B$ 1

(ii) Hence or otherwise find: 2

$$\int \sin 4x \ \cos 2x \ dx$$

c) (i) Show that $\frac{d}{dx}(x - \tan^{-1}x) = \frac{x^2}{1+x^2}$ 1

(ii) Hence or otherwise find the exact value of 1

$$\int_0^1 \frac{x^2}{1+x^2}\,dx$$

d) Given $A(-2,3)$ and $B(4,7)$ find the coordinates of the point which divides the interval AB externally in the ratio $3:1$ 1

e) If α, β and γ are the roots of $x^3 - 2x^2 + 4x - 7 = 0$ evaluate $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ 2

Question 12 continued on next page

Question 12 (cont'd) Marks

f) A particle moving in a straight line has an acceleration given by $\ddot{x} = x^2$ where its displacement is x metres from the origin. If initially the particle is at rest when $x = 2$, find its velocity when $x = 4$. 2

g) The normal at $P(2ap,\ ap^2)$ to the parabola $x^2 = 4ay$ meets the curve again at $Q(2aq,\ aq^2)$

(i) Given that the equation of the normal at P is $x + py = ap^3 + 2ap$ 1

show that $q = -\frac{(2+p^2)}{p}$

(ii) Find a value for p so that the lines OP and OQ are at right angles, where O is the origin. 2

Question 13 – Start A New Booklet – (15 marks) Marks

a) If $3n^2 - 7n + 5 \equiv An(n-1) + Bn + C$ find A, B and C 2

b) Evaluate, leaving your answer in exact form 3

$$\int_{\frac{1}{8}}^{\frac{\sqrt{3}}{8}} \frac{dx}{\sqrt{1 - 16x^2}}$$

c) If α and β are the roots of $x^2 + bx + c = 0$, form the equation, in general form, whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ 2

d) A curve is defined by the parametric equations $x = t - 3,\ y = t^2 - 9$

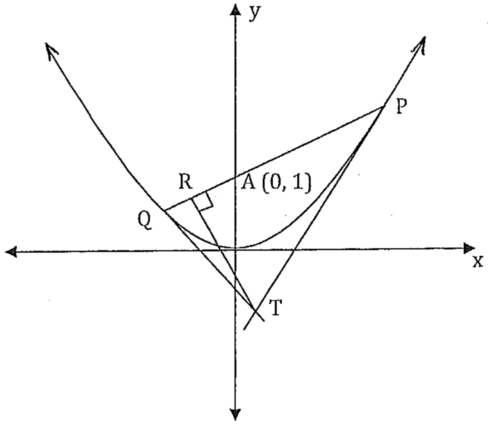
(i) Find $\frac{dy}{dx}$ in terms of t 1

(ii) Find the equation of the tangent to the curve at the point where $t = -3$ 2

Question 13 continued on next page

Question 13 (cont'd) Marks

e)



PQ is a chord of the parabola $x^2 = 8y$ passing through the point $A(0,1)$ where P is $(4p, 2p^2)$ and Q is ~~$(4q, 2p^2)$~~ $(4q, 2q^2)$

The tangents to the parabola at P and Q meet at the point T.

R is a point on the chord PQ with $RT \perp PQ$

(i) Write down the equations of the tangents at P and Q and hence find the coordinates of T **2**

(ii) Show that the equation of the chord PQ is given by **1**

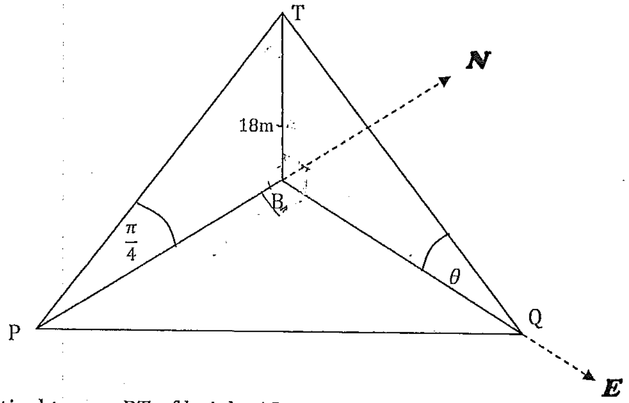
$$2y = (p+q)x - 4pq$$

(iii) Show that $pq = -\frac{1}{2}$ **1**

(iv) Find the equation of RT **1**

Question 14 – Start A New Booklet – (15 marks) Marks

a)



A vertical tower BT of height 18 metres stands with its base B on horizontal grounds. B is due North of a fixed point P and the angle of elevation from P to the top of the tower T is $\frac{\pi}{4}$ radians. Q is a moving point on the ground due East of B and the angle of elevation from Q to T is θ radians where $0 < \theta < \frac{\pi}{2}$. The size of the angle θ is increasing at a constant rate of 0.02 radians per minute.

(i) Show that $PQ = 18 \operatorname{cosec} \theta$ **2**

(ii) Find the rate at which the length PQ is changing when $\theta = \frac{\pi}{3}$ **2**

b) A person hits a ball off the ground with a bat, projecting the ball at a velocity of 50 m/s at an angle of projection θ such that $\tan\theta = \frac{3}{4}$ **1**

(i) Taking the origin as the point of projection and $g = 10$ m/s show that $\dot{x} = 40$ and $\dot{y} = -10t + 30$ and then find x and y in terms of t **2**

(ii) A tall building is 100 m from where the ball is hit on horizontal ground. If the ball passes through a small open window find the height of the window above the ground. **2**

(iii) Find the velocity and angle that the ball makes with the horizontal as it passes through the window. **2**

Question 14 continued on next page

Question 14 (cont'd) **Marks**

c) Find the general solution in radians of the equation $\sin 2x = \cos x$ **2**

d) By considering the expansion of both sides of the identity **2**

$(1+x)^{m+n} = (1+x)^m(1+x)^n$, where m and n are positive integers, show that

$$\binom{m+n}{3} = \binom{m}{3} + \binom{m}{2}\binom{n}{1} + \binom{m}{1}\binom{n}{2} + \binom{n}{3}$$

End of Paper

TRIAL HSC EXTENSION 1 2012 ST GEORGE GIRLS HIGH

Question 11

(a) (i) $\frac{d\log_e(\cos^2 x)}{dx} = \frac{2\cos x(-\sin x)}{\cos^2 x}$ $(\cos x \neq 0)$ OR $\frac{-2\sin 2x}{1+\cos 2x}$

$= -2\tan x$ OR equivalent

(ii) $\int_0^1 \frac{x^2}{x^3+1}\,dx = \frac{1}{3}\int_0^1 \frac{3x^2}{x^3+1}\,dx$

$= \frac{1}{3}\left[\log_e(x^3+1)\right]_0^1$

$= \frac{1}{3}(\log_e 2 - \log_e 1)$

$= \frac{1}{3}\log_e 2$

(b) $T_{k+1} = {}^{10}C_k (x^2)^k \left(\frac{2}{x}\right)^{10-k}$ OR ${}^{10}C_k (x^2)^{10-k}\left(\frac{2}{x}\right)^k$

$= {}^{10}C_k\, x^{2k}\, 2^{10-k}\, x^{k-10}$ $\quad = {}^{10}C_k\, x^{20-2k}\, 2^k x^{-k}$

$= {}^{10}C_k\, 2^{10-k}\, x^{3k-10}$ $\quad = {}^{10}C_k\, 2^k\, x^{20-3k}$

If $3k-10=2$ $\quad 20-3k=2$

$k=4$ $\quad k=6$

Coeff of $x^2 = {}^{10}C_4 \times 2^6$ $\quad {}^{10}C_6 \times 2^6$ $\quad (=13440)$

(c) Let $P(x) = ax^2+bx+14$

$P(1) = -12$

$a+b+14=-12$

$a+b=-26$ ①

$P(-2)=0$

$4a-2b+14=0$

$4a-2b=-14$

$2a-b=-7$ ②

① + ② $\quad 3a=-33$

$a=-11$

Subst in ①

$-11+b=-26$

$b=-15$

$a=-11 \quad b=-15$

(d) $y = 5-x$ $\quad \sqrt{3}y = x+1$

$m_1 = -1$ $\quad m_2 = \frac{1}{\sqrt{3}}$

Let θ be the acute angle between the lines

$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$

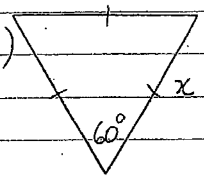
$= \left|\frac{-1-\frac{1}{\sqrt{3}}}{1+-1\times\frac{1}{\sqrt{3}}}\right|$

$= \left|\frac{-\sqrt{3}-1}{\sqrt{3}-1}\right|$

$= \frac{\sqrt{3}+1}{\sqrt{3}-1}$

$\therefore \theta = 75^\circ$

(e) (i)



$A = \frac{1}{2}x \times x \sin 60^\circ$

$= \frac{1}{2}x^2 \times \frac{\sqrt{3}}{2}$

$= \frac{\sqrt{3}}{4}x^2$

(ii) $\frac{dx}{dt} = 0.5$ cm/s $\quad \frac{dA}{dt} = \frac{dA}{dx}\frac{dx}{dt}$ $\quad \frac{dx}{dt} = 5$ mm/s

$= \frac{\sqrt{3}}{2}x \cdot 0.5$ $\quad$ OR $\frac{\sqrt{3}}{2}x \times 5$

When $x=10$ $\quad \frac{dA}{dt} = \frac{\sqrt{3}}{2}\times 10 \times 0.5$ $\quad$ When $x=100$

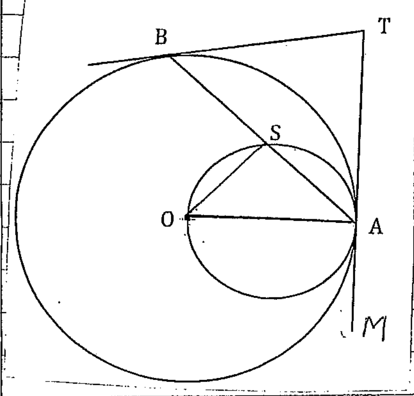
$\frac{dA}{dt} = \frac{\sqrt{3}}{2}\times 100\times 5$

$= \frac{5\sqrt{3}}{2}$ $\quad = 250\sqrt{3}$

$\therefore$ Area is increasing at a rate of $\frac{5\sqrt{3}}{2}$ cm²/s when $x=10$

(OR $250\sqrt{3}$ mm²/s) when $x=100$.

(f)



(i) Join OA, OS. Let M be on TA produced.

OA is a radius of larger circle

$\therefore \angle OAT = 90^\circ$ (angle between radius and tangent at point of contact)

(*)

$\therefore \angle OAM = 180^\circ - 90^\circ$ (straight angle)

$= 90^\circ$

$\therefore \angle OSA = \angle OAM$ (angle between chord and tangent equals angle in alternate segment)

$= 90^\circ$

(**) ie $OS \perp AB$

Since O is centre of larger circle and AB is a chord of the larger circle

OS bisects AB (line from centre of circle perpendicular to a chord bisects the chord.

ie AB is bisected at S

OR From (*) Let $\angle TAB = x^\circ$

$\therefore \angle AOS = x^\circ$ (angle between chord and tangent = angle in alternate segment

$\angle OAS = \angle OAT - \angle BAT$

$= 90^\circ - x^\circ$

$\therefore \angle OSA = 180^\circ - (x^\circ + 90^\circ - x^\circ)$ (angle sum of triangle)

$= 90^\circ$

(then from (**))

OR From (*) If C is the centre of the smaller circle then A, C and O are collinear (centres of touching circles and point of contact are collinear)

$\therefore$ C lies on OA ie OA is a diameter of smaller circle

$\therefore \angle OSA = 90^\circ$ (angle in a semicircle is 90°)

(then from (**))

(ii) In Δs ATS, BTS

AS = BS (proved in (i))

AT = BT (tangents from external point to circle are equal in length)

ST is common

$\therefore \angle AST = \angle BST$ (corresponding angles in congruent triangles)

$\angle AST + \angle BST = 180^\circ$ (straight angle)

$\therefore \angle AST = \angle BST = 90^\circ$

(***) Hence $\angle AST + \angle ASO = 90^\circ + 90^\circ$

ie $\angle TSO = 180^\circ$

ie T, S, O lie on a staight line and hence are collinear

Question 12

(a) Aim: To prove 7^n+2 is divisible by 3
ie $7^n+2 = 3A$ where A is an integer

For $n=1$ $7^1+2 = 9 = 3\times 3$
$\therefore$ Proposition true for $n=1$

Assume proposition is true for $n=k$ where k is a positive integer
ie $7^k+2 = 3B$ where B is an integer

Aim to show that proposition is then true for $n=k+1$

$$\begin{aligned}7^{k+1}+2 &= 7\times 7^k+2\\ &= 7\times(3B-2)+2 \quad \text{(by inductive hypothesis)}\\ &= 7\times 3B-14+2\\ &= 3\times 7B-12\\ &= 3(7B-4)\\ &= 3C \quad \text{(C is an integer since integers closed under mult}^{\text{n}}\text{ and subtraction)}\end{aligned}$$

ie proposition is true for $n=k+1$ if true for $n=k$.
Hence by induction proposition is true for all positive integers n.

(b) (i) $\sin(A+B)+\sin(A-B) = \sin A\cos B+\cos A\sin B + \sin A\cos B-\cos A\sin B$
$= 2\sin A\cos B$

(ii) $\sin 4x\cos 2x = \frac{1}{2}\left(\sin(4x+2x)+\sin(4x-2x)\right)$
$= \frac{1}{2}(\sin 6x+\sin 2x)$

$\therefore \int \sin 4x\cos 2x\,dx = \frac{1}{2}\int(\sin 6x+\sin 2x)\,dx$
$= \frac{1}{2}\left(-\frac{1}{6}\cos 6x-\frac{1}{2}\cos 2x\right)+C$
$= -\frac{1}{12}\cos 6x-\frac{1}{4}\cos 2x+C$

(c) (i) $\frac{d}{dx}(x-\tan^{-1}x) = 1-\frac{1}{1+x^2}$
$= \frac{1+x^2-1}{1+x^2}$
$= \frac{x^2}{1+x^2}$

(ii) $\int_0^1 \frac{x^2}{1+x^2}\,dx = \left[x-\tan^{-1}x\right]_0^1$
$= (1-\tan^{-1}1)-(0-\tan^{-1}0)$
$= 1-\frac{\pi}{4}-0$
$= \frac{4-\pi}{4}$

(d) $(-2,3)$ $(4,7)$
$-3:1$
$\left(\frac{1\times -2+-3\times 4}{-3+1}, \frac{1\times 3+-3\times 7}{-3+1}\right) = \left(\frac{-14}{-2}, \frac{-18}{-2}\right)$
Point is $(7,9)$

(e) $x^3-2x^2+4x-7=0$ has roots α,β,γ

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma} \qquad \frac{4}{1},\ \frac{--7}{1}$$

$$=\frac{4}{7}$$

OR If $P(x)=x^3-2x^2+4x-7$

then $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ are roots of $P\left(\frac{1}{x}\right)=0$

ie $\left(\frac{1}{x}\right)^3-2\left(\frac{1}{x^2}\right)+4\left(\frac{1}{x}\right)-7=0$

$1-2x+4x^2-7x^3=0$

has roots $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$

$$\therefore \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{-4}{-7}$$

$$=\frac{4}{7}$$

(f) $\ddot{x}=x^2$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx}=x^2$$

$$\frac{1}{2}v^2=\frac{x^3}{3}+c_1$$

When $t=0$ $v=0$ $x=2$

$$0=\frac{8}{3}+c_1$$

$$c_1=-\frac{8}{3}$$

$$\frac{1}{2}v^2=\frac{x^3}{3}-\frac{8}{3}$$

$$v^2=\frac{2x^3}{3}-\frac{16}{3}$$

When $x=4$

$$v^2=\frac{2\times 64}{3}-\frac{16}{3}$$

$$=\frac{112}{3}$$

$$v=\pm\sqrt{\frac{112}{3}}$$

But $v\geq 0$

$$\therefore v=\sqrt{\frac{112}{3}}$$

(g) (i) $x+py=ap^3+2ap$

meets $x^2=4ay$

$$x+p.\frac{x^2}{4a}=ap^3+2ap$$

$$\frac{p}{4a}.x^2+x-(ap^3+2ap)=0$$

has roots $2ap$ and $2aq$

$$2ap+2aq=\frac{-1}{p/4a}$$

$$=\frac{-4a}{p}$$

$$2aq=\frac{-4a}{p}-2ap$$

$$q=-\frac{2}{p}-p$$

$$=\frac{-2-p^2}{p}$$

$$=\frac{-(2+p^2)}{p}$$

OR

$$\text{Grad } PQ=\frac{ap^2-aq^2}{2ap-2aq}$$

$$=\frac{p+q}{2}$$

Grad of normal $=-\frac{1}{p}$

$$\therefore -\frac{1}{p}=\frac{p+q}{2}$$

$$-2=p^2+pq$$

$$pq=-2-p^2$$

$$q=\frac{-(2+p^2)}{p}$$

(ii) $\text{Grad } OP=\frac{ap^2-0}{2ap-0}$

$$=\frac{p}{2}$$

$$\text{Grad } OQ=\frac{q}{2}$$

If $OP\perp OQ$ then $\frac{p}{2}\times\frac{q}{2}=-1$

$$pq=-4$$

$$\therefore q=\frac{-4}{p}$$

$\therefore \quad \frac{-4}{p} = \frac{-(2+p^2)}{p}$

$2+p^2 = 4$

$p^2 = 2$

$p = \pm\sqrt{2}$

OR $\quad \frac{p}{2} \times \frac{q}{2} = -1$

$\frac{p}{2} \times \frac{-(2+p^2)}{p} \times \frac{1}{2} = -1$

$\frac{-(2+p^2)}{4} = -1$

$-(2+p^2) = -4$

$2+p^2 = 4$

$p^2 = 2$

$p = \pm\sqrt{2}$

Question 13

(a) $\quad 3n^2 - 7n + 5 \equiv An(n-1) + Bn + C$

Let $n=0 \quad 5 = 0 + 0 + C \qquad C = 5$

Coeff $n^2 \quad 3 = A \qquad A = 3$

Let $n=1 \quad 3-7+5 = 0 + B + C$

$1 = B + 5$

$B = -4$

(b) $\quad \int_{\frac{1}{8}}^{\frac{\sqrt{3}}{8}} \frac{dx}{\sqrt{1-16x^2}} = \int_{\frac{1}{8}}^{\frac{\sqrt{3}}{8}} \frac{dx}{4\sqrt{\frac{1}{16}-x^2}}$

$= \frac{1}{4}\left[\sin^{-1}\frac{x}{1/4}\right]_{\frac{1}{8}}^{\frac{\sqrt{3}}{8}}$

$= \frac{1}{4}\left[\sin^{-1} 4x\right]_{\frac{1}{8}}^{\frac{\sqrt{3}}{8}}$

$= \frac{1}{4}\left(\sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{1}{2}\right)$

$= \frac{1}{4}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$

$= \frac{\pi}{24}$

(c) $\quad x^2 + bx + c = 0$

$\alpha + \beta = -b$

$\alpha\beta = \frac{c}{1}$

$$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta} \qquad \frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha}=1$$

$$=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$$

$$=\frac{b^2-2c}{c}$$

Eq^n is $\left(x-\frac{\alpha}{\beta}\right)\left(x-\frac{\beta}{\alpha}\right)=0$

$$x^2-\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)x+\frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha}=0$$

$$x^2-\frac{(b^2-2c)}{c}x+1=0$$

$$cx^2-(b^2-2c)x+c=0$$

(d) Eq^n of tangents at P and Q

$$y=px-2p^2 \quad ①$$
$$y=qx-2q^2 \quad ②$$

Subst ① in ②

$$px-2p^2=qx-2q^2$$
$$(p-q)x=2(p^2-q^2)$$
$$x=\frac{2(p-q)(p+q)}{p-q} \quad (p\neq q)$$
$$=2(p+q)$$

$$y=p.2(p+q)-2p^2$$
$$=2pq$$

$\therefore$ T is point $(2(p+q),2pq)$

(ii) Grad PQ $=\frac{2p^2-2q^2}{4p-4q}$

$$=\frac{2(p-q)(p+q)}{4(p-q)}$$
$$=\frac{p+q}{2}$$

Eq^n of chord PQ is

$$y-2p^2=\frac{p+q}{2}(x-4p)$$
$$=\frac{p+q}{2}\cdot x-2p(p+q)$$
$$=\frac{p+q}{2}x-2p^2-2pq$$
$$y=\frac{p+q}{2}x-2pq$$
$$2y=(p+q)x-4pq$$

(iii) Since PQ passes through A(0,1)

$$2=0-4pq$$
$$pq=-\frac{1}{2}$$

(iv) RT $\perp$ PQ

$\therefore$ Grad RT $=\frac{-2}{p+q}$

$\therefore$ Eq^n of RT is

$$y-2pq=-\frac{2}{p+q}(x-2(p+q))$$
$$y-2\times-\frac{1}{2}=-\frac{2}{p+q}x+4$$
$$y=-\frac{2}{p+q}x+3$$

(e) (i) $x = t-3 \qquad y = t^2-9$

$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = 2t$

$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt}$

$= \frac{2t}{1}$

(ii) When $t = -3$

$x = -6$

$y = 0$

$\frac{dy}{dx} = -6$

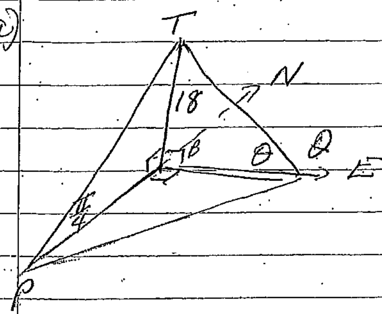
Eqⁿ of tangent is

$y - 0 = -6(x - -6)$

$y = -6x - 36$

Question 14

(a)



(i) $\frac{BQ}{18} = \cot\theta$

$BQ = 18\cot\theta$

$\frac{BP}{18} = \tan\frac{\pi}{4}$

$= 1$

$BP = 18$

In $\triangle PBQ$ $\quad PQ^2 = BP^2 + BQ^2$

$= 18^2 + 18^2\cot^2\theta$

$= 18^2(1+\cot^2\theta)$

$= 18^2\operatorname{cosec}^2\theta$

$PQ = 18\operatorname{cosec}\theta$

(ii) $\frac{d\,PQ}{dt} = \frac{d\,PQ}{d\theta} \cdot \frac{d\theta}{dt}$

$= -18\operatorname{cosec}\theta\cot\theta \cdot \frac{d\theta}{dt}$

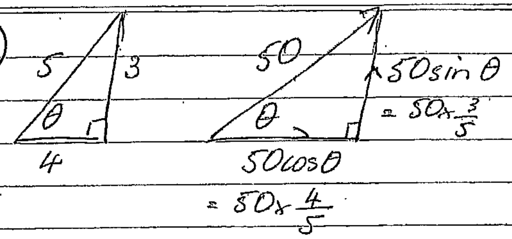
$= -18\operatorname{cosec}\theta\cot\theta \cdot 0.02$

$= -0.36 \times \operatorname{cosec}\frac{\pi}{3} \cdot \cot\frac{\pi}{3}$

$= -0.36 \times \frac{2}{\sqrt{3}} \times \frac{1}{\sqrt{3}}$

$= -0.24$

(b) (i)



$\ddot{x} = 0$ $\qquad\qquad$ $\ddot{y} = -10$

$\dot{x} = c_1$ $\qquad\qquad$ $\dot{y} = -10t + c_2$

When $t=0$ $\dot{x} = 50\cos\theta$ $\qquad$ When $t=0$ $\dot{y} = 50\sin\theta$

$= 40$ $\qquad\qquad$ $= 30$

$\therefore$ $= c_1$ $\qquad\qquad$ $= c_2$

$\dot{x} = 40$ $\qquad\qquad$ $\dot{y} = -10t + 30$

$x = 40t + c_3$ $\qquad\qquad$ $y = -5t^2 + 30t + c_4$

When $t=0$ $x=0$ $\qquad$ When $t=0$ $y=0$ $\therefore$

$\therefore c_3 = 0$ $\qquad\qquad$ $\therefore c_4 = 0$

$x = 40t$ $\qquad\qquad$ $y = -5t^2 + 30t$

(ii) If $x = 100$ then

$100 = 40t$

$t = \frac{5}{2}$

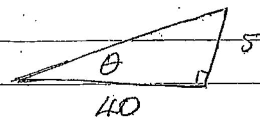
$y = -5\times\left(\frac{5}{2}\right)^2 + 30\times\frac{5}{2}$

$= \frac{175}{4}$

$\therefore$ Height of window is 43.75 m

(iii) When $t = \frac{5}{2}$ $\quad \dot{x} = 40$

$\dot{y} = -10\times\frac{5}{2} + 30 = 5$



$v^2 = 40^2 + 5^2$ $\qquad\qquad$ $\tan\theta = \frac{5}{40}$

$= 1600 + 25$ $\qquad\qquad$ $= \frac{1}{8}$

$= 1625$

$v = \sqrt{1625}$ $\qquad\qquad$ $\theta = \tan^{-1}\left(\frac{1}{8}\right)$

$= 5\sqrt{65}$ $\qquad\qquad$ $= 7°8'$

Velocity is $5\sqrt{65}$ m/s and angle ball's path makes with the horizontal is $7°8'$

(c) $\sin 2x = \cos x$

$2\sin x\cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0$ or $\sin x = \frac{1}{2}$

$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$ $\qquad$ $x = \frac{\pi}{6} + 2k\pi,\ \pi - \frac{\pi}{6} + 2k\pi$

$x = \frac{(2k+1)\pi}{2}, \frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi \quad (k \in \mathbb{Z})$

(d) $(1+x)^{m+n} = (1+x)^m(1+x)^n$

On LHS coeff $x^3 = \binom{m+n}{3}$

RHS $= \left(1 + \binom{m}{1}x + \binom{m}{2}x^2 + \binom{m}{3}x^3 + \ldots\right)$

$\times\left(1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots\right)$

Term in $x^3 = 1\times\binom{n}{3}x^3 + \binom{m}{1}x\times\binom{n}{2}x^2 + \binom{m}{2}x^2\times\binom{n}{1}x$

$+ 1\times\binom{m}{3}x^3$

$\therefore$ Coeff $x^3 = \binom{n}{3} + \binom{m}{1}\times\binom{n}{2} + \binom{m}{2}\times\binom{n}{1} + 1\times\binom{m}{3}$

$= \binom{m+n}{3}$