St George Girls High School

Year 11

Common Test – 2

2011

Mathematics Extension 1

Marks: 72

Instructions

1. Working time – 70 minutes
2. All questions should be attempted.
3. Show all working.
4. **Start each question on a new page.**
5. Marks will be deducted for careless work or poorly presented solutions.
6. On the cover sheet of the answer booklet clearly show:

   a) your name
   b) your mathematics class and teacher

Question 1: (18 Marks) – Start A New Page Marks

a) Simplify $64^{x+1} \div 8^{x-1}$ 1

b) If the distance from $(4, k)$ to $(8, 2)$ is 5. Find all possible values of k. 2

c) (i) By differentiation from first principles show that if $f(x) = x^2 - 2x$ then $f'(x) = 2x - 2$ 2

(ii) Find the gradient of the tangents at the points where the curve crosses the x axis. 2

d) Write in index form $x = \log_2 7$ 1

e) Find $\frac{dy}{dx}$ if

(i) $y = x\sqrt{x}$ 1

(ii) $y = \frac{1-x}{(4+x)^2}$ 3

f) Express as a trigonometric ratio of angle A.

(i) $\cos(180^\circ + A)$ 1

(ii) $\sin(90^\circ + A)$ 1

g) Sketch $y = |x^2 + 2x - 3|$ showing all relevant features. 2

h) Write down, in terms of k, the co-ordinates of the point A that divides the interval joining $P(1, 4)$ and $Q(-4, 7)$ internally in the ratio $k:1$ 2

Question 2: (18 Marks) – Start A New Page **Marks**

a) If $y = (4x+3)^5$ use the substitution $u = 4x+3$ and the chain rule to show $\frac{dy}{dx} = 20(4x+3)^4$ 2

b) If $\tan\theta = \frac{1}{3}$ and $180^\circ \le \theta \le 360^\circ$ find the exact value of $\sin\theta$ 2

c) Given $P(2ap, ap^2)$ and $R(2ar, ar^2)$, $p \neq r$

(i) Find in its simplest form the gradient of PR 2

(ii) Show that the equation of PR is $y = \left(\frac{p+r}{2}\right)x - apr$ 2

d) Find the gradient of the normal to $y = \frac{1}{x^2}$ at the point $(1, 1)$ 2

e) The sum of the first six terms of an arithmetic series is 45 and the sum of the first 12 terms is -18. Find the first term and the common difference. 3

f) Solve

(i) $3 \le |2x+1| \le 7$ 2

(ii) $\frac{4x}{3x-2} \ge 1$ 3

Question 3: (18 Marks) – Start A New Page **Marks**

a) The line l makes an angle of 135° with the positive x axis. If l passes through $(-2, -1)$ write the equation of l in general form. 2

b) Simplify

(i) $\sin^3\theta + \sin\theta\cos^2\theta$ 2

(ii) $(1 - \sin^2\alpha)(1 - \tan^2\alpha)$ 2

c) Solve $|2x - 7| = 3x + 2$ 2

d) Prove

(i) $\sec\theta + \tan\theta = \frac{1+\sin\theta}{\cos\theta}$ 1

(ii) $\frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta} = 2\sec^2\theta$ 2

e) Simplify $\frac{6^x + 2^x}{2^{2x}}$ 1

Question 3 (cont'd) **Marks**

f) For a certain series, the sum of the first n terms is given by $S_n = 2n^2 - 5n$

(i) Find an expression for the sum of the first $(n-1)$ terms. 1

(ii) Deduce an expression for the n^{th} term. 1

g) (i) Show that $y = \frac{4x}{x^2+16}$ is an odd function. 1

(ii) Find where the graph of the function crosses the co-ordinate axes. 1

(iii) Show the x axis is the horizontal asymptote. 1

(iv) Hence sketch the curve. 1

Question 4: (18 Marks) – Start A New Page **Marks**

a) Find the distance between the parallel lines $x - 2y + 1 = 0$ and $3x - 6y + 7 = 0$ 2

b) Simplify $\log_{\frac{1}{2}} 16^{-x}$ 2

c) On attached sheet (I), shade the region satisfying the following inequations simultaneously 3

$$\begin{cases} x \geq 1 \\ y \leq \sqrt{9 - x^2} \\ y \geq x^2 - 3x \end{cases}$$

d) Without using your calculator determine which of $12 - \sqrt{5}$ and $3 + 3\sqrt{5}$ is greater [show all working] 2

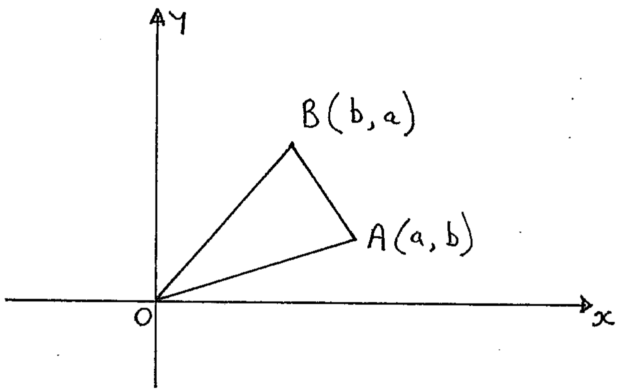
e) Use the principle of mathematical induction to prove for all positive integers n 3

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}$$

f) Find the equation of the circle with centre at the origin that has the line $3x - 4y = 30$ as a tangent. 3

Question 4 (cont'd) **Marks**

g)



ΔOAB has its vertices as $O(0,0)$, $A(a,b)$ and $B(b,a)$

(i) Show that ΔOAB is isosceles. 1

(ii) Find the midpoint of AB and the gradient of AB and use them to show that the equation of the perpendicular bisector of AB passes through O 2

End of Paper

SOLUTIONS ext. 1.

ST GEORGE G.H.S - 2011

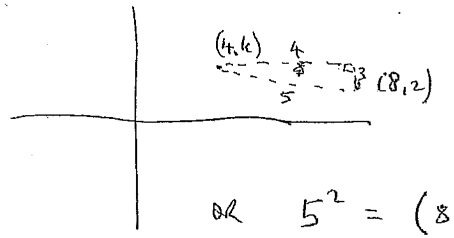
Question 1

a) $64^{x+1} \div 8^{x-1} = (2^6)^{x+1} \div (2^3)^{x-1}$

$= 2^{6x+6-3x+3}$

$= 2^{3x+9} \quad [\text{or } 8^{x+3}]$

b)



$k = 2+3$ or $k = 2-3$

$= 5 \qquad = -1$

OR $5^2 = (8-4)^2 + (2-k)^2$

$25 = 16 + (2-k)^2$

$9 = (2-k)^2$

$2-k = \pm 3$

$k = 2 \pm 3$

$= 5$ or -1

c) i) $f(x) = x^2 - 2x$

$f'(x) = \lim_{h \to 0} \dfrac{[(x+h)^2 - 2(x+h)] - (x^2 - 2x)}{h}$

$= \lim_{h \to 0} \dfrac{\cancel{x^2} + 2xh + h^2 - \cancel{2x} - 2h - \cancel{x^2} + \cancel{2x}}{h}$

$= \lim_{h \to 0} 2x + h - 2$

$= 2x - 2$

ii) $x^2 - 2x = 0$ when $x = 0$ or 2

at $x = 0 \quad f'(x) = -2$

$x = 2 \quad f'(x) = 2$

d) $x = \log_2 7$

$2^x = 7$

e) (i) $y = x\sqrt{x}$

$= x^{3/2}$

$\dfrac{dy}{dx} = \dfrac{3}{2} x^{1/2}$

(ii) $y = \dfrac{1-x}{(4+x)^2}$

$\dfrac{dy}{dx} = \dfrac{-1(4+x)^2 - (1-x)(2(4+x))}{(4+x)^4}$

$= \dfrac{-(4+x)^2 - 2(1-x)(4+x)}{(4+x)^4}$

$= \dfrac{-(4+x) - 2(1-x)}{(4+x)^3}$

$= \dfrac{-4 - x - 2 + 2x}{(4+x)^3}$

$= \dfrac{-6 + x}{(4+x)^3}$

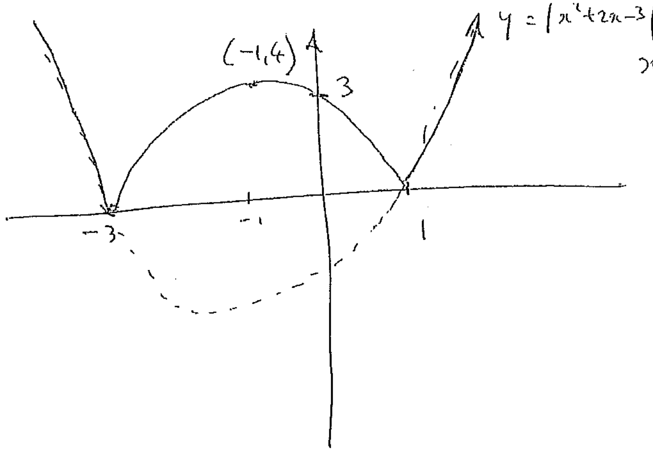
f) (i) $\cos(180^\circ + A) = -\cos A$

(ii) $\sin(90^\circ + A) = \sin(90^\circ - (-A))$

$= \cos(-A)$

$= \cos A$

3)



x^2+2x-3
$= (x+3)(x-1)$

h) $P(1,4)$ $\quad Q(-4,7)$

$k:1$

$x = \dfrac{1\times 1 + -4\times k}{k+1}$ $\qquad y = \dfrac{4\times 1 + 7\times k}{k+1}$

$= \dfrac{1-4k}{k+1}$ $\qquad = \dfrac{4+7k}{k+1}$

Question 2

a) $y = (4x+3)^5$

Let $u = 4x+3$

$\dfrac{du}{dx} = 4$

$y = u^5$

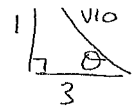
$\dfrac{dy}{du} = 5u^4$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= 5u^4 \times 4$

$= 20(4x+3)^4$

b) $\tan\theta = \frac{1}{3}$



$\sin\theta = \dfrac{1}{\sqrt{10}}$

As $180^\circ \leq \theta \leq 270$ $\qquad \sin\theta < 0$

$\therefore \sin\theta = -\frac{1}{\sqrt{10}}$

c) $P(2ap, ap^2)$ $\quad R(2ar, ar^2)$

(i) $M_{PR} = \dfrac{ar^2 - ap^2}{2ar - 2ap}$

$= \dfrac{a(r^2-p^2)}{2a(r-p)}$

$= \dfrac{(r-p)(r+p)}{2(r-p)}$ $\quad$ as $p \neq r$

$= \dfrac{r+p}{2}$

(ii) Equation of PR

$y - ap^2 = \frac{p+r}{2}(x - 2ap)$

$y - ap^2 = \left(\frac{p+r}{2}\right)x - ap(p+r)$

$y - \cancel{ap^2} = \left(\frac{p+r}{2}\right)x - \cancel{ap^2} - apr$

$y = \left(\frac{p+r}{2}\right)x - apr$

d) $y = x^{-2}$

$\dfrac{dy}{dx} = -2x^{-3}$

at $x=1$ $\quad \frac{dy}{dx} = -2$ $\qquad \therefore$ gradient normal $= \frac{1}{2}$

e) T_1 to T_6 $\quad S_6 = \frac{6}{2}(2a+5d)$

$45 = 3(2a+5d) \Rightarrow 15 = 2a+5d \quad (1)$

T_1 to T_{12} $\quad S_{12} = \frac{12}{2}(2a+11d)$

$-18 = 6(2a+11d) \Rightarrow -3 = 2a+11d \quad (2)$

$(2)-(1)$. $\quad -18 = 6d$

$\underline{d = -3} \qquad \underline{a = 15}$

f) (i) $3 \leq |2x+1| \leq 7$

$3 \leq 2x+1 \leq 7 \qquad -7 \leq 2x+1 \leq -3$

$2 \leq 2x \leq 6 \qquad -8 \leq 2x \leq -4$

$1 \leq x \leq 3 \qquad -4 \leq x \leq -2$

(ii) $\frac{4x}{3x-2} \geq 1 \qquad x \neq 2/3$

$(3x-2)^2 \frac{4x}{3x-2} \geq 1(3x-2)^2$

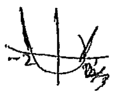
$4x(3x-2) \geq (3x-2)^2$

$0 \geq (3x-2)^2 - 4x(3x-2)$

$0 \geq (3x-2)(3x-2-4x)$

$(3x-2)(-2-x) \leq 0$

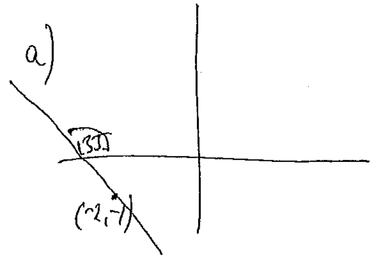
$(3x-2)(x+2) \geq 0$



$x \leq -2$ or $x > 2/3$

Question 3

a)



$m = \tan 135°$

$m = -1$

$y - -1 = -1(x - -2)$

$y+1 = -x-2$

$x+y+3 = 0$

b) (i) $\sin^3\theta + \sin\theta\cos^2\theta$

$= \sin\theta(\sin^2\theta + \cos^2\theta)$

$= \sin\theta$

(ii) $(1-\sin^2\alpha)(1-\tan^2\alpha)$

$= \cos^2\alpha \cdot \left(1 - \frac{\sin^2\alpha}{\cos^2\alpha}\right)$

$= \cos^2\alpha - \sin^2\alpha$

c) $|2x-7| = 3x+2$

$2x-7 = 3x+2 \qquad 2x-7 = -(3x+2)$

$-9 = x \qquad 2x-7 = -3x-2$

No solution $\qquad 5x = 5$

$\underline{x = 1}$

d) 1) $\sec\theta + \tan\theta = \frac{1+\sin\theta}{\cos\theta}$

LHS $= \sec\theta + \tan\theta$

$= \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$

$$= \frac{1+\sin\theta}{\cos\theta}$$

$= RHS$

ii) $LHS = \frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta}$

$$= \frac{1-\sin\theta}{1-\sin^2\theta} + \frac{1+\sin\theta}{1-\sin^2\theta}$$

$$= \frac{2}{\cos^2\theta}$$

$$= 2\sec^2\theta$$

$= RHS.$

e) $\frac{6^x+2^x}{2^{2x}} = \frac{3^x \cdot 2^x + 2^x}{2^x \cdot 2^x}$

$$= \frac{3^x+1}{2^x}$$

f) i) $S_{n-1} = 2(n-1)^2 - 5(n-1)$

$= (n-1)(2n-2-5)$

$= (n-1)(2n-7)$

ii) $T_n = S_n - S_{n-1}$

$= 2n^2-5n - [2n^2-9n+7]$

$= 4n-7$

g) i) $y = \frac{4x}{x^2+16}$

$x = a \quad y = \frac{4a}{a^2+16}$

$x = -a \quad y = \frac{4(-a)}{(-a)^2+16}$

$$= \frac{-4a}{a^2+16}$$

$\therefore$ ODD FUNCTION

ii) $y=0 \quad 4x=0$ only one intercept $x=0$

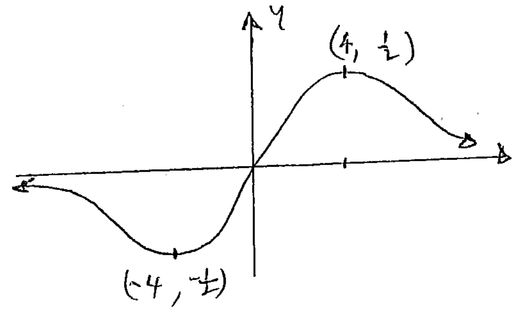
(iii) $x \to \infty \quad y = \frac{\quad}{x^2+16}$

$$= \frac{\frac{4}{x}}{1+\frac{16}{x^2}}$$

$y \to 0^+$

$x \to -\infty \quad y \to 0^-$

(iv)



QUESTION 4

a) $x-2y+1=0$

$3x-6y+7=0$

As $(1,1)$ lies on $x-2y+1=0$

$$d = \frac{|3(1)-6(1)+7|}{\sqrt{9+36}}$$

$$= \frac{4}{\sqrt{45}}$$

$$= \frac{4}{3\sqrt{5}}$$

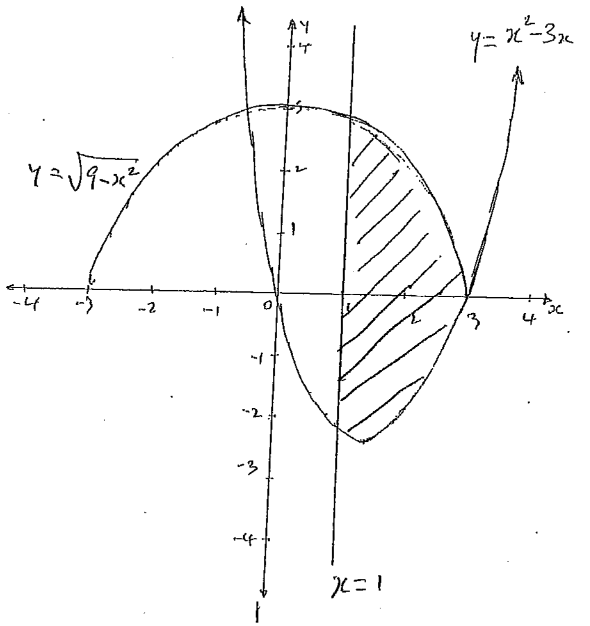
$$= \frac{4\sqrt{5}}{15}$$

b) $\log_{\frac{1}{2}} 16^{-x} = -x \log_{\frac{1}{2}} 16$

$= -x \log_{\frac{1}{2}} \left(\frac{1}{2}\right)^{-4}$

$= 4x$

c) Question 4 (c)



d) $12 - \sqrt{5} - (3 + 3\sqrt{5}) = 9 - 4\sqrt{5}$

$= \sqrt{81} - \sqrt{80}$

> 0

$\therefore 12 - \sqrt{5} > 3 + 3\sqrt{5}$

e) $1 + r + r^2 + \dots + r^{n-1} = \dfrac{1-r^n}{1-r}$

For $n=1$ LHS $=1$ RHS $= \dfrac{1-r}{1-r}$

Assume true for $n=k$.

$1 + r + r^2 + \dots + r^{k-1} = \dfrac{1-r^k}{1-r}$

For $n = k+1$

$1 + r + r^2 + \dots + r^{k-1} + r^k$

$= \dfrac{1-r^k}{1-r} + r^k$

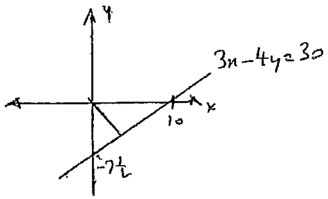
$= \dfrac{1 - r^k + r^k(1-r)}{1-r}$

$= \dfrac{1 - r^k + r^k - r^{k+1}}{1-r}$

$= \dfrac{1 - r^{k+1}}{1-r}$ as required

As true for $n=k+1$ when true for $n=k$, and as true for $n=1$ then true for all positive integers n.

f)



Gradient $3x - 4y = 30$ is $\frac{3}{4}$

$M_\perp$ is $-\frac{4}{3}$

Eq^n of normal $y = -\frac{4}{3}x$

$3y = -4x$

$3x - 4\left(-\frac{4}{3}x\right) = 30$

$9x + 16x = 90$

$x = \frac{18}{5}$

$\left(\frac{18}{5}, \frac{-24}{5}\right)$

$$d = \sqrt{\left(\frac{18}{5}\right)^2 + \left(-\frac{24}{5}\right)^2}$$

$$= \sqrt{\frac{324 + 576}{25}}$$

$$= \sqrt{\frac{900}{25}}$$

$$= \sqrt{36}$$

$= 6$ $\therefore$ Eq^n $x^2 + y^2 = 36$

or

$$d = \frac{|3(0) + 4(0) - 30|}{\sqrt{3^2 + (-4)^2}}$$

$$= \frac{|-30|}{5}$$

$= 6$ $\therefore$ $x^2 + y^2 = 36$

9) (i) $OA = \sqrt{a^2+b^2}$

$OB = \sqrt{b^2+a^2}$

$= OA$ $\therefore$ $\Delta\, OAB$ is isosceles

(ii) Midpoint AB $\left(\frac{a+b}{2}, \frac{a+b}{2}\right)$

$M_{AB} = \frac{b-a}{a-b}$

$= -1$

$M_\perp = 1$

$\therefore$ Eq^n $\perp$ through midpoint

$y - \frac{a+b}{2} = 1\left(x - \frac{a+b}{2}\right)$

$y = x$

which passes through origin