



Mathematics Extension 2

Time Allowed: 75 Minutes

Instructions to Candidates

1. All 3 questions may be attempted.
2. All necessary working must be shown.
3. Start each question on a new page.

Question 1 – (20 marks) – Start a New Page

Marks

a) Find these integrals:

8

(i) $\int \cot x \operatorname{cosec}^2 x \, dx$

(ii) $\int \frac{dx}{\sqrt{4x^2-9}}$

(iii) $\int \sec^6 x \, dx$

(iv) $\int \frac{4x-1}{x^2-4} \, dx$

b) (i) Evaluate $\int_0^1 u e^u \, du$

4

and (ii) hence, using a suitable substitution, evaluate $\int_0^1 x^5 e^{x^3} \, dx$

c) Find the value of $\int_0^{\frac{\pi}{2}} \frac{dx}{5+4 \cos x}$, using the 't' method.

4

d) (i) Express $\frac{7t}{(3-t)(3t-2)}$ as the sum of its partial fractions.

4

and (ii) hence, find $\int \frac{7t}{-3t^2+11t-6} \, dt$

Question 2 - (20 marks) Polynomials – Start a New Page

Marks

- a) (i) If α, β and γ are the roots of the cubic equation $x^3 + mx + n = 0$, find in terms of m and n , the values of:

1. $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

3

2. $\alpha^2 + \beta^2 + \gamma^2$

2

- (ii) Determine the cubic equation whose roots are α^2, β^2 and γ^2

3

- b) The polynomial $P(x)$ has equation $P(x) = x^4 - 2x^3 - x^2 + 2x + 10$

4

Given that $x - 2 + i$ is a factor of $P(x)$, express $P(x)$ as a product of two real quadratic functions.

- c) Given that $Q(x) = x^4 - 5x^3 + 4x^2 + 3x + 9$ has a zero of multiplicity 2, solve the equation $Q(x) = 0$ over the complex field.

4

~~$x < 0$~~ or < 0

- d) Prove that $2x + \frac{x^2}{2} - \frac{2x^3}{3} - \frac{x^4}{4} + e = 0$ has no real roots if $e < \frac{-19}{12}$

4

How many real roots are there if $e \leq \frac{-19}{12}$?

Question 3 - (20 marks) – Start a New Page

Marks

- a) Using suitable substitutions, or otherwise, find:

4

(i) $\int \frac{e^{3x}}{\sqrt{4-e^{6x}}} dx$

(ii) $\int \sec^9 x \tan x dx$

- b) (i) By considering the expansion of $(\cos \theta + i \sin \theta)^3$ and by using De Moivre's Theorem show that $\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta$ and $\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta$

2

(ii) Show that $\tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$

1

- (iii) Find the general solution of the equation $\tan 3\theta = \sqrt{3}$

2

- (iv) Use the substitution $x = \tan \theta$ to express the equation $\frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} = \sqrt{3}$ as a polynomial in terms of x

1

- (v) Hence, show that $\tan \frac{\pi}{9} + \tan \frac{4\pi}{9} + \tan \frac{7\pi}{9} = 3\sqrt{3}$

2

- (vi) Find the polynomial of least degree that has zeros

2

$(\cot \frac{\pi}{9})^2, (\cot \frac{4\pi}{9})^2, (\cot \frac{7\pi}{9})^2$

- c) (i) Show that $\int \sec x dx = \ln(\sec x + \tan x) + c$

4

- (ii) Use the method of integration by parts to find: $\int \sec^3 x dx$

- d) Find $\int \sqrt{\frac{2+x}{3-x}} dx$

2

SOLUTIONS.

QUESTION I

(a) (i) $\int \cot x \operatorname{cosec}^2 x \, dx$

$u = \cot x$
 $du = -\operatorname{cosec}^2 x \, dx$

$$= \int u \cdot (-1) \, du$$

$$= -\frac{u^2}{2} + c$$

$$= -\frac{1}{2} \cot^2 x + c$$

(ii) $\int \frac{dx}{\sqrt{4x^2-9}}$

$$= \int \frac{dx}{2\sqrt{x^2-\frac{9}{4}}}$$

$$= \frac{1}{2} \log \left| x + \sqrt{x^2 - \frac{9}{4}} \right| + c$$

(iii) $\int \sec^6 x \, dx$

$$= \int \sec^2 x \cdot \sec^4 x \, dx$$

$$= \int \sec^2 x \cdot (\tan^2 x + 1)^2 \, dx \quad \text{let } u = \tan x$$

$$du = \sec^2 x \, dx$$

$$= \int (u^2 + 1)^2 \, du$$

$$= \int (u^4 + 2u^2 + 1) \, du$$

$$= \frac{u^5}{5} + \frac{2u^3}{3} + u + c$$

$$= \frac{1}{5} \tan^5 x + \frac{2}{3} \tan^3 x + \tan x + c$$

(iv) $\int \frac{4x-1}{(x+2)(x-2)} \, dx$

$$= \int \left(\frac{\frac{9}{4}}{x+2} + \frac{\frac{7}{4}}{x-2} \right) dx$$

$$= \frac{9}{4} \ln|x+2| + \frac{7}{4} \ln|x-2| + c$$

$$\text{let } \frac{4x-1}{(x+2)(x-2)} = \frac{a}{x+2} + \frac{b}{x-2}$$

$$\Rightarrow 4x-1 = a(x-2) + b(x+2)$$

$$x=2 \Rightarrow 7 = 4b$$

$$b = \frac{7}{4}$$

$$x=-2 \Rightarrow -9 = -4a$$

$$a = \frac{9}{4}$$

(b) (i) $\int_0^1 \underbrace{u}_{p} \underbrace{e^u}_{\frac{dq}{du}} \, du$

$$= pq - \int q \frac{dp}{dx} \, dx$$

$$= [ue^u]_0^1 - \int_0^1 e^u \, du$$

$$= [ue^u - e^u]_0^1$$

$$= (e - e) - (0 - 1)$$

$$= 1$$

(ii) $\int_0^1 x^5 e^{x^3} \, dx$

$$= \frac{1}{3} \int_0^1 3x^2 \cdot x^3 \cdot e^{x^3} \, dx$$

let $u = x^3$
 $du = 3x^2 \, dx$

$$= \frac{1}{3} \int_0^1 u e^u \, du$$

$$= \frac{1}{3} \cdot 1$$

$$= \frac{1}{3}$$

(c) $\int_0^{\frac{\pi}{4}} \frac{dx}{5+4\cos x}$

$t = \tan \frac{x}{2}$
 $dx = \frac{2dt}{1+t^2}$

$$= \int_0^1 \frac{2dt}{5+4 \cdot \frac{1-t^2}{1+t^2}}$$

$$= \int_0^1 \frac{2dt}{5(1+t^2)+4-4t^2}$$

$$= \int_0^1 \frac{2dt}{9+t^2}$$

$$= \frac{2}{3} \tan^{-1} \left(\frac{t}{3} \right) \Big|_0^1$$

3.

$$(d) \quad (i) \quad \text{let } \frac{7t}{(3-t)(3t-2)} \equiv \frac{a}{3-t} + \frac{b}{3t-2}$$

$$\Rightarrow 7t = a(3t-2) + b(3-t)$$

$$t=3 \Rightarrow 21 = 7a \\ a = 3$$

$$\text{Constant term} \Rightarrow 0 = -2a + 3b \\ = -6 + 3b \\ b = 2$$

$$\therefore \frac{7t}{(3-t)(3t-2)} = \frac{3}{3-t} + \frac{2}{3t-2}$$

$$(ii) \quad \int \frac{7t}{(3-t)(3t-2)} dt = \int \frac{3}{3-t} dt + \int \frac{2}{3t-2} dt \\ = -3 \ln|3-t| + \frac{2}{3} \ln|3t-2| + c$$

QUESTION 2:

$$(a) \quad (i) \quad \frac{1}{x} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$$

$$= \frac{m}{-n}$$

$$= -\frac{m}{n}$$

$$\therefore x^2 + \beta^2 + \gamma^2 = (x + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$$

$$= 0 - 2(m)$$

$$= -2m$$

$$(ii) \quad y = x^2 \Rightarrow x = \pm\sqrt{y}$$

$$\therefore P(\sqrt{x}) = 0 \text{ has roots } \alpha^2, \beta^2, \gamma^2$$

$$\text{i.e. } (\sqrt{x})^3 + m\sqrt{x} = -n$$

$$x\sqrt{x} + m\sqrt{x} = -n$$

$$\sqrt{x}(x+m) = -n$$

$$\Rightarrow x(x^2 + 2mx + m^2) = -n^2$$

$$\text{i.e. } x^3 + 2mx^2 + m^2x - n^2 = 0$$

4.

$$(b) \quad P(x) = x^4 - 2x^3 - x^2 + 2x + 10$$

$x = 2-i$ is a zero $\Rightarrow x = 2+i$ is a zero as all co-efficients are real

\therefore One factor is $x^2 - [(2-i) + (2+i)]x + (2-i)(2+i)$

$$\text{i.e. } x^2 - 4x + 5$$

Hence $P(x) = (x^2 - 4x + 5)(x^2 + kx + 2)$ for some k ①

Equating co-efficients of x^3 LHS/RHS of ①

$$\Rightarrow -2 = k - 4$$

$$\therefore k = 2$$

$$\therefore P(x) = (x^2 - 4x + 5)(x^2 + 2x + 2)$$

$$(c) \quad Q(x) = x^4 - 5x^3 + 4x^2 + 3x + 9 \quad \alpha, \alpha, \beta, \delta$$

$$Q'(x) = 4x^3 - 15x^2 + 8x + 3 \quad \alpha, \delta, \epsilon$$

$$Q'(1) = 4 - 15 + 8 + 3$$

$$= 0 \Rightarrow x-1 \text{ is a factor of } Q'(x)$$

$$\therefore Q'(x) = (x-1)(4x^2 + kx - 3)$$

$$\text{co-eff of } x \Rightarrow 8 = -3 - k$$

$$\therefore k = -11$$

$$\therefore Q'(x) = (x-1)(4x^2 - 11x - 3)$$

$$= (x-1)(x-3)(4x+1)$$

$$\text{now } Q(1) = 1 - 5 + 4 + 3 + 9 \neq 0$$

$$Q(3) = 81 - 135 + 36 + 9 + 9$$

$$= 0$$

Hence $x=3$ is the root of multiplicity 2

$$\text{Hence } Q(x) = (x-3)^2(x^2 + kx + 1)$$

$$\text{co-eff of } x \Rightarrow 3 = -6 + 9k$$

$$k = 1$$

$$\therefore Q(x) = (x-3)^2(x^2 + x + 1)$$

$$Q(x) = (x-3)^2 \left[(x+\frac{1}{2})^2 - \frac{3}{4}i^2 \right]$$

$$= (x-3)^2 \left(x+\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) \left(x+\frac{1}{2} - \frac{\sqrt{3}}{2}i \right)$$

Hence $Q(x) = 0$

$$\Rightarrow x = 3, 3, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

(d) Consider $y = 2x + \frac{x^2}{2} - \frac{2x^3}{3} - \frac{x^4}{4} + e$ ——— ①

$$\frac{dy}{dx} = 2 + x - 2x^2 - x^3$$

Stationary points at $\frac{dy}{dx} = 0$

$$\text{i.e. } x^3 + 2x^2 - x - 2 = 0$$

$$x^2(x+2) - 1(x+2) = 0$$

$$(x+2)(x+1)(x-1) = 0$$

$$x = -2, -1, 1$$

$$\frac{d^2y}{dx^2} = 1 - 4x - 3x^2$$

$$x = -2 \Rightarrow \frac{d^2y}{dx^2} = 1 + 8 - 12$$

$$< 0 \Rightarrow \text{rel. max at } x = -2$$

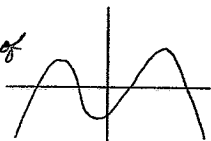
$$x = -1 \Rightarrow \frac{d^2y}{dx^2} = 1 + 4 - 3$$

$$> 0$$

$$x = 1 \Rightarrow \frac{d^2y}{dx^2} = 1 - 4 - 3$$

$$< 0 \Rightarrow \text{rel. max. at } x = 1$$

Since this graph has a basic shape of



we need only check the y -values of the relative maximum turning points

$$\text{when } x = -2, y = -4 + 2 + \frac{16}{3} - 4 + e$$

$$= -\frac{2}{3} + e$$

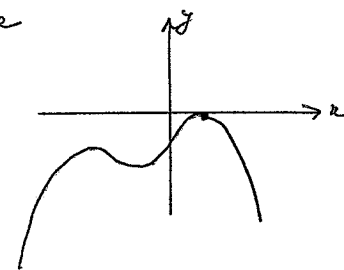
$$x = 1, y = 2 + \frac{1}{2} - \frac{2}{3} - \frac{1}{4} + e$$

$$= \frac{19}{12} + e$$

There are no real roots if $\frac{19}{12} + e < 0$

$$\text{i.e. } e < -\frac{19}{12}$$

If $e = -\frac{19}{12}$ graph has the shape



Hence there is a double root at $x = 1$ for $e = -\frac{19}{12}$

and no real roots for $e < -\frac{19}{12}$

QUESTION 3:

$$(a) (i) \int \frac{e^{3x}}{\sqrt{4 - e^{6x}}} dx \quad \text{let } u = e^{3x}$$

$$du = 3e^{3x}$$

$$= \frac{1}{3} \int \frac{3e^{3x}}{\sqrt{4 - e^{6x}}} dx$$

$$= \frac{1}{3} \int \frac{du}{\sqrt{4 - u^2}}$$

$$= \frac{1}{3} \sin^{-1}\left(\frac{u}{2}\right) + C$$

$$= \frac{1}{3} \sin^{-1}\left(\frac{e^{3x}}{2}\right) + C$$

$$\begin{aligned}
 \text{(ii)} \quad & \int \sec^9 x \tan x \, dx \\
 &= \int \sec^8 x \cdot \sec x \tan x \, dx \quad u = \sec x \\
 & \quad \quad \quad du = \sec x \tan x \, dx \\
 &= \int u^8 \, du \\
 &= \frac{u^9}{9} + c \\
 &= \frac{1}{9} \sec^9 x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) (i)} \quad & (\cos \theta + i \sin \theta)^3 = \cos^3 \theta + 3\cos^2 \theta (i \sin \theta) \\
 & \quad + 3\cos \theta (i \sin \theta)^2 + (i \sin \theta)^3 \\
 &= \cos^3 \theta + (3\cos^2 \theta \sin \theta) i \\
 & \quad - 3\cos \theta \sin^2 \theta - (\sin^3 \theta) i \\
 &= (\cos^3 \theta - 3\cos \theta \sin^2 \theta) \\
 & \quad + i(3\cos^2 \theta \sin \theta - \sin^3 \theta) \quad \text{--- (1)} \\
 &= \cos 3\theta + i \sin 3\theta \quad \text{--- (2)}
 \end{aligned}$$

by De Moivre's theorem

equating real parts of (1) and (2)

$$\Rightarrow \cos 3\theta = \cos^3 \theta - 3\cos \theta \sin^2 \theta \quad \text{--- (3)}$$

equating imaginary parts of (1) and (2)

$$\Rightarrow \sin 3\theta = 3\cos^2 \theta \sin \theta - \sin^3 \theta \quad \text{--- (4)}$$

$$\begin{aligned}
 \text{(ii)} \quad \text{Hence} \quad \tan 3\theta &= \frac{3\cos^2 \theta \sin \theta - \sin^3 \theta}{\cos^3 \theta - 3\cos \theta \sin^2 \theta} \div \frac{\cos \theta}{\cos \theta} \\
 &= \frac{3\tan^2 \theta - \tan^3 \theta}{1 - 3\tan^2 \theta} \quad \text{--- (5)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad \tan 3\theta &= \sqrt{3} \\
 \Rightarrow 3\theta &= \frac{\pi}{3} + n\pi \quad n \text{ integer} \\
 &= \frac{\pi(1+3n)}{3} \\
 \therefore \theta &= \frac{\pi}{9}(3n+1)
 \end{aligned}$$

(iv) let $x = \tan \theta$ in (5)

$$\begin{aligned}
 \therefore \frac{3x - x^3}{1 - 3x^2} &= \sqrt{3} \\
 \text{ie } 3x - x^3 &= \sqrt{3} - 3\sqrt{3}x^2 \\
 \text{ie } x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} &= 0 \quad \text{--- (6)}
 \end{aligned}$$

(v) The solutions of $\frac{3\tan \theta - \tan^3 \theta}{1 - 3\tan^2 \theta} = \sqrt{3}$

$$\text{are } \theta = \frac{\pi}{9}(3n+1)$$

$$= \frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{10\pi}{9}, \frac{13\pi}{9}, \dots$$

Then the solutions of (6) are

$$x = \tan \frac{\pi}{9}, \tan \frac{4\pi}{9}, \tan \frac{7\pi}{9}, \tan \frac{\pi}{9}, \tan \frac{4\pi}{9}, \dots$$

$$\text{ie } x = \tan \frac{\pi}{9}, \tan \frac{4\pi}{9} \text{ and } \tan \frac{7\pi}{9}$$

$$\sum \text{roots in (6)} \Rightarrow \tan \frac{\pi}{9} + \tan \frac{4\pi}{9} + \tan \frac{7\pi}{9} = 3\sqrt{3}$$

$$\text{(vi)} \quad y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$$

$$P(x) = x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0$$

$$\text{Then } P\left(\frac{1}{y}\right) = \frac{1}{y^3} - 3\sqrt{3} \cdot \frac{1}{y} - \frac{3}{y} + \sqrt{3} = 0$$

$$\text{ie } 1 - 3\sqrt{3} \cdot \sqrt{y} - 3x + \sqrt{3}x\sqrt{y} = 0$$

$$\sqrt{y}(\sqrt{3}x - 3\sqrt{3}) = 3x - 1$$

$$\text{ie } x(3x^2 - 18x + 27) = 9x^2 - 6x + 1$$

$$\text{ie } 3x^3 - 27x^2 + 33x - 1 = 0$$

$$\begin{aligned} \text{(c) (i) } \int \sec x \, dx &= \int \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} \, dx \\ &= \ln |\sec x + \tan x| + c \end{aligned}$$

$$\begin{aligned} \text{(ii) } I &= \int \sec^3 x \, dx \\ &= \int \underbrace{\sec x}_u \cdot \underbrace{\sec^2 x}_{dv} \, dx \\ &= \sec x \tan x - \int \sec x \tan x \cdot \tan x \, dx \\ &= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx \\ &= \sec x \tan x - I + \int \sec x \, dx \\ \therefore 2I &= \sec x \tan x + \ln |\sec x + \tan x| + c \\ \therefore I &= \frac{1}{2} [\sec x \tan x + \ln |\sec x + \tan x|] + c \end{aligned}$$

$$\begin{aligned} \text{(d) } \int \sqrt{\frac{2+x}{3-x}} \, dx &= \int \frac{\sqrt{2+x}}{\sqrt{3-x}} \cdot \frac{\sqrt{2+x}}{\sqrt{2+x}} \, dx \\ &= \int \frac{2+x}{\sqrt{6+x-x^2}} \, dx \\ &= \frac{-1}{2} \int \frac{1-2x}{\sqrt{6+x-x^2}} \, dx + \frac{5}{2} \int \frac{dx}{\sqrt{\frac{25}{4} - (x-\frac{1}{2})^2}} \end{aligned}$$

$$= -\frac{1}{2} \int \frac{du}{\sqrt{u}} + \frac{5}{2} \sin^{-1} \left(\frac{x-\frac{1}{2}}{\frac{5}{2}} \right) + c$$

$$= -\frac{1}{2} \cdot 2\sqrt{u} + \frac{5}{2} \sin^{-1} \left(\frac{2x-1}{5} \right) + c$$

$$= -\sqrt{6+x-x^2} + \frac{5}{2} \sin^{-1} \left(\frac{2x-1}{5} \right) + c$$