St George Girls' High School

Trial Higher School Certificate Examination

2003

Mathematics Extension 1

Total Marks – 84

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Attempt ALL questions.
- Begin each question on a new page
- Write your student number on each page
- All necessary working must be shown.
- Diagrams are not to scale.
- Board-approved calculators may be used.
- The mark allocated for each question is listed at the side of the question.

Question	Mark
Q1	/12
Q2	/12
Q3	/12
Q4	/12
Q5	/12
Q6	/12
Q7	/12
Total	/84

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Question 1 – (12 marks) – Start a new page **Marks**

a) Find the exact value of $\int_2^3 \frac{x^2}{x^3-7}\,dx$ 2

b) Solve for x: $\frac{2}{x-1} \leq 1$ 3

c) $P(19, -15)$ is the point which divides the line interval 'AB' externally in the ratio 3:2. Find the coordinates of $B(x, y)$ given $A(-2, 3)$. 3

d) (i) Find $\frac{d}{dx}(\tan^{-1}x + x)$

(ii) Hence, evaluate $\int_0^1 \frac{x^2+2}{x^2+1}\,dx$ 4

(leave in <u>exact</u> form).

Question 2 – (12 marks) – Start a new page **Marks**

a) The equation $x^3 - mx + 2 = 0$ has two of its roots equal. 4

(i) Write down expressions for the sum of the roots and for the product of the roots.

(ii) Hence, find the value of m.

b) The polynomial equation $8x^3 - 36x^2 + 22x + 21 = 0$ has roots which form an arithmetic progression. 3

Find the roots of the polynomial.

c) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola with the equation $x^2 = 4ay$.

It is given that the chord PQ has equation $y = \left(\frac{p+q}{2}\right)x - apq$ 3

(i) Show that the gradient of the tangent at P is p.

(ii) Prove that if PQ passes through the focus, then the tangent at P is parallel to the normal at Q.

d) (i) Write down the equation for the inverse function of $y = 2^x$, write your response with y as the subject. 2

(ii) Write down the domain of the inverse function from part (i).

Question 3 – (12 marks) – Start a new page **Marks**

a) Find the term independent of x in the expansion of $\left(x-\frac{2}{x^3}\right)^{12}$ 3

b) Find the greatest coefficient in the expansion of $(2+3x)^{14}$ 4

c) (i) Show that $\sqrt{12}\sin x+2\cos x\equiv 4\cos\left(x-\frac{\pi}{3}\right)$ 5

(ii) Hence, solve the equation $\sqrt{12}\sin x+2\cos x=-2\sqrt{2}$ for $0\le x\le 2\pi$

[Give all answers correct to two decimal places]

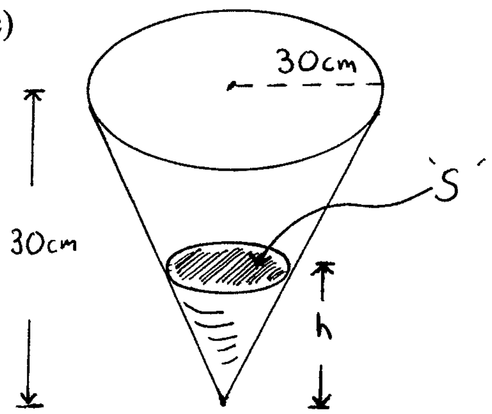
<u>Question 4</u> – (12 marks) – Start a new page **Marks**

a) The region bounded by the curve $y = \sin x$, the x-axis and the lines $x = \frac{\pi}{12}$ and $x = \frac{\pi}{4}$ is rotated through one complete revolution about the x-axis. Find the volume of the solid so formed. 4

[Give your answer in terms of π].

b) Use Mathematical induction to show that the expression $7^n + 5$ is divisible by 6 for all positive integers n.

c)



Water is poured into a conical vessel at a constant rate of 24cm^3 per second. The depth of water is h cm at any time t seconds. 4

What is the rate of increase of the area of the surface 'S' of the water when the depth is 16cm?

[NOT TO SCALE]

<u>Question 5</u> – (12 marks) – Start a new page **Marks**

a) Newton's Law of Cooling states that when an object at temperature $T°$ is placed in an environment at a temperature of $R°$, then the rate of temperature loss is given by the equation

$$\frac{dT}{dt} = k(T - R)$$

where t is the time in seconds and k is a constant.

(i) Show that $T = R + Ae^{kt}$ is a solution to the equation. 1

(ii) A packet of peas, initially at 24°C is placed in a snap-freeze refrigerator in which the internal temperature is maintained at –40°C.

After 5 seconds the temperature of the packet is 19°C. How long will it take for the packet's temperature to reduce to 0°C? 3

b) Consider the function $y = \log_e\left(\frac{2x}{2+x}\right)$

(i) Show that the domain of the function is: $x < -2, \ x > 0$ 2

(ii) Find the value of x for which $y = 0$ 1

(iii) Show that $\frac{dy}{dx} = \frac{2}{x(2+x)}$ and hence show that the function is increasing for all x in the domain. 2

(iv) ~~Find any possible points of inflexion.~~ Find value(s) of x for which $\frac{d^2y}{dx^2} = 0$. 1

(v) Find $\lim_{x \to \infty} \left[\log_e\left(\frac{2x}{2+x}\right)\right]$ 1

(vi) Sketch the graph of the function. 1

Question 6 – (12 marks) – Start a new page **Marks**

a) By noting that $(1+x)^n = \sum_{r=0}^{n} {}^nC_r \; x^r$,

prove that

(i) $\sum_{r=0}^{n} {}^nC_r = 2^n$ 1

(ii) $\sum_{r=1}^{n} r.\,{}^nC_r = n.2^{n-1}$ 2

b) Evaluate $\int_1^3 \frac{dx}{(1+x)\sqrt{x}}$ using the substitution $u=\sqrt{x}$, give the EXACT value. 4

c) A particle moving in Simple Harmonic Motion starts from rest at a distance 10 metres to the right of its centre of oscillation O. The period of the motion is 2 seconds.

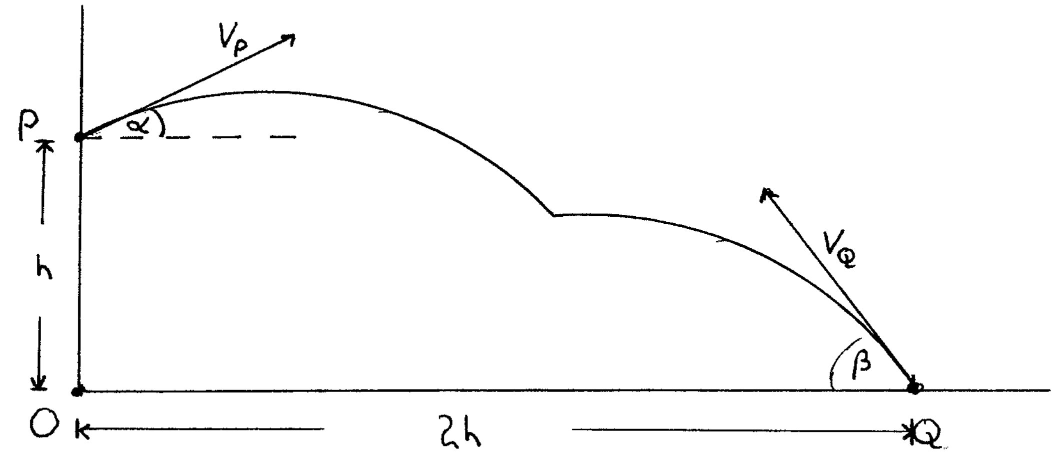
(i) Find the speed of the particle when it is 4 metres from its starting point. 5

(ii) Find the time taken by the particle to first reach the point 4 metres from its starting point, in seconds correct to two decimal points.

Question 7 – (12 marks) – Start a new page **Marks**

a) O and Q are two points $2h$ metres apart on horizontal ground. P is a point h metres directly above O. 6

A particle is projected from P towards Q with speed $V_P\ ms^{-1}$ at an angle 'α' above the horizontal. At the same time another particle is projected from Q towards P with speed $V_Q\ ms^{-1}$ at an angle of 'β' above the horizontal. The two particles collide 'T' seconds after projection.



(i) For the projectile travelling from P towards Q the equations of motion are $\ddot{x} = 0$ and $\ddot{y} = -g$.

Use calculus to show that at time t seconds, its horizontal distance x_P from O and its vertical height y_P from O are given by $x_P = (V_P \cos\alpha)t$ and $y_P = (V_P \sin\alpha)t - \frac{1}{2}gt^2 + h$

(ii) For the particle going from Q towards P, write down expressions for the horizontal distance x_Q from Q and its vertical height y_Q from Q at time t seconds.

(iii) Hence, show that $\dfrac{V_P}{V_Q} = \dfrac{2\sin\beta - \cos\beta}{2\sin\alpha + \cos\alpha}$

Question 7 (cont'd) **Marks**

b) (i) Write down an expression for $\sin(x-y)$ 6

(ii) If $\sin\alpha = c$ and $\sin(60^\circ - \alpha) = d$, prove that $c^2 + cd + d^2 = \frac{3}{4}$

(iii) If ΔABC is equilateral and D is any point on the side BC and if a and b are the lengths of the perpendiculars from D to AB and AC respectively, prove

$$AD = \frac{2}{\sqrt{3}}\sqrt{a^2 + ab + b^2}$$

EXTENSION 1 – SOLUTIONS.

QUESTION 1: (12 MARKS)

(a) $\displaystyle\int_2^3 \frac{x^2}{x^3-7}\,dx = \frac{1}{3}\Big[\ln(x^3-7)\Big]_2^3$

$= \frac{1}{3}(\ln 20 - \ln 1)$

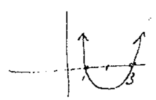
$= \frac{1}{3}\ln 20$ —

(b) $\dfrac{2}{x-1} \leq 1 \qquad x \neq 1$

$2(x-1) \leq (x-1)^2$

$0 \leq (x-1)^2 - 2(x-1)$

$0 \leq (x-1)(x-3)$



$\Rightarrow \quad x \leq 1 \quad \underline{\text{OR}} \quad x \geq 3$

but $x \neq 1$

$\therefore x < 1 \quad \underline{\text{OR}} \quad x \geq 3$ — 3

(c) $\underline{A(-2,3) \qquad\qquad B(x,y)}$

$3 : -2$

Hence $(19,-15) = \left(\dfrac{3x+4}{1}, \dfrac{3y-6}{1}\right)$

$\Rightarrow \left.\begin{array}{l} x = 5 \\ y = -3 \end{array}\right\}$ — 3

(d) (i) $\dfrac{d}{dx}(\tan^{-1}x + x) = \dfrac{1}{1+x^2} + 1$

d.

(ii) $\displaystyle\int_0^1 \left(\frac{x^2+2}{x^2+1}\right)dx = \int_0^1\left(\frac{x^2+1}{x^2+1} + \frac{1}{x^2+1}\right)dx$

$\displaystyle = \int_0^1\left(1 + \frac{1}{x^2+1}\right)dx$

$= \Big[x + \tan^{-1}x\Big]_0^1$

$= (1 + \tan^{-1}1) - (0+0)$

$= 1 + \frac{\pi}{4}$

— 4

QUESTION 2: (12 MARKS)

(a) Let roots be α, α, β — 4 MARKS.

(i) $2\alpha + \beta = 0$ — ①

$\alpha^2\beta = -2$ — ②

(ii) from ①, $\beta = -2\alpha$ and in ②

$\alpha^2 \times (-2\alpha) = -2$

$\alpha^3 = 1$

$\left.\begin{array}{l} \alpha = 1 \\ \therefore \beta = -2 \end{array}\right\}$

ie Roots are $1, 1, -2$

now $\Sigma\alpha\beta \Rightarrow \alpha^2 + 2\alpha\beta = -m$

ie $1 - 4 = -m$

$\therefore \underline{m = 3}$

(b) Let roots be $\alpha - d, \alpha, \alpha + d$ — 3 MARKS

$\Sigma\alpha \Rightarrow (\alpha - d) + \alpha + (\alpha + d) = \frac{36}{8}$

$3\alpha = \frac{36}{8}$

$\alpha = \frac{3}{2}$

$\Sigma\alpha\beta \Rightarrow \frac{3}{2}\left(\frac{3}{2}-d\right)+\left(\frac{3}{2}-d\right)\left(\frac{3}{2}+d\right)+\frac{3}{2}\left(\frac{3}{2}+d\right)=\frac{11}{4}$

$\frac{9}{4}-\frac{3d}{2}+\frac{9}{4}-d^2+\frac{9}{4}+\frac{3}{2}d = \frac{11}{4}$

ie $\frac{27}{4}-d^2 = \frac{11}{4}$

$27-4d^2 = 11$

$4d^2 = 16$

$d^2 = 4$

$d = 2, -2$

$d=2 \Rightarrow$ roots are $-\frac{1}{2}, \frac{3}{2}, \frac{7}{2}$

$d=-2 \Rightarrow$ roots are $\frac{7}{2}, \frac{3}{2}, -\frac{1}{2}$

ie Roots are $-\frac{1}{2}, \frac{3}{2}, \frac{7}{2}$

——— 3 MARKS.

(c) (i) $x^2 = 4ay$

$\Rightarrow y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

at $P(2ap, ap^2)$: $\frac{dy}{dx} = \frac{2ap}{2a}$

$= p$

(ii) PQ through focus $(0, a)$

$\Rightarrow a = 0 - apq$

ie $pq = -1$

normal at Q has gradient $-\frac{1}{q}$

tangent at Q has gradient q

$\therefore pq=-1 \Rightarrow$ tangent at P $\perp$ tangent at Q

ie tangent at P // normal at Q

(d) (i) $f: y = 2^x$ D: all real x

R: $y > 0$

$f^{-1}: x = 2^y$

$\therefore \log_2 x = y$

ie $f^{-1}: y = \log_2 x$

(ii) D: $x > 0$

D: $x > 0$

R: all real y

QUESTION 3: (12 MARKS)

(a) $\left(x-\frac{2}{x^3}\right)^{12}$: $T_{k+1} = {}^{12}C_k\, x^{12-k}\left(-\frac{2}{x^3}\right)^k$

$= {}^{12}C_k \cdot x^{12-k}\cdot(-2)^k\cdot x^{-3k}$

$= {}^{12}C_k\,(-2)^k\cdot x^{12-4k}$

Independent of $x \Rightarrow 12-4k=0$

$k=3$

$\therefore$ Term is $T_4 = {}^{12}C_3\,(-2)^3$

$= -1\ 760$ ——— 3 MARKS.

(b) $(2+3x)^{14}$, $T_{k+1} = {}^{14}C_k\cdot 2^{14-k}\cdot(3x)^k$

$\therefore T_k = {}^{14}C_{k-1}\cdot 2^{14-(k-1)}\cdot(3x)^{k-1}$

$= {}^{14}C_{k-1}\cdot 2^{15-k}\cdot(3x)^{k-1}$

Then co-efficients P_k, P_{k+1}

$\Rightarrow \frac{P_{k+1}}{P_k} = \frac{{}^{14}C_k\, 2^{14-k}\cdot 3^k}{{}^{14}C_{k-1}\, 2^{15-k}\, 3^{k-1}}$

$= \frac{\cancel{14!}}{(14-k)!\,k!}\times\frac{(15-k)!\,(k-1)!}{\cancel{14!}}\times\frac{3}{2}$

$= \frac{(15-k)\times 3}{2k}$

$= \frac{45-3k}{2k}$

Then $\dfrac{P_{k+1}}{P_k} > 1 \quad \Rightarrow \quad P_{k+1} > P_k$

Thus $\dfrac{45-3k}{2k} > 1 \quad \Rightarrow \quad 45 > 5k$

$k < 9 \quad \Rightarrow \quad P_{k+1} > P_k$

ie $P_9 > P_8 \ldots\ldots P_3 > P_2 > P_1$

$\therefore$ P_9 is largest co-efficient

and $P_9 = {}^{14}C_8 \cdot 2^6 \cdot 3^8$

$= 1\,260\,971\,712$ —— 4 MARKS.

(c) (i) $4\cos\left(x - \frac{\pi}{3}\right) = 4\left[\cos x \cos\frac{\pi}{3} + \sin x \sin\frac{\pi}{3}\right]$

$= 4\left[\cos x \cdot \frac{1}{2} + \sin x \cdot \frac{\sqrt{3}}{2}\right]$

$= 2\cos x + \sqrt{12}\sin x$

(ii) $\sqrt{12}\sin x + 2\cos x = -2\sqrt{2}$

$\Rightarrow 4\cos\left(x - \frac{\pi}{3}\right) = -2\sqrt{2}$

$\cos\left(x - \frac{\pi}{3}\right) = \dfrac{-\sqrt{2}}{2}$

$0 \le x \le 2\pi$

$\Rightarrow -\frac{\pi}{3} \le x - \frac{\pi}{3} \le \frac{5\pi}{3}$

$\left(x - \frac{\pi}{3}\right)_{\text{acute}} = \frac{\pi}{4}$

$\therefore \quad x - \frac{\pi}{3} = \frac{3\pi}{4}, \frac{5\pi}{4}$

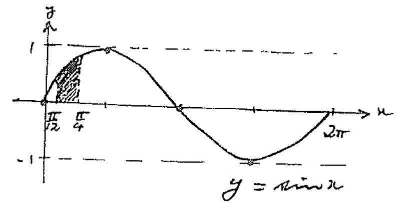
$\therefore x = \frac{13\pi}{12}, \frac{19\pi}{12}$

$= 3.403\ldots, \; 4.974\ldots$

$= 3.40, \; 4.97$ (correct to 2 dec places)

QUESTION 4: (12 MARKS)

(a)



—— 4 MARKS

$$V = \pi\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} y^2\,dx$$

$$= \pi\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \sin^2 x\,dx$$

$$= \frac{\pi}{2}\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} (1 - \cos 2x)\,dx$$

$$= \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_{\frac{\pi}{12}}^{\frac{\pi}{4}}$$

$$= \frac{\pi}{2}\left[\left(\frac{\pi}{4} - \frac{1}{2}\right) - \left(\frac{\pi}{12} - \frac{1}{2}\cdot\frac{1}{2}\right)\right]$$

$$= \frac{\pi}{2}\left[\frac{\pi}{6} - \frac{1}{4}\right]$$

$\therefore$ Volume is $\frac{\pi}{2}\left(\frac{\pi}{6} - \frac{1}{4}\right)$ units^3

(b) Proposition: $7^n + 5$ is divisible by 6 for all integers $n \ge 1$

(i) Test for $n = 1$: $7^1 + 5 = 12$

$\therefore$ True for $n = 1$

(ii) Assume proposition is true for some integer $n = k$

ie $7^k + 5 = 6M$, M integer —— ①

Then $7^{k+1} + 5 = 7(7^k) + 5$

$= 7(6M - 5) + 5$ from ①

$= 42M - 30$

$= 6(7M - 5)$

$= 6N \quad N$ integer

$\therefore$ If proposition is true for $n = k$ then it is also true for $n = k+1$

But true for $n=1 \Rightarrow$ true for $n=2$

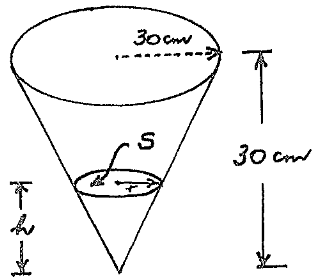
and hence by the Principle of Mathematical Induction it is true for all integers $n \geq 1$.

(c) $\frac{dV}{dt} = 24 \text{ cm}^3/s$

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi h^3$

$\frac{dV}{dh} = \pi h^2$



By similar triangles

$\frac{r}{h} = \frac{30}{30}$

$\therefore r = h$

$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$

$= \frac{1}{\pi h^2} \cdot 24 \text{ cm/s}$

when $h = 16$

$\frac{dh}{dt} = \frac{1}{\pi . 16^2} \cdot 24 \text{ cm/s}$

$= \frac{3}{32\pi} \text{ cm/s}$

$\frac{dS}{dt} = \frac{dS}{dh} \cdot \frac{dh}{dt}$ where $S = \pi r^2$

$= \pi h^2$

$\frac{dS}{dh} = 2\pi h$

$= 2\pi h \times \frac{3}{32\pi} \text{ cm}^2/s$ —— 4 MARKS

$h=16 \Rightarrow \frac{dS}{dt} = 3 \text{ cm}^2/s$

OR $\frac{dS}{dt} = \frac{dS}{dh} \times \frac{dh}{dV} \times \frac{dV}{dt}$

$= 2\pi h \times \frac{1}{\pi h^2} \times 24 \text{ cm}^2/s$

$h=16 \Rightarrow \frac{dS}{dt} = 32\pi \times \frac{1}{256\pi} \times 24 \text{ cm}^2/s$

$= 3 \text{ cm}^2/s$

QUESTION 5 (12 MARKS)

(a) $\frac{dT}{dt} = k(T-R)$ —— ①

(i) $T = R + Ae^{kt}$ —— ②

$\Rightarrow \frac{dT}{dt} = kAe^{kt}$

$= k(T-R)$ from ② —— 1 MARK

(ii) $\frac{dT}{dt} = k(T - -40)$

$= k(T+40)$

$\Rightarrow T = -40 + Ae^{kt}$

$\left.\begin{matrix} t=0 \\ T=24 \end{matrix}\right\} \Rightarrow 24 = -40 + A$

$\therefore A = 64$

$\therefore T = -40 + 64e^{kt}$

$\left.\begin{matrix} t=5 \\ T=19 \end{matrix}\right\}$ $19 = -40 + 64e^{5k}$

$59 = 64e^{5k}$

$\frac{59}{64} = e^{5k}$

$5k = \ln\left(\frac{59}{64}\right)$

$\therefore k = \frac{1}{5}\ln\left(\frac{59}{64}\right)$

$= -0.016269\ldots$

$T=0 \Rightarrow 0 = -40 + 64e^{kt}$

$40 = 64e^{kt}$

$e^{kt} = \frac{5}{8}$

$kt = \ln\left(\frac{5}{8}\right)$

$t = \frac{\ln\left(\frac{5}{8}\right)}{k}$

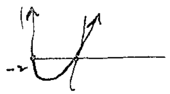
$= 28.889\ldots$

$= 29$ (to nearest whole)

$\therefore$ It will take 29 s.

(b) $y = \log\left(\frac{2x}{2+x}\right)$

(i) must have $\frac{2x}{2+x} > 0$



$2x(2+x) > 0$

$\therefore x < -2$ OR $x > 0$ —— 2 MARKS

$\therefore$ Domain is $x < -2$ OR $x > 0$

(ii) $y = 0 \Rightarrow \log\left(\frac{2x}{2+x}\right) = 0$

$$\frac{2x}{2+x} = 1$$

$$2x = 2 + x \quad \text{—— 1 MARK}$$

$$\therefore x = 2$$

(iii) $y = \log\left(\frac{2x}{2+x}\right)$

$$= \log 2x - \log(2+x)$$

$$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{2+x}$$

$$= \frac{2+x-x}{x(2+x)}$$

$$= \frac{2}{x(2+x)} \quad \text{—— 2 MARKS}$$

now $2x(2+x) > 0$,

$\therefore x(2+x) > 0$

$\therefore \frac{dy}{dx} > 0$ for all x in D

$\therefore$ fn is increasing

(iv) $\frac{dy}{dx} = \frac{2}{2x+x^2}$

$$\frac{d^2y}{dx^2} = \frac{(2x+x^2).0 - 2(2+2x)}{(2x+x^2)^2}$$

$$= \frac{-4-4x}{(2x+x^2)^2}$$

Possible pt of inflexion when $y'' = 0$

$$-4-4x = 0$$

$$x = -1$$

But this is outside the domain

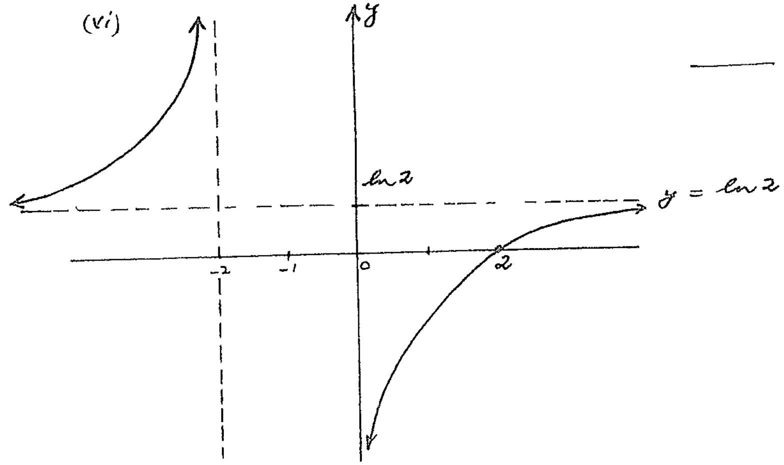
$\therefore$ There are no points of inflexion —— 1 MARK

(v) $\lim_{x\to\infty}\left[\log\left(\frac{2x}{2+x}\right)\right]$

$$= \lim_{x\to\infty}\left[\log\left(\frac{2}{\frac{2}{x}+1}\right)\right] \quad \text{—— 1 MARK}$$

$$= \log 2$$

(vi) —— 1 MARK



QUESTION 6: (12 MARKS)

(a) $(1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$ —— ①

(i) sub $x = 1$ in ①

$\Rightarrow 2^n = \sum_{r=0}^{n} {}^nC_r$ —— 1 MARK

(ii) now $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_k x^k + \dots + {}^nC_n x^n$

Differentiating with respect to x:

$$n(1+x)^{n-1} = {}^nC_1 + 2\,{}^nC_2 x + \dots + k.{}^nC_k x^{k-1} + \dots + n.{}^nC_n x^{n-1}$$

$$= \sum_{r=1}^{n} {}^nC_r . r . x^{r-1}$$

let $x = 1$

$\Rightarrow n.2^{n-1} = \sum_{r=1}^{n} r.\,{}^nC_r$ —— 2 MARKS

(b) $\displaystyle\int_1^3 \frac{dx}{(1+x)\sqrt{x}}$

$u = \sqrt{x}$
$= x^{\frac{1}{2}}$
$\frac{du}{dx} = \frac{1}{2} x^{-\frac{1}{2}}$
$du = \frac{dx}{2\sqrt{x}}$

$$= \int_1^{\sqrt{3}} \frac{2\,du}{1+u^2}$$

$$= 2\left[\tan^{-1} u\right]_1^{\sqrt{3}}$$

$$= 2\left[\tan^{-1}\sqrt{3} - \tan^{-1} 1\right]$$

$$= 2\left[\frac{\pi}{3} - \frac{\pi}{4}\right]$$

$$= \frac{\pi}{6}$$

(c) $T = \frac{2\pi}{n} = 2$

$\therefore n = \pi$

$\left.\begin{array}{l} t=0 \\ v=0 \\ x=10 \end{array}\right\}$

(i) The motion can be described by

$$x = 10\cos \pi t$$
$$\dot{x} = -10\pi \sin \pi t$$
$$(\dot{x})^2 = 100\pi^2 \sin^2 \pi t$$
$$= \pi^2 (100 \sin^2 \pi t)$$

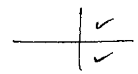
ie $v^2 = \pi^2(100 - x^2)$

when $x = 6$
$$v^2 = \pi^2 \times 64$$
$$v = \pm 8\pi$$

$\therefore$ Speed is 8π m/s

(ii) when $x = 6$
$$6 = 10\cos \pi t$$
$$0.6 = \cos \pi t$$

$(\pi t)_{acute} = 0.927\ldots$



First time $\Rightarrow \therefore \pi t = 0.92729\ldots$
$$\therefore t = \frac{0.92729\ldots}{\pi}$$
$$= 0.2951\ldots$$
$$= 0.30 \quad (\text{correct to 2 dec pl})$$

$\therefore$ Time taken is 0.30 s

QUESTION 7 :

— 5 MARKS

(i) P:

$\ddot{x}_p = 0$ —— (1)
$\dot{x} = C$
at $t=0$, $\dot{x}_p = V_p \cos\alpha$
$\therefore C = V_p\cos\alpha$
$\therefore \dot{x}_p = V_p\cos\alpha$ —— (2)
$\Rightarrow x = (V_p\cos\alpha)t + C$
at $t=0$, $x=0$ $\therefore C=0$
$\therefore x_p = (V_p\cos\alpha)t$ —— (3)

$\ddot{y}_p = -g$ —— (4)
$\dot{y} = -gt + C$
at $t=0$, $\dot{y} = V_p\sin\alpha$
$\therefore C = V_p\sin\alpha$
$\therefore \dot{y}_p = -gt + V_p\sin\alpha$ —— (5)
$\Rightarrow y = -\frac{1}{2}gt^2 + (V_p\sin\alpha)t + C$
at $t=0$, $y=h$ $\therefore C = h$
$\therefore y_p = -\frac{1}{2}gt^2 + (V_p\sin\alpha)t + h$ —— (6)

(ii) $x_Q = (V_Q\cos\beta)t$

$y_Q = -\frac{1}{2}gt^2 + (V_Q\sin\beta)t$

(iii) at collision, $t = T$, $x_p + x_Q = 2h$ and $y_p = y_Q$

now $y_p = y_Q$

$$\Rightarrow -\cancel{\tfrac{1}{2}g}T^2 + (V_Q\sin\beta)T = -\cancel{\tfrac{1}{2}g}T^2 + (V_p\sin\alpha)T + h$$

$$\therefore T = \frac{h}{V_Q\sin\beta - V_p\sin\alpha}$$

Then $x_P + x_Q = 2R$

$$\Rightarrow (V_P \cos\alpha)\cdot\frac{R}{V_Q\sin\beta - V_P\sin\alpha} + (V_Q\cos\beta)\cdot\frac{R}{V_Q\sin\beta - V_P\sin\alpha} = 2R$$

$$\therefore V_P\cos\alpha + V_Q\cos\beta = 2V_Q\sin\beta - 2V_P\sin\alpha$$

$$V_P(\cos\alpha + 2\sin\alpha) = V_Q(2\sin\beta - \cos\beta)$$

$$\therefore \frac{V_P}{V_Q} = \frac{2\sin\beta - \cos\beta}{2\sin\alpha + \cos\alpha}$$

— 6 MARKS

(b) (i) $\sin(x-y) = \sin x\cos y - \cos x \sin y$

(ii) $c^2 + cd + d^2$

$$= \sin^2\alpha + \sin\alpha\left[\sin(60-\alpha)\right] + \left[\sin(60-\alpha)\right]^2$$

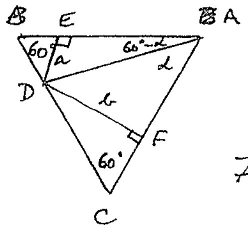
$$= \sin^2\alpha + \sin\alpha\left[\frac{\sqrt3}{2}\cos\alpha - \frac12\sin\alpha\right] + \left(\frac{\sqrt3}{2}\cos\alpha - \frac12\sin\alpha\right)^2$$

$$= \sin^2\alpha + \frac{\sqrt3}{2}\sin\alpha\cos\alpha - \frac12\sin^2\alpha + \frac34\cos^2\alpha - \cancel{\frac{\sqrt3}{2}\sin\alpha\cos\alpha} + \frac14\sin^2\alpha$$

$$= \frac34\sin^2\alpha + \frac34\cos^2\alpha$$

$$= \frac34$$

(iii)



In $\triangle AED$ $\sin(60°-\alpha) = \frac{a}{AD} = d$

In $\triangle AFD$ $\sin\alpha = \frac{b}{AD} = c$

From (ii) above,

$$c^2 + cd + d^2 = \frac34$$

$$\Rightarrow \frac{a^2}{(AD)^2} + \left(\frac{a}{AD}\right)\cdot\left(\frac{b}{AD}\right) + \frac{b^2}{(AD)^2} = \frac34$$

$$a^2 + ab + b^2 = \frac34 (AD)^2$$

$$\therefore AD = \frac{2}{\sqrt3}\left(a^2 + ab + b^2\right)$$