St George Girls High School

Trial Higher School Certificate Examination

2005

# Mathematics Extension 1

**Total Marks – 84**

**General Instructions**

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Attempt ALL questions.
- Begin each question on a new booklet
- Write your student number on each page
- All necessary working must be shown.
- Diagrams are not to scale.
- Board-approved calculators may be used.
- The mark allocated for each question is listed at the side of the question.

**Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.**



**Question 1** (12 marks) **Marks**

a) Find the coordinates of the point $P$ that divides $AB$ internally in the ratio 2 : 3 where $A$ is $(-3, 5)$ and $B$ is $(-6, -10)$ **2**

b) Find the possible values of $a$ if the lines $2x + 3y - 5 = 0$ and $ax + 2y + 3 = 0$ are inclined to each other at 45° **4**

c) Solve for $x$: $\dfrac{2}{x-1} > 3$ **3**

d) Find $\int \dfrac{x}{\sqrt{x-1}}\,dx$ using the substitution $x = u + 1$ **3**

**Question 2** (12 marks)

a) (i) Express $\sqrt{3}\,\sin x + \cos x$ in the form $R\sin(x + \alpha)$ 2

(ii) Hence, sketch the graph of $y = \sqrt{3}\,\sin x + \cos x$ for $0 \le x \le 2\pi$ 2

b) (i) Show that $f(x) = 2\log_e x + 2x$ has a zero between $x = 0.5$ and $x = 1$ 1

(ii) Starting with $x = 0.5$, use one application of Newton's method to find a better approximation for this zero. Write your answer correct to three significant figures 3

c) Find $\int \frac{dx}{\sqrt{9 - 4x^2}}$ 2

d) Find $\int \cos^2 4x \; dx$ 2



**Question 3** (12 marks) **Marks**

a) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are points on the parabola $4ay = x^2$ such that the chord $PQ$ subtends a right angle at the vertex $O$

(i) Show that $pq = -4$ 2

(ii) Find the locus of the mid-point of $PQ$ 3

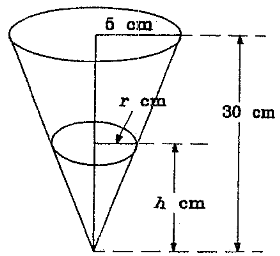
b) Show that $\int_0^3 \left( \frac{x}{x^2 + 9} + \frac{1}{x^2 + 9} \right) dx = \log_e \sqrt{2} + \frac{\pi}{12}$ 3

c) If the roots of the equation $x^3 + bx^2 + cx + d = 0$ are in geometric progression show that $c^3 = b^3 d$ 4

**Question 4** (12 marks)

a) A container is in the shape of an inverted right circular cone of base radius 5cm and height 30cm. Water is poured into the container at a rate of 2cm³/min

(i) Show that $r = \frac{h}{6}$ 1




(ii) Find the rate at which the level of water is rising when the water is 10cm deep 3

b) (i) State the domain and range of $y = 2\cos^{-1}\left(\frac{x}{3}\right)$ 2

(ii) Hence sketch $y = 2\cos^{-1}\left(\frac{x}{3}\right)$ 1

c) Given $f(x) = \sqrt[3]{x-1}$ for $x > 1$

(i) Show that the function is monotonic increasing for all $x$ in the given domain 2

(ii) State the domain and range of $f^{-1}(x)$ 1

(iii) Find $f^{-1}(x)$ and explain why the inverse is a function 2



**Question 5** (12 marks) **Marks**

a) By induction show that $7^n - 3^n$ is divisible by 4 for all integers $n \geq 1$ 3

b) The velocity $v$ and position $x$ of a particle moving in a straight line are connected by the relation $v = 3 + 5x$. Show that the acceleration $a$ of the particle is $5v$ 2

c) Find the term independent of $x$ in the expansion of $(3-x)^4\left(1+\frac{2}{x}\right)^7$ 4

d) Evaluate $\cos\left(2\tan^{-1}\frac{3}{4}\right)$ without the use of a calculator 3

**Question 6** (12 marks)

a) The cooling rate of a body is proportional to the difference between the temperature of the body and that of a surrounding medium ie. $\frac{dT}{dt} = -k(T - T_1)$ where $T$ is the temperature of the cooling body and $T_1$ is the temperature of the surrounding medium

(i) Show that $T - T_1 = Ae^{-kt}$ satisfies this equation 2

(ii) A cup of coffee cools from 80° to 40° in 10 minutes when placed in a room with temperature 18°. How long will it take for the coffee's temperature to fall to 20°? 4

b) A particle is moving in a straight line such that its acceleration at time $t$ seconds is $\ddot{x} = -4x$, where $x$ is the displacement in metres from the origin. The particle is initially 6m to the right of the origin.

(i) Find its displacement in terms of time 3

(ii) Find the position and time when the particle first obtains a velocity of 6m/s 3



**Question 7** (12 marks) **Marks**

a) (i) Differentiate $x(1 + x)^n$ 1

(ii) Write the binomial expansion for $x(1 + x)^n$ 1

(iii) Hence show that $\sum_{r=0}^{n} (r + 1)\ {}^nC_r = (n + 2)\, 2^{n-1}$ 3

b) A particle is projected from a point O with an initial velocity of 60m/s at an angle of 30° to the horizontal. At the same instant a second particle is projected in the opposite direction with an initial velocity of 50m/s from a point level with O and 100m from O.

(i) Show that the horizontal and vertical displacement equations of the first particle are given by: 2

$x = 60\cos 30° t$ and $y = 60\sin 30° t - \frac{1}{2} gt^2$
where $g$ is acceleration due to gravity

(ii) Find the angle of projection of the second particle if they collide 3

(iii) Find the time at which the two particles collide 2

**End of Paper**

**1) a)** $A(-3,5)$ $B(-6,-10)$ Ratio $2:3$

$x = \frac{3\times -3 + 2\times -6}{2+3} = -\frac{21}{5}$ $\quad y = \frac{3\times 5 + 2\times -10}{2+3} = -1$

$\therefore P$ is $(-4\frac{1}{5}, -1)$

**b)** $2x+3y-5=0$ $\quad ax+2y+3=0$

$m_1 = -\frac{2}{3}$ $\quad m_2 = -\frac{a}{2}$

$\therefore \tan 45 = 1 = \left|\frac{-\frac{2}{3}+\frac{a}{2}}{1+ -\frac{2}{3}\times -\frac{a}{2}}\right|$

$1 = \left|\frac{\frac{-4+3a}{6}}{\frac{6+2a}{6}}\right|$

$\therefore |6+2a| = |3a-4|$

$\therefore 6+2a = 3a-4$ or $6+2a = 4-3a$

$10 = a$ $\quad 5a = -2$

$a = -\frac{2}{5}$

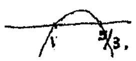
**c)** $\frac{2}{x-1} > 3$ $\quad x \neq 1$

$2(x-1) > 3(x-1)^2$

$2(x-1) - 3(x-1)^2 > 0$

$(x-1)[2-3(x-1)] > 0$

$(x-1)(5-3x) > 0$




$\therefore 1 < x < \frac{5}{3}$

**d)** $\int \frac{x}{\sqrt{x-1}}\,dx$ $\quad x = u+1$, $dx = du$

$= \int \frac{u+1}{\sqrt{u}}\,du$

$= \int \sqrt{u} + \frac{1}{\sqrt{u}}\,du$

$= \int u^{\frac{1}{2}} + u^{-\frac{1}{2}}$

$= \frac{2}{3}u^{3/2} + 2u^{1/2} + c$

$= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + c$

**2) a) i)** $\sqrt{3}\sin x + \cos x \equiv R\sin(x+\alpha)$

$\sqrt{3}\sin x + \cos x \equiv R\sin x\cos\alpha + R\cos x\sin\alpha$

$\therefore R\cos\alpha = \sqrt{3}$ $\quad R\sin\alpha = 1.$

$\therefore R^2(\cos^2\alpha + \sin^2\alpha) = 3+1$

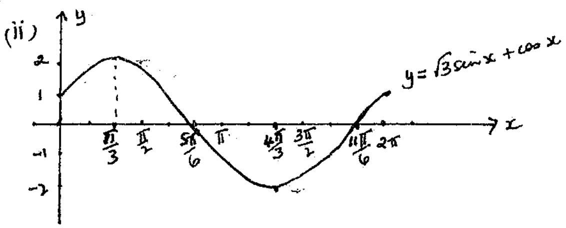
$\therefore R = 2$ $\quad R > 0.$

and $\frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{\sqrt{3}}$

$\therefore \tan\alpha = \frac{\pi}{6}.$

$\therefore \sqrt{3}\sin x + \cos x = 2\sin(x+\frac{\pi}{6})$

(ii)




**b) i)** $f(x) = 2\log_e x + 2x.$

$f(0.5) \doteqdot -0.386$

$f(1) = 2.$

$\therefore$ Since sign change a zero lies between $\frac{1}{2}$ and 1.

**ii)** $f'(x) = \frac{2}{x} + 2.$

If $z_1 = 0.5$

then $z_2 = 0.5 - \frac{f(0.5)}{f'(0.5)}$

$= 0.5 - \frac{(2\ln 0.5 + 1)}{6}$

$\doteqdot 0.56438\ldots$

$\doteqdot 0.564$ (to 3 sig. fig)

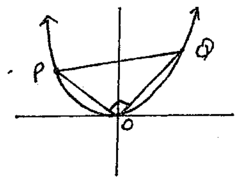
**c)** $\int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\sin^{-1}\frac{2x}{3} + c$

**d)** $\int \cos^2 4x\,dx = \frac{1}{2}\int 1 + \cos 8x\,dx$

$= \frac{1}{2}\left(x + \frac{\sin 8x}{8}\right) + c$

$= \frac{x}{2} + \frac{\sin 8x}{16} + c$

**3) a)** $P(2ap, ap^2)$ $\quad Q(2aq, aq^2)$




**i)** m of $OP = \frac{ap^2-0}{2ap-0} = \frac{p}{2}$ $\quad$ m of $OQ = \frac{aq^2-0}{2aq-0} = \frac{q}{2}$

Since $P\hat{O}Q = 90^0$

$\frac{p}{2}\times\frac{q}{2} = -1$

$\therefore pq = -4$

**ii)** midpt $PQ = \left(\frac{2ap+2aq}{2}, \frac{ap^2+aq^2}{2}\right)$

$= \left(a(p+q), \frac{a(p^2+q^2)}{2}\right)$

$x = a(p+q)$

$\therefore p+q = \frac{x}{a}$

$y = \frac{a(p^2+q^2)}{2}$

$\frac{2y}{a} = (p+q)^2 - 2pq$

$= \left(\frac{x}{a}\right)^2 - 2\times -4$

$\frac{2y}{a} = \frac{x^2}{a^2} + 8$

$2ay = x^2 + 8a^2$

$x^2 = 2a(y-4a)$

**b)** $\int_0^3 \frac{x}{x^2+9} + \frac{1}{x^2+9}\,dx$

$= \left[\frac{1}{2}\ln(x^2+9) + \frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^3$

$= \left(\frac{1}{2}\ln 18 + \frac{1}{3}\tan^{-1}1\right) - \left(\frac{1}{2}\ln 9 + 0\right)$

$= \frac{1}{2}\ln 2 + \frac{1}{3}\times\frac{\pi}{4}$

$= \ln\sqrt{2} + \frac{\pi}{12}$ as req.

**c)** $x^3 + bx^2 + cx + d = 0$

Let roots be $\frac{\alpha}{r}, \alpha, \alpha r$

$\therefore \frac{\alpha}{r} + \alpha + \alpha r = -b$ — (1)

$\frac{\alpha^2}{r^2} + \alpha^2 + \alpha^2 r = c$ — (2)

$\frac{\alpha}{r}\times\alpha\times\alpha r = -d$ — (3)

$\therefore \alpha^3 = -d$

From (1): $\alpha\left(\frac{1}{r} + 1 + r\right) = -b$

From (2): $\alpha^2\left(\frac{1}{r} + 1 + r\right) = c$

$\therefore \frac{1}{\alpha} = -\frac{b}{c}$

$\alpha = \frac{c}{-b}$

$\therefore \left(\frac{c}{-b}\right)^3 = -d$

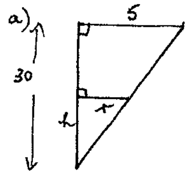
$\frac{c^3}{-b^3} = -d$

$\therefore c^3 = b^3 d$

as req.

$\frac{\frac{1}{\alpha}}{\beta} \times \frac{1}{\beta}$

4 a) i) Since Δ's are similar




$\frac{r}{5} = \frac{h}{30}$

$\therefore r = \frac{5h}{30} = \frac{h}{6}$

ii) Vol of cone $= \frac{1}{3}\pi r^2 h$

$\therefore V = \frac{1}{3}\pi \times \left(\frac{h}{6}\right)^2 \times h$

$= \frac{\pi h^3}{108}$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$2 = \frac{3\pi h^2}{108} \times \frac{dh}{dt}$

$\therefore$ when $h = 10$ : $2 = \frac{300\pi}{108} \times \frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \frac{216}{300\pi}$

$= \frac{18}{25\pi}$

$\therefore$ water is rising at $\frac{18}{25\pi}$ cm/min

(b) i) $y = 2\cos^{-1}\frac{x}{3}$

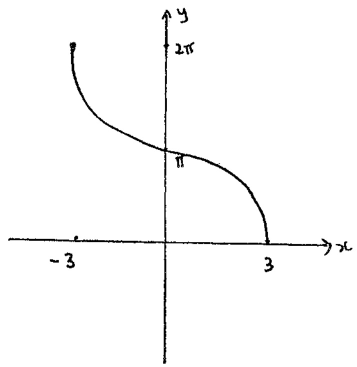
D: $-1 \le \frac{x}{3} \le 1$

$-3 \le x \le 3$

R: $0 \le \frac{y}{2} \le \pi$

$0 \le y \le 2\pi$

ii)




c) $f(x) = \sqrt[3]{x-1}$ $\quad x > 1$

i) $f(x) = (x-1)^{1/3}$

$f'(x) = \frac{1}{3}(x-1)^{-2/3}$

$= \frac{1}{3\sqrt[3]{(x-1)^2}}$

Since $(x-1)^2$ is positive for all $x$

$\sqrt[3]{(x-1)^2} > 0$

$\therefore \frac{1}{3\sqrt[3]{(x-1)^2}} > 0$ for all $x$

$\therefore f(x)$ is montonic increasing

(ii) For $f(x)$: D: $x > 1$

R: $y > 0$

$\therefore$ For $f^{-1}(x)$: D: $x > 0$

R: $y > 1$

(iii) $y = (x-1)^{1/3}$

For inverse: $x = (y-1)^{1/3}$

$x^3 = y - 1$

$\therefore y = x^3 + 1$

Since f(x) is montonic increasing and it passes horizontal line test inverse will also be a function

5. a) Assertion: that $7^n - 3^n$ is divisible by 4 for $n \ge 1$

For $n=1$: $7^1 - 3^1 = 4$ which is divisible by 4

$\therefore$ Assertion is true for $n = 1$.

Assume assertion is true for $n = k$

i.e. that $7^k - 3^k$ is divisible by 4

i.e. $7^k - 3^k = 4M$ (where M is a positive integer)

We need to prove that:

$7^{k+1} - 3^{k+1}$ is also divisible by 4.

$7^{k+1} - 3^{k+1} = 7^k.7 - 3^k.3$

$= (8-1).7^k - (4-1).3^k$

$= 8.7^k - 7^k - 4.3^k + 3^k$

$= 8.7^k - 4.3^k - (7^k - 3^k)$

$= 8.7^k - 4.3^k - 4M$ using assumption

$= 4(2.7^k - 3^k - M)$

$= 4J$ where J is a positive integer)

$\therefore 7^{k+1} - 3^{k+1}$ is divisible by 4.

$\therefore$ If statement is true for $n=k$, it is true for $n=k+1$.

$\therefore$ Since statement is true for $n=1$, it is true for $n=2$ and by induction it is true for all $n \ge 1$.

b) $v = 3 + 5x$

Since $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \ddot{x}$

$\frac{d}{dx}\left(\frac{1}{2}(3+5x)^2\right) = 2 \times \frac{1}{2}(3+5x) \times 5$

$= 5(3+5x)$

$= 5v$

$\therefore$ acceleration $= 5v$.

c) $(3-x)^4\left(1+\frac{2}{x}\right)^7$

$(3-x)^4 = {}^4C_0 3^4 - {}^4C_1 3^3 x + {}^4C_2 3^2 x^2 - {}^4C_3 3x^3 + {}^4C_4 x^4$

$\left(1+\frac{2}{x}\right)^7 = {}^7C_0 + {}^7C_1 \frac{2}{x} + {}^7C_2 \frac{4}{x^2} + {}^7C_3 \frac{8}{x^3} + {}^7C_4 \frac{16}{x^4} \ldots$

$\therefore$ Term Independent of $x = 3^4 \times {}^7C_0 - {}^4C_1 27 \times {}^7C_1 \times 2 + {}^4C_2 9 \times {}^7C_2 \times 4 - {}^4C_3 3 \times {}^7C_3 \times 8 + {}^7C_4 16$

$= 81 - 1512 + 4536 - 3360 + 560$

$= 305$

d) next page

6 a) i) $\frac{dT}{dt} = -k(T - T_1)$

$T - T_1 = Ae^{-kt}$

$\therefore T = T_1 + Ae^{-kt}$

LHS $= \frac{dT}{dt}$

$= -k \times Ae^{-kt}$

RHS $= -k(T - T_1)$

$= -k(T_1 + Ae^{-kt} - T_1)$

$= -k \cdot Ae^{-kt}$

$=$ LHS

$\therefore T - T_1 = Ae^{-kt}$ satifies eqn.

ii) $T_1 = 18$

$t = 0$ : $T = 80$

$\therefore 80 = 18 + A \times 1$

$\therefore A = 62$

$\therefore T = 18 + 62e^{-kt}$

$t = 10$, $T = 40$

$40 = 18 + 62e^{-10k}$

$\frac{22}{62} = e^{-10}$

$\therefore k = \frac{\ln \frac{11}{31}}{-10}$.

$T = 20$:

$20 = 18 + 62e^{-kt}$

$\frac{2}{62} = e^{-kt}$

$t = \frac{\ln \frac{1}{31}}{-\left(\frac{\ln \frac{11}{31}}{-10}\right)}$

$= 33.14$ min (2dp).

b) i) Since $\ddot{x} = -4x$ particle is moving in SHM about origin

$\therefore x = a\cos(nt + \alpha)$

$= 6\cos(2t + \alpha)$

$t = 0$, $x = 6$ :

$\therefore 6 = 6\cos\alpha$

$\cos\alpha = 1$

$\therefore \alpha = 0$

$\therefore x = 6\cos 2t$

ii) $v = -12\sin 2t$

$v = 6$ : $6 = -12\sin 2t$

$\therefore \sin 2t = -\frac{1}{2}$

$2t = \frac{7\pi}{6}$

$t = \frac{7\pi}{12}$

$t = \frac{7\pi}{12}$ : $x = 6\cos 2 \times \frac{7\pi}{12}$

$= 6\cos\frac{7\pi}{6}$

$= 6 \times -\frac{\sqrt{3}}{2}$

$= -3\sqrt{3}$

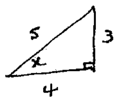
$\therefore$ Particle first reaches 6m/sec after $\frac{7\pi}{12}$ secs., $3\sqrt{3}$ metres to left of origin.

5d

$\cos\left(2\tan^{-1}\frac{3}{4}\right) = \cos 2x$

where $x = \tan^{-1}\frac{3}{4}$

$\therefore \tan x = \frac{3}{4}$




$\therefore \cos 2x = 2\cos^2 x - 1$

$= 2\left(\frac{4}{5}\right)^2 - 1$

$= \frac{32}{25} - 1$

$= \frac{7}{25}$

7 a) i) $\frac{d}{dx}\left(x(1+x)^n\right) = (1+x)^n \times 1 + x \times n(1+x)^{n-1}$

$= (1+x)^n + nx(1+x)^{n-1}.$

ii) $x(1+x)^n = x\,{}^nC_0 + x\,{}^nC_1 x + x\,{}^nC_2 x^2 + \ldots + x\,{}^nC_n x^n$

$= {}^nC_0 x + {}^nC_1 x^2 + {}^nC_2 x^3 + \ldots + {}^nC_n x^{n+1}$

iii) $\sum_{r=0}^{n} (r+1)\,{}^nC_r = {}^nC_0 + 2\,{}^nC_2 + 3\,{}^nC_3 + \ldots + (n+1)\,{}^nC_r$

from (ii): $\frac{d}{dx} x(1+x)^n = {}^nC_0 + 2\,{}^nC_1 x + 3\,{}^nC_2 x^2 + \ldots + (n+1)\,{}^nC_n x^n$

$\therefore$ from (i): $(1+x)^n + nx(1+x)^{n-1} = {}^nC_0 + 2\,{}^nC_1 x + 3\,{}^nC_2 x^2 + \ldots + (n+1)\,{}^nC_n x^n$

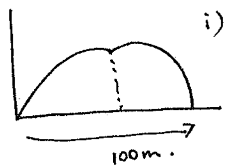
Let $x = 1$:

$2^n + n(2)^{n-1} = {}^nC_0 + 2\,{}^nC_1 + 3\,{}^nC_2 + \ldots + (n+1)\,{}^nC_n$

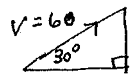
$\therefore 2^{n-1}(2+n) = {}^nC_0 + 2\,{}^nC_1 + 3\,{}^nC_2 + \ldots + (n+1)\,{}^nC_n.$

$\therefore \sum_{r=0}^{n} (r+1)\,{}^nC_r = 2^{n-1}(n+2)$

b)




i) First particle.




$\ddot{x} = 0$

$\dot{x} = C = 60\cos 30$

$x = \int 60\cos 30 \cdot dt$

$= 60\cos 30 t + k$

$t = 0$ $x = 0$ $\therefore k = 0$

$\therefore x = 60\cos 30 t$

$\ddot{y} = -g$

$\dot{y} = -gt + M$

$t = 0$ $\dot{y} = 60\sin 30$

$\therefore \dot{y} = -gt + 60\sin 30$

$y = \int \dot{y}\, dt$

$= -\frac{gt^2}{2} + 60\sin 30 t$

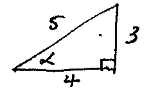
$t = 0$ $y = 0$ $\therefore N = 0$

$\therefore y = -\frac{gt^2}{2} + 60\sin 30 t$

ii) When particles collide height above ground is same

$\therefore 60\sin 30 t - \frac{1}{2}gt^2 = 50\sin\alpha t - \frac{1}{2}gt^2$

$30t = 50\sin\alpha t$




$\therefore \sin\alpha = \frac{3}{5}$

$\therefore \alpha = 36°52'$

iii) $x$ values add to 100 m.

$\therefore 60\cos 30 t + 50\cos 36°52' t = 100$

$30\sqrt{3}t + 50 \times \frac{4}{5}t = 100$

$t = \frac{100}{\ldots} = 1.087$ secs (to 3dp)