St George Girls High School

Year 11

End of Preliminary Course Examination

2004

Mathematics Extension 1

Time Allowed: 2 hours
(plus 5 minutes reading time)

Instructions

1. All questions should be attempted.
2. All necessary working must be shown.
3. Begin each question on a **new page**.
4. Marks will be deducted for careless work or poorly presented solutions.

Question 1 (16 marks) – Start a New Page **Marks**

a) Find the derivatives of:

(i) $\dfrac{2x-3}{3x-1}$ 2

(ii) $\left(x^2+3\right)^5$ 1

(iii) $\dfrac{5}{3x^2}$ 2

b) Solve for $x : \dfrac{2-x}{2+x} \geq 1$ 3

c) In what ratio does the point $A(-2, -5)$ divide the interval joining $B(3, 10)$ to $C(1, 4)$? 2

d) Prove that $\sin\theta + \cot\theta\cos\theta \equiv \operatorname{cosec}\theta$ 2

e) Find the values of a, b, c if $a(x+1)^2 + b(x+1) + c \equiv 3x^2 + 4x + 6$ 2

f) Sketch $y = \dfrac{1-x}{x}$ clearly showing intercepts and asymptotes 2

Question 2 (16 marks) – Start a New Page **Marks**

a) Evaluate each of the following limits:

(i) $\lim\limits_{x \to 2} \dfrac{x-2}{x^2-4}$ 2

(ii) $\lim\limits_{x \to \infty} \dfrac{3x^2-5}{4-5x^2}$ 2

(iii) $\lim\limits_{x \to c} \dfrac{f(x)-f(c)}{x-c}$ where $f(x)=x^2-2x$ 3

b) The excavation of a tunnel becomes increasingly difficult as the work progresses. In the first week 200 tonnes were removed, in the second week 160 tonnes and in the third week 128 tonnes. In any week only 80% of the previous weeks excavation was achieved. This continued for many weeks.

(i) Find the amount excavated in the 10^{th} week. 1

(ii) Find the total excavated in the first 10 weeks. 2

(iii) How many weeks would it take to excavate a total of 850 tonnes? 2

c) Find k if kx^2-8x+2 is positive definite. 2

d) Find the Cartesian relationship between x and y if 2

$x=\sec\theta-1$

$y=2+\tan\theta$

Question 3 (16 marks) – Start a New Page **Marks**

a) Consider the function $f(x) = 10^x$

(i) Sketch $y = f(x)$ 1

(ii) State the domain and range of $f(x)$ 1

(iii) Clearly explain why $f(x)$ has an inverse 1

(iv) Prove that $f^{-1}(x) = \log_{10} x$ 1

(v) State the domain and range of $f^{-1}(x)$ 1

(vi) Evaluate $f^{-1}[f(3)]$ 1

b) Solve for θ where $0^\circ \le \theta \le 360^\circ$

(i) $2\cos^2\theta = 2 - \sin\theta$ 4

(ii) $\cos(2\theta + 30^\circ) = +\frac{1}{2}$ 3

c) Find the vertex and focus of the parabola $y^2 = 10y + 36x + 83$ 3

Question 4 (16 marks) – Start a New Page **Marks**

a) The roots of $ax^2+bx+c=0$ are α and β

(i) Write down expressions for $\alpha+\beta$ and $\alpha\beta$ 1

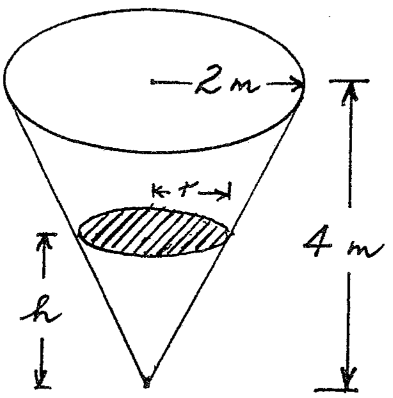
(ii) Prove that $\alpha^3+\beta^3=\dfrac{3abc-b^3}{a^3}$ 3

b) Find the value of k if the line $y=2x+k$ is tangential to the circle $x^2+y^2=9$ 3

c) Prove by Mathematical Induction that $\dfrac{1}{1\times 2}+\dfrac{1}{2\times 3}+\ldots+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$ 4

for all integers $n\geq 1$

d)



The conical container shown is being filled with water at 0.1m^3/min.

At any time, the radius of the surface of the water is r metres and the depth of water is h metres where $h=2r$. The container was initially empty.

(i) Show that the volume of water in the container at any time is given by $V=\dfrac{2}{3}\pi r^3$ 1

(ii) When the radius is 0.6m find the rate at which the

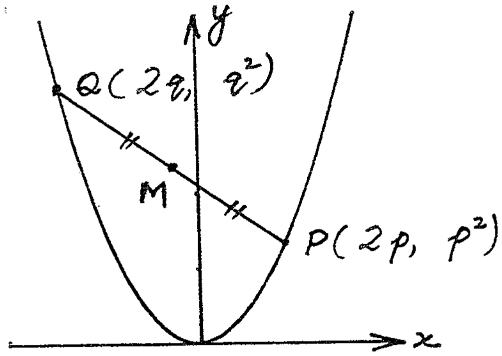
(α) radius 3

(β) depth 1

is increasing

Question 5 (16 marks) – Start a New Page **Marks**

a)



The points $P(2p, p^2)$ and $Q(2q, q^2)$ are variable points on the parabola $x^2 = 4y$. M is the mid-point of PQ.

(i) Show that the equation of the chord PQ is $y - \frac{1}{2}(p+q)x + pq = 0$ 2

(ii) Find the coordinates of M. 1

(iii) If the chord PQ always passes through the point $(0, 4)$ prove that $pq = -4$, and, hence find the locus of M. 3

b) Consider the function defined by

$$f(x) = \begin{cases} x+1 & \text{for} \quad x \le 1 \\ a - b(x-2)^2 & \text{for} \quad x > 1 \end{cases}$$

(i) Show that for $f(x)$ to be continuous at $x = 1$ then $a - b = 2$ 1

(ii) Find the values of a and b such that $f(x)$ is continuous and differentiable at $x = 1$. 2

c) A curve is defined by the parametric equations

$$\left.\begin{aligned} x &= t^3 - 3t \\ y &= 3 - t^2 \end{aligned}\right\}$$

(i) Show that $\dfrac{dy}{dx} = -\dfrac{2t}{3t^2 - 3}$ 2

(ii) Find the equation of the tangent to this curve at the point where $t = 2$ 3

d) Find the acute angle between these two lines: 2

$y = x + 1$ and $y = 3x + 1$

Extension 1

Yearly Exam

Question 1

a.) (i) $\frac{dy}{dx} = \frac{(3x-1).2 - (2x-3).3}{(3x-1)^2}$

$= \frac{6x-2-6x+9}{(3x-1)^2}$

$= \frac{7}{(3x-1)^2}$

(ii) $f'(x) = 5(x^2+3)^4 . 2x$

$= 10x(x^2+3)^4$

(iii) $y = \frac{5}{3}x^{-2}$

$\frac{dy}{dx} = \frac{5}{3} . -2x^{-3}$

$= \frac{-10}{3x^3}$

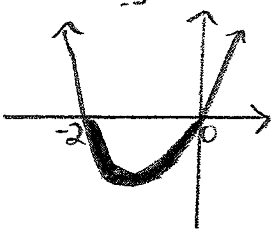
b.) $\frac{2-x}{2+x} . (2+x)^2 \geq (2+x)^2$ $\quad x \neq -2.$

$(2+x)^2 - (2-x)(2+x) \leq 0.$

$(2+x)[(2+x) - (2-x)] \leq 0.$

$(2+x).2x \leq 0.$

$\therefore -2 < x \leq 0.$



c.) $-2 = \frac{m.1 + n.3}{m+n}$, $-5 = \frac{4m + 10n}{m+n}$

$-2m - 2n = m + 3n$ $\quad -5m - 5n = 4m + 10n$

$-3m = 5n$ $\quad -9m = 15n.$

$\therefore$ A divides BC externally in the ratio 5 : 3.

d.) $\sin\theta + \cot\theta . \cos\theta = \operatorname{cosec}\theta$

$LHS = \sin\theta + \frac{\cos\theta}{\sin\theta} . \cos\theta .$

$= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta} .$

$= \frac{1}{\sin\theta}$

$= \operatorname{cosec}\theta$

$= RHS .$

e.) $a(x+1)^2 + b(x+1) + c \equiv 3x^2 + 4x + 6$

$LHS = ax^2 + 2ax + a + bx + b + c .$

$= ax^2 + (2a+b)x + (a+b+c)$

$\therefore a = 3$

$\therefore 2a + b = 4 .$

sub. $a = 3$, $b = -2$.

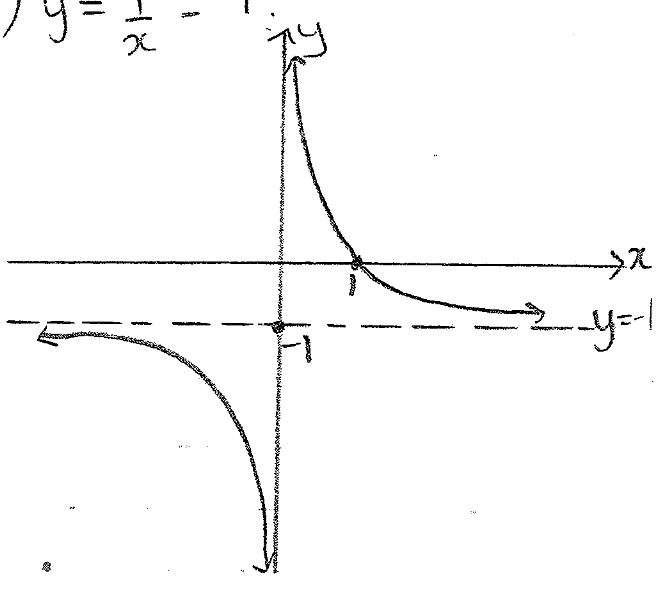
$\therefore a + b + c = 6$

sub. $a = 3$, $b = -2$

$c = 5$.

$\therefore a = 3, b = -2, c = 5.$

f.) $y = \frac{1}{x} - 1.$



Question 2

a.) (i) $\lim_{x\to 2} \frac{x-2}{(x-2)(x+2)}$

$= \lim_{x\to 2} \frac{1}{x+2}$

$= \frac{1}{4}$

(ii) $\lim_{x\to\infty} \frac{3-5/x^2}{4/x^2-5} = -\frac{3}{5}$

(iii) $\lim_{x\to c} \frac{x^2-2x-(c^2-2c)}{x-c}$

$= \lim_{x\to c} \frac{\cancel{(x-c)}(x+c)-2\cancel{(x-c)}}{\cancel{x-c}}$

$= \lim_{x\to c} x+c-2$

$= 2c-2.$

b.) $200, 200(0.8), 200(0.8)^2, \ldots$

(i) $T_{10} = 200(0.8)^9$

$= 26.8$ (2 d.p.)

$\therefore$ 26.8t was excavated in the 10th week.

(ii) $S_{10} = \frac{200(1-(0.8)^{10})}{1-0.8}$

$= 892.6$ (2 d.p.)

$\therefore$ 892.6t was excavated in the first 10 weeks.

(iii) $850 = \frac{200(1-(0.8)^n)}{0.2}$

$\frac{170}{200} = 1-0.8^n$

$0.8^n = \frac{30}{200}$

$\log 0.8^n = \log \frac{30}{200}$

$n = \frac{\log 30/200}{\log 0.8}$

$= 8.5$ (1 d.p.)

$\therefore$ it would take 9 weeks to excavate 850t.

c.) $\Delta = 64 - 4.k.2$

$= 64 - 8k.$

positive definite when $\Delta < 0$, $k > 0$.

$64 - 8k < 0.$

$-8k < -64$

$\therefore k > 8$

d.) $x+1 = \sec\theta$ (1)

$y-2 = \tan\theta$. (2)

$(1)^2 - (2)^2$

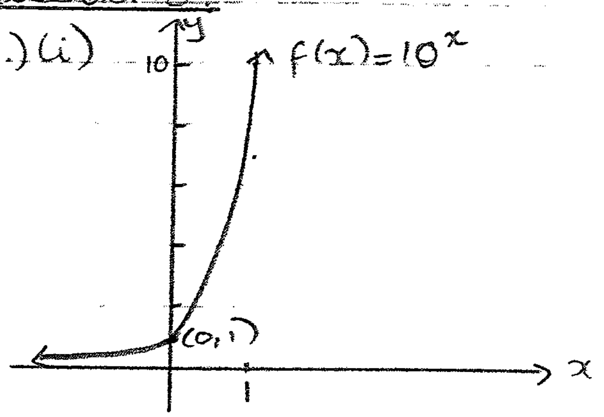
$(x+1)^2 - (y-2)^2 = \sec^2\theta - \tan\theta$

$= \tan^2\theta + 1 - \tan\theta$

$(x+1)^2 - (y-2)^2 = 1.$

Question 3

a.) (i)



(ii) domain: all real x.

range: $y > 0$.

(iii) $f(x)$ has an inverse because the horizontal line test holds true ie. when we draw a horizontal line through $f(x)$ at any point it only cuts the graph once, indicating that there is only one x-value for every y-value.

(iv.) $y = 10^x$.

$f^{-1}(x)$: $x = 10^y$.

$\log_{10} x = \log_{10} 10^y$.

$y = \log_{10} x$.

(v.) domain: $x > 0$

range: all real y.

(vi) $f(3) = 10^3$

$f^{-1}[f(3)] = \log_{10} 10^3$

$= 3$.

b.) (i) $2(1-\sin^2\theta) - 2 + \sin\theta = 0$

$-2\sin^2\theta + \sin\theta = 0$.

$\sin\theta(1 - 2\sin\theta) = 0$.

$\therefore \sin\theta = 0$ or $\sin\theta = \frac{1}{2}$

$\theta = 0°, 180°, 360°$

$\theta = 30°, 150°$

$\therefore \theta = 0°, 30°, 150°, 180°, 360°$.

(ii) $\cos(2\theta + 30) = \frac{1}{2}$

$30° \leq 2\theta + 30 \leq 750°$

$2\theta + 30 = 60°, 300°, 420°, 660°$

$2\theta = 30°, 270°, 390°, 630°$

$\therefore \theta = 15°, 135°, 195°, 315°$.

c.) $y^2 - 10y = 36x + 83$.

$y^2 - 10y + 25 = 36x + 83 + 25$

$(y-5)^2 = 36(x+3)$.

$4a = 36$

$a = 9$.

vertex is $(-3, 5)$

focus is $(6, 5)$.

Question 4

a.) $ax^2+bx+c=0$.

(i) $\alpha+\beta=-\frac{b}{a}$ $\quad \alpha\beta=\frac{c}{a}$

(ii) $\alpha^3+\beta^3=(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)$

$=(\alpha+\beta)[(\alpha+\beta)^2-2\alpha\beta-\alpha\beta]$

$=(\alpha+\beta)[(\alpha+\beta)^2-3\alpha\beta]$

$=\frac{-b}{a}[(-\frac{b}{a})^2-3.(\frac{c}{a})]$

$=-\frac{b}{a}\left(\frac{b^2}{a^2}-\frac{3c}{a}\right)$

$=\frac{-b}{a}\cdot\frac{b^2-3ac}{a^2}$

$=\frac{-b^3+3a}{a^3}$

$=\frac{3a-b^3}{a^3}$.

b.) $x^2+(2x+k)^2=9$.

$x^2+4x^2+4xk+k^2-9=0$.

$5x^2+4xk+(k^2-9)=0$.

If tangential to the circle, it only touches once ie. $\Delta=0$

$\Delta=(4k)^2-4.5.(k^2-9)$

$=16k^2-20k^2+180$.

$=180-4k^2$.

when $\Delta=0$.

$180=4k^2$

$k^2=45$.

$\therefore k=\pm\sqrt{45}$

c.) test statement true for $n=1$.

LHS $=\frac{1}{1(1+1)}$ $\quad$ RHS $=\frac{1}{1+1}$

$=\frac{1}{2}$ $\quad =\frac{1}{2}$

LHS = RHS

$\therefore$ true for $n=1$.

Assume statement is true for $n=k$.

ie. $\frac{1}{1\times2}+\frac{1}{2\times3}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$

Prove statement is true for $n=k+1$.

ie. $\frac{1}{1\times2}+\frac{1}{2\times3}+\ldots+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$.

$S_{k+1}=S_k+T_{k+1}$

$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$

$=\frac{k^2+2k+1}{(k+1)(k+2)}$

$=\frac{(k+1)^2}{(k+1)(k+2)}$

$=\frac{k+1}{k+2}$

$=$ RHS.

$\therefore$ If statement is true for $n=k$, then it is true for $n=k+1$.

Since the statement is true for $n=1$, it is true for $n=2$ and by induction it is true for all d.) $\frac{dV}{dt} = 0.1$, $h = 2r$.

(i) $V = \frac{1}{3}\pi r^2 h$.

sub. $h = 2r$.

$V = \frac{1}{3}\pi r^2 (2r)$.

$= \frac{2}{3}\pi r^3$

(ii)(α) $\frac{dr}{dt} = \frac{dV}{dt} \cdot \frac{dr}{dV}$.

$\frac{dV}{dr} = 2\pi r^2$.

$\frac{dr}{dt} = \frac{1}{10} \cdot \frac{1}{2\pi r^2}$

$= \frac{1}{20\pi r^2}$

when $r = 0.6$.

$\frac{dr}{dt} = \frac{1}{20\pi(0.6)^2}$

$= \frac{5}{36\pi}$

∴ the radius is increasing at $\frac{5}{36\pi}$ m/min.

(β) $\frac{dh}{dt} = \frac{dh}{dr} \cdot \frac{dr}{dt}$

$\frac{dh}{dr} = 2$.

$\frac{dh}{dt} = 2 \cdot \frac{5}{36\pi}$

$= \frac{5}{18\pi}$

∴ the height is increasing at a rate of $\frac{5}{18\pi}$ m/min.

Question 5.

a) $P(2p, ap^2)$ $Q(2q, aq^2)$ $x^2 = 4y$

(i) $m = \frac{q^2 - p^2}{2q - 2p}$

$= \frac{(q-p)(q+p)}{2(q-p)}$

$= \frac{p+q}{2}$

$y - p^2 = \frac{p+q}{2}(x - 2p)$.

$y - p^2 = \frac{1}{2}(p+q)x - p^2 - pq$

$y - \frac{1}{2}(p+q)x + pq = 0$

(ii) $x = \frac{2p+2q}{2}$, $y = \frac{p^2+q^2}{2}$

$= p+q$

$m\left(p+q, \frac{1}{2}(p^2+q^2)\right)$.

(iii) sub. $(0,4)$ in PQ.

$4 - 0 + pq = 0$.

$\therefore pq = -4$.

$x = p+q$ (1) $\quad y = \frac{1}{2}(p^2+q^2)$ (2)

$= \frac{1}{2}\left[(p+q)^2 - 2pq\right]$

sub. (1) in (2).

$y = \frac{1}{2}\left[x^2 - 2pq\right]$.

sub. $pq = -4$,

$y = \frac{1}{2}(x^2 - 8)$.

$2y = x^2 - 8$.

$\therefore x^2 = 2(y-4)$ is locus of

b) $f(x) = \begin{cases} x+1 & , x \le 1 \\ a - b(x-2)^2 & , x > 1 \end{cases}$.

(i) $f(1) = 2$.

$\lim_{x \to 1^+} a - b(x-2)^2 = 2$.

$\therefore a - b = 2$.

(ii) $f'(x) = \begin{cases} 1 & , x \quad 1 \\ -2b(x-2) & , x > 1 \end{cases}$

$f'(1) = 1$.

$\lim_{x \to 1^+} -2b(x-2) = 1$.

$-2b(1-2) = 1$.

$2b = 1$.

$b = \frac{1}{2}$

sub. $b = \frac{1}{2}$.

$a - \frac{1}{2} = 2$.

$a = 2\frac{1}{2}$.

$\therefore a = 2\frac{1}{2}$, $b = \frac{1}{2}$.

c.) $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$= -2t \cdot \frac{1}{3t^2-3}$

$= \frac{-2t}{3t^2-3}$

(ii) when $t=2$, $\frac{dy}{dx} = -\frac{4}{9}$

when $t=2$, $x = 2$

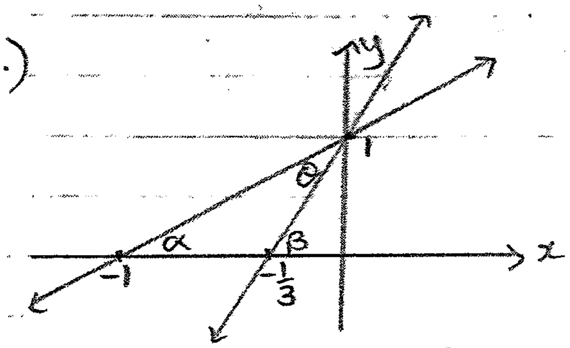
$y = -1$

$y + 1 = -\frac{4}{9}(x-2)$

$9y + 9 = -4x + 8$

$\therefore 4x + 9y + 1 = 0$ is the equation of the tangent at $t = 2$.

d.)



$$\alpha + \theta = \beta.$$

$\tan \alpha = 1$ $\qquad$ $\tan \beta = 3.$

$\alpha = 45^\circ$ $\qquad$ $\beta = 71^\circ 34'$

$$\theta = 71^\circ 34' - 45^\circ$$

$$= 26^\circ 34' \text{ (nearest min.)}$$