St George Girls High School

Year 11 – Higher School Certificate Course

Assessment Task 1

2005

Mathematics

General Instructions

- Reading time – 5 minutes.
- Working time – 75 minutes.
- Write using black or blue pen.
- Attempt all questions.
- Start each question on a new page.
- Show ALL working.
- Marks for each question are shown in right column
- Complete cover sheet clearly showing
  - your name
  - mathematics class and teacher

Question 1 (12 marks) – Start a New Page **Marks**

a) Find the second derivative of $y = \sqrt{x}$ **2**

b) Find the equation of the locus of a point $P(x, y)$ such that it is equidistant from $A(-4, 3)$ and $B(6, 5)$ **3**

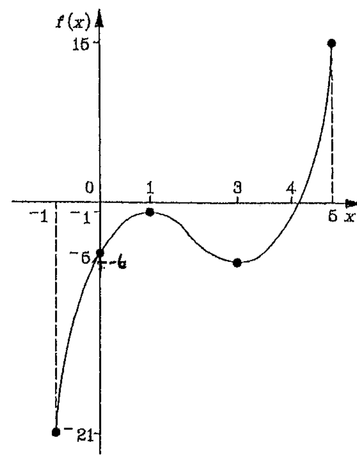
c) Express $x^2 - 3x + 2$ in the form $A(x-1)^2 + B(x-1) + C$ **3**

d) What values of m will make the expression $x^2 + 6x + m$ positive definite? **2**

e) Use the graph below to identify the co-ordinates of the: **2**

(i) global maximum

(ii) local minimum



Marks

Question 2 (11 marks) – Start a New Page

a) Find the equation of the parabola which has the co-ordinates of the focus $(-1, 2)$ and equation of the directrix $y = 4$. **3**

b) Explain why the roots of $3x^2 - 7x - 3 = 0$ can be described as being irrational and distinct. **2**

c) For what intervals of x is the function $f(x) = 2x^3 + 3x^2 - 12x - 4$ increasing? **3**

d) Draw a neat sketch of the function where: **1**

$f(-1) = 2$ $\quad f(4) = -3$

$f'(-1) = 0$ $\quad f'(4) = 0$

$f''(-1) < 0$ $\quad f''(4) > 0$

e) Find the primitive function of: **2**

(i) $\dfrac{6}{x^2}$ (ii) $(3x-5)^6$

Marks

Question 3 (12 marks) – Start a New Page

a) Given that α and β are the roots of the equation $x^2 - 3x - 5 = 0$, find the values of: **10**

(i) $\alpha + \beta$

(ii) $\alpha\beta$

(iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$

(iv) $\alpha^2 + \beta^2$

(v) $\left(\alpha - \dfrac{1}{\beta}\right)\left(\beta - \dfrac{1}{\alpha}\right)$

(vi) $(\alpha - \beta)^2$

b) Explain why $y = -3x + 2$ is decreasing for all x. **2**

Question 4 (12 marks) – Start a New Page **Marks**

a) Given the curve $y = x^4 - 8x^3 + 18x^2 + 1$, **10**

(i) Show that there are stationary points at $(0, 1)$ and $(3, 28)$ and determine their nature.

(ii) Find any points of inflexion.

(iii) Hence, sketch the curve $y = x^4 - 8x^3 + 18x^2 + 1$

b) If one root of the equation $x^2 + bx + c = 0$ is twice the other root, show that $2b^2 = 9c$ **2**

Question 5 (11 marks) – Start a New Page

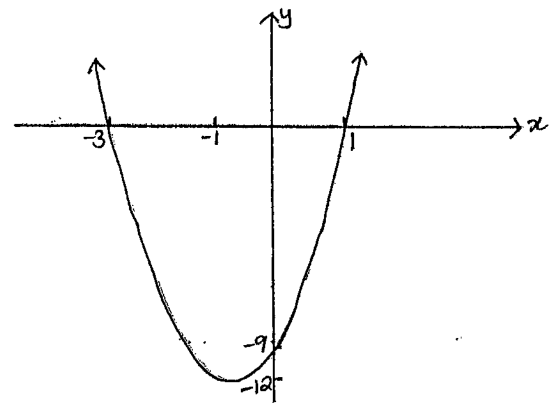
a) Solve $\frac{1}{x^4} + \frac{2}{x^2} - 3 = 0$ by using a suitable substitution. **3**

b) Find the values of k for which $x^2 - (k+3)x + 4k = 0$ has equal roots. **3**

c) The gradient function of a curve is $x^2 + 2x - 15$. Find the equation of the curve if it passes through the point $(3, -20)$. **3**

d) The graph below is of $y = f'(x)$. Copy the diagram into your answer booklet. **2**

On the same axes sketch a possible curve for $y = f(x)$

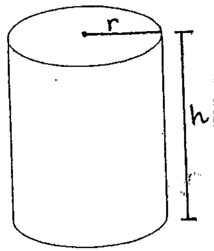


Question 6 (12 marks) – Start a New Page **Marks**

a) A cylindrical can with a base and no top is to be made from $300\pi\ \text{cm}^2$ of sheet metal. 6

(i) Show that the volume of the can is given by:

$$V = 150\pi r - \frac{\pi r^3}{2}$$



(ii) If the volume of the can is to be a maximum find the radius of the base.

b) Find the centre and radius of the circle $x^2 + 2y + y^2 - 8x + 13 = 0$ 3

c) For the equation $kx^2 - (k-1)x - (k+4) = 0$, find the value of k for which the roots are reciprocals of one another. 3

End of Paper

① a) $y = x^{\frac{1}{2}}$

$y' = \frac{1}{2}x^{-1/2}$

$y'' = -\frac{1}{4}x^{-1\frac{1}{2}}$

b) $A(-4,3)$ $B(6,5)$

midpt AB $= (1,4)$

gradient of AB $= \frac{5-3}{6--4}$

$= \frac{1}{5}$

∴ m of ⊥ bisector $= -5$

∴ Eqn of locus:

$y-4 = -5(x-1)$

$y-4 = -5x+5$

$5x+y-9=0$

c) $x^2-3x+2 = A(x-1)^2 + B(x-1) + C$

$x^2-3x+2 = Ax^2-2Ax+A+Bx-B+C$

∴ $A=1$ $\quad -2A+B=-3 \quad A-B+C=2$

∴ $B=-1 \quad 1+1+C=2$

∴ $C=0$

∴ $x^2-3x+2 = (x-1)^2-(x-1)$

d) x^2+6x+m

For positive definite $a>0$ $\Delta<0$

$a=1$ ∴ $a>0$

$\Delta = b^2-4ac$

$= 36-4\times1\times m$

$= 36-4m$

∴ $36-4m<0$

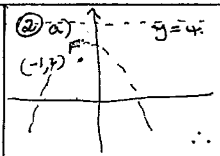
$36<4m$

$m>9$

e) i) global maximum is $(5,15)$

ii) local minimum is $(3,-6)$

② a)



Vertex $(-1,3)$

$a=1$

∴ Eqn is: $(x+1)^2 = -4(y-3)$

(b) $3x^2-7x-3=0$

$\Delta = b^2-4ac$

$= 49-4\times3\times-3$

$= 49+36$

$= 85$

Since $\Delta>0$ and it is not a perfect square, the roots are irrational and distinct.

(c) $f(x) = 2x^3+3x^2-12x-4$

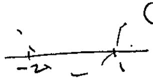
$f'(x) = 6x^2+6x-12$

For increasing function $f'(x)>0$

∴ $6x^2+6x-12>0$

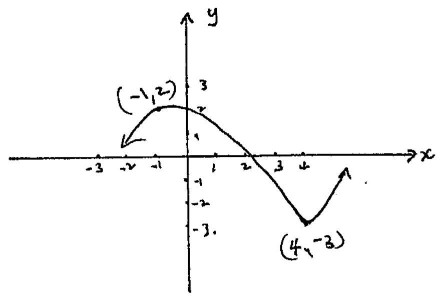
$x^2+x-2>0$

$(x+2)(x-1)>0$



∴ $x<-2$ or $x>1$

d)



e) i) $\int 6x^{-2}\,dx = \frac{6x^{-1}}{-1} + c$

$= -\frac{6}{x} + c$

ii) $\int (3x-5)^6\,dx = \frac{(3x-5)^7}{7\times3} + c$

$= \frac{(3x-5)^7}{21} + c$

③ a) $x^2-3x-5=0$

i) $\alpha+\beta = -\frac{b}{a}$

$= 3$

ii) $\alpha\beta = \frac{c}{a}$

$= -5$

iii) $\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\beta+\alpha}{\alpha\beta}$

$= \frac{3}{-5}$

iv) $\alpha^2+\beta^2 = (\alpha+\beta)^2-2\alpha\beta$

$= 3^2-2\times-5$

$= 19$

v) $(\alpha-\frac{1}{\beta})(\beta-\frac{1}{\alpha})$

$= \alpha\beta-1-1+\frac{1}{\alpha\beta}$

$= -5-2+\frac{1}{-5}$

$= -7\frac{1}{5}$

vi) $(\alpha-\beta)^2 = \alpha^2-2\alpha\beta+\beta^2$

$= (\alpha+\beta)^2-2\alpha\beta-2\alpha\beta$

$= 9-4\times-5$

$= 29$

b) $y=-3x+2$

$y' = -3$

Since derivative is a negative constant the curve is decreasing for all values of x.

④ a) $y = x^4-8x^3+18x^2+1$

i) $y' = 4x^3-24x^2+36x$

$y'' = 12x^2-48x+36$

Stat. points when $y'=0$

∴ $4x^3-24x^2+36x=0$

$x^3-6x^2+9x=0$

$x(x^2-6x+9)=0$

∴ $x(x-3)^2=0$

∴ $x=0 \qquad x=3$

$y=1 \qquad y=28$

$x=0$: $y''=36>0$

∴ min t. pt at $(0,1)$

$x=3$: $y'' = 12\times9-48\times3+36$

$=0$

∴ inconclusive

(ii) Inflexions when $y''=0$ & sign change

∴ $12x^2-48x+36=0$

$x^2-4x+3=0$

$(x-3)(x-1)=0$

$x=3 \qquad x=1$

$y=28 \qquad y=12$.

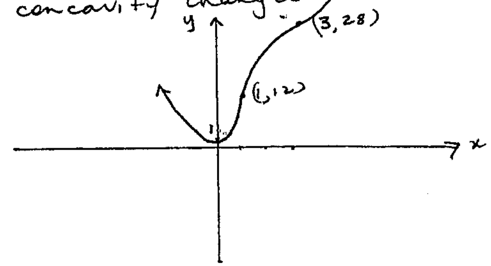
Check for sign change

x	$2\frac{1}{2}$	3	$3\frac{1}{2}$
y''	-9	0	15

x	$\frac{1}{2}$	1	$1\frac{1}{2}$
y''	15	0	-9

∴ $(3,28)$ is point of inflexion since concavity changes but since $y'=0$ for $x=3$, this point is a horizontal point of inflexion

∴ $(1,12)$ is a point of inflexion since concavity changes.



b) $x^2+bx+c=0$

Let roots be α and 2α

$\alpha+2\alpha = b \qquad \alpha\times2\alpha = c$

$3\alpha = b \qquad 2\alpha^2 = c$

∴ $\alpha = \frac{b}{3} \qquad$ ∴ $2\times\left(\frac{b}{3}\right)^2 = c$

$\frac{2b^2}{9} = c$

5 a) $\frac{1}{x^4} + \frac{2}{x^2} - 3 = 0$

Let $m = \frac{1}{x^2}$

$\therefore m^2 + 2m - 3 = 0.$

$(m+3)(m-1) = 0$

$\therefore m = -3 \quad \therefore m = 1$

$\therefore \frac{1}{x^2} = -3 \qquad \frac{1}{x^2} = 1$

$1 = -3x^2 \qquad \therefore x = \pm 1$

$x^2 = \frac{1}{-3}$

∴ no real solutions

b) $x^2 - (k+3)x + 4k = 0$

Equal roots $\Delta = 0$

$[-(k+3)]^2 - 4 \times 1 \times 4k = 0$

$k^2 + 6k + 9 - 16k = 0.$

$k^2 - 10k + 9 = 0$

$(k-9)(k-1) = 0$

$\therefore k = 9$ or $k = 1$

c) $y = \int (x^2 + 2x - 15)\, dx$

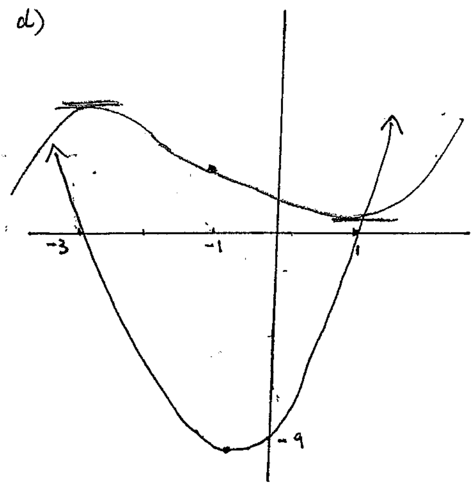
$y = \frac{x^3}{3} + x^2 - 15x + c$

$-20 = \frac{27}{3} + 9 - 45 + c$

$\therefore c = 7$

$\therefore \quad y = \frac{x^3}{3} + x^2 - 15x + 7$

d)



6 a) S.A. $= \pi r^2 + 2\pi r h = 300\pi$ 1

$\therefore r^2 + 2rh = 300$

$h = \frac{300 - r^2}{2r}$ 1

Volume $= \pi r^2 h$ ½

$= \pi r^2 \left(\frac{300 - r^2}{2r} \right)$ 1

$= \frac{\pi r}{2}(300 - r^2)$

$= 150\pi r - \frac{\pi r^3}{2}$ ½

ii) $V' = 150\pi - \frac{3\pi r^2}{2}$

$V'' = -\frac{6\pi r}{2} = -3\pi r$

Max when $V' = 0$ 1

$\therefore 150\pi - \frac{3\pi r^2}{2} = 0$

$300\pi = 3\pi r^2$

$r^2 = 100$ 1

$\therefore r = \pm 10.$

$r > 0$ since length $\therefore r = 10$

Test for max:

$r = 10 \quad V'' = -3\pi \times 10 < 0$ 1

∴ Max. volume when radius is 10 cm.

b) $x^2 + 2y + y^2 - 8x + 13 = 0$

$x^2 - 8x + 16 + y^2 + 2y + 1 = -13 + 16 + 1$ 1

$(x-4)^2 + (y+1)^2 = 4$ 1

∴ centre is $(4, -1)$ $r = 2$. 1 ½ ea.

c) $kx^2 - (k-1)x - (k+4) = 0$

If reciprocals product is 1. 1

$\therefore \frac{c}{a} = 1$ 1

$\frac{-(k+4)}{k} = 1$ ← ①

$-k - 4 = k$

$-4 = 2k$

$\therefore k = -2$ 1

2unit.

Revise

Quadratic sum + product

Quadratic → discriminant.

Locus

Max + Min

1 ✓
2 ✓
3
4 ✓

5
6 ✓
7
8 ✓