St George Girls High School

Year 11

Common Test 2

June 2006

Mathematics Extension 1

General Instructions

1. Working time – 75 minutes
2. Use only black or blue pens.
3. Board-approved calculators may be used.
4. All necessary working must be shown in all questions.
5. Start each question on a new page.
6. All 6 questions may be attempted.

Question 1 – 10 marks – (Start a new page) **Marks**

a) Find the gradient of the line perpendicular to $\frac{x}{3}+\frac{y}{4}=1$ 2

b) Solve the equation $2^x \cdot 4^{x+1}=\frac{1}{2}$ 2

c) State whether the functions are odd, even or neither. 2

(i) $f(x)=\frac{x^3}{x^2+1}$ (ii) $f(x)=2^x-2^{-x}$

d) For each of the following, T_n is the n^{th} term of a series. Determine whether it is arithmetic, geometric or neither. 2

(i) $T_n=\frac{2-3n}{2}$ (ii) $Tn=(\log_{10}2)^n$

e) Graph the solution to $x^2-4<0$ on the number line. 2

Question 2 – 10 marks – (Start a new page) **Marks**

a) A sequence has terms $2, \frac{5}{2}, \ldots$

(i) Find the next term if the sequence is arithmetic. 1

(ii) Find the previous term if the sequence is geometric. 1

b) In the sequence 20, 18.8, 17.6 ...

(i) Show that this sequence is arithmetic. 1

(ii) Find the first negative term. 2

(iii) Find the least number of terms to be added to make a negative sum. 2

c) For a certain series, the sum of the first n terms is given by $S_n = 2n^2 - 5n$

(i) Find an expression for the sum of the first $(n-1)$ terms. 1

(ii) Deduce an expression for the n^{th} term. 2

Question 3 – 10 marks – (Start a new page) **Marks**

a) Find the distance between the parallel lines $2x-3y-10=0$ and $4x-6y+5=0$ 2

b) If $A \equiv (-2, 3)$, $B \equiv (4, -3)$ and $C \equiv (p, 2)$, find the value of p given that A, B, C are collinear. 2

c) $A(1, 1)$, $B(3, 5)$ and $C(9, -1)$ are points on the number plane. The points M and N divide BC and CA respectively in the same ratio $k:1$

(i) Find the coordinates of M and N in terms of k. 2

(ii) Deduce the gradient of MN is $(3-k)/(4k-3)$ 2

(iii) Hence find the value of k if MN is perpendicular to CA. 2

Question 4 – 10 marks – (Start a new page) **Marks**

a) Consider the function $f(x)=\dfrac{x}{x^2-1}$

(i) Prove this is an odd function. 1

(ii) Find the zero(s) of $f(x)$ 1

(iii) State the equations of the vertical asymptotes. 1

(iv) Explain what happens to the curve $y=f(x)$ as $x\to\pm\infty$ 1

(v) Draw a neat sketch of $y=f(x)$ 2

b) Solve for x:

(i) $\dfrac{2x}{x-1}<1$ 2

(ii) $\left|\dfrac{2x}{x-1}\right|=2$ 2

Question 5 – 10 marks – (Start a new page) **Marks**

a) Prove the following:

$1\times2+2\times2^2+3\times2^3+\cdots+n\times2^n=(n-1)2^{n+1}+2,\ n$ a positive integer. 4

b) Simplify:

(i) $\dfrac{25^{3n}\times\left(\frac{1}{5}\right)^{2n-1}}{5^{1-2n}}$ 2

(ii) $\dfrac{3^m+9^m}{9^m+27^m}$ 2

(iii) $\log_{\frac{1}{2}}\left(8^{-x}\right)$ 2

2^{3-x}

$-x \log_{\frac{1}{2}} 8$

$= -x \log$

Question 6 – 10 marks – (Start a new page) **Marks**

a) The series $1+\frac{1-p}{p}+\left(\frac{1-p}{p}\right)^2+\left(\frac{1-p}{p}\right)^3+\cdots,\ p>0$ has a limiting sum.

Find:

(i) the range of values of p for which the series has a limiting sum.

(ii) an expression for the limiting sum in terms of p. 1

b) Find x if:

(i) $x,\ 2x-1,\ 2x+2$ is an arithmetic sequence. 2

(ii) $x,\ 2x-3,\ 3x-2$ is a geometric sequence. 2

c) Evaluate $\sum_{r=1}^{20}\left(2r+2^r\right)$ leaving the answer in index form. 3

End of Paper

Extension 1
Year 11 CT #2
June 2006
Solutions

Q1 a) $\frac{x}{3}+\frac{y}{4}=1$

$\therefore 4x+3y=12$

$3y=-4x+12$

$y=-\frac{4}{3}x+4$

(2) $\therefore$ slope $=-4/3$

$\therefore$ line perpendicular to this line has slope of $3/4$.

b) $2^x \cdot 4^{x+1}=\frac{1}{2}$

$2^x \cdot (2^2)^{x+1}=2^{-1}$

$2^{x+2x+2}=2^{-1}$

$2^{3x+2}=2^{-1}$

$3x+2=-1$

(2) $3x=-3$

$x=-1$

c) (i) $f(x)=\frac{x^3}{x^2+1}$

$f(-x)=\frac{(-x)^3}{(-x)^2+1}$

(1) $=-\frac{x^3}{x^2+1}$

$=-f(x)$

$\therefore$ odd

(ii) $f(x)=2^x-2^{-x}$

$f(-x)=2^{-x}-2^{--x}$

$=-(2^x-2^{-x})$

(1) $=-f(x)$

d) (i) $T_n = \frac{2-3n}{2}$

A.P.

Proof: $T_n - T_{n-1} = \frac{2-3n}{2} - \cancel{2(} \frac{2-3(n-1)}{2}$

$= -3/2$

(1) $\therefore$ common difference of $-3/2$

OR $T_1 = -\frac{1}{2}$, $T_2 = -2$, $T_3 = -3\frac{1}{2}$ $\therefore$ common difference of $-3/2$.

(ii) $T_n = (\log_{10}2)^n$

$T_1 = \log_{10}2$

$T_2 = (\log_{10}2)^2$

(1) $T_3 = (\log_{10}2)^3$ $\therefore$ G P with $r = \log_{10}2$.

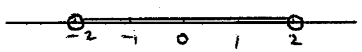
OR $\frac{T_n}{T_{n-1}} = \frac{(\log_{10}2)^n}{(\log_{10}2)^{n-1}} = \log_{10}2$

e) $x^2 - 4 < 0$

$\therefore x^2 < 4$

$|x| < 2$

(2)



Question 2

(1) a) (i) 3 $(d = \frac{1}{2})$

(ii) $r = 5/2 \div 2$

$= 5/4$

$\therefore x \times 5/4 = 2$

(1) $\therefore x = 8/5$

ie previous term is $8/5$.

b) (i) $18.8 - 20 = 17.6 - 18.8 = -1.2$

(1) AP with $d = -1.2$

(ii) $T_n = a + (n-1)d$

$= 20 + (n-1)(-1.2)$

$\therefore 20 - 1.2n + 1.2 < 0$

$21.2 < 1.2n$

$n > \frac{21.2}{1.2}$

(2) $n > 17\frac{2}{3}$

$\therefore n = 18$

$T_{18} = 20 + 17 \times -1.2$

$= -0.4$

(iii) $S_n = \frac{n}{2}(2a + (n-1)d) < 0$

$\therefore \frac{n}{2}(40 + (n-1)(-1.2)) < 0$

$n(40 - 1.2n + 1.2) < 0$

$n(41.2 - 1.2n) < 0$

(2) $\therefore n \not< 0$, $n > \frac{41.2}{1.2}$

$n > 34\frac{1}{3}$

$\therefore$ at least 35 terms required to give a negative partial sum.

c) (i) $S_n = 2n^2 - 5n$

$\therefore S_{n-1} = 2(n-1)^2 - 5(n-1)$

$= 2n^2 - 4n + 2 - 5n + 5$

(1) $= 2n^2 - 9n + 7$.

(ii) $T_n = S_n - S_{n-1}$

(2) $= 2n^2 - 5n - (2n^2 - 9n + 7)$

Question 3

a) $2x - 3y - 10 = 0$ ①

$4x - 6y + 5 = 0.$ ②

(5,0) lies on line ①

∴ find distance from (5,0) to ②

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$$

$$= \frac{|4\times 5 - 6\times 0 + 5|}{\sqrt{4^2+(-6)^2}}$$

(2)

$$= \frac{25}{\sqrt{52}}$$

$$= \frac{25\sqrt{52}}{52} \text{ units.}$$

b) A, B, C collinear

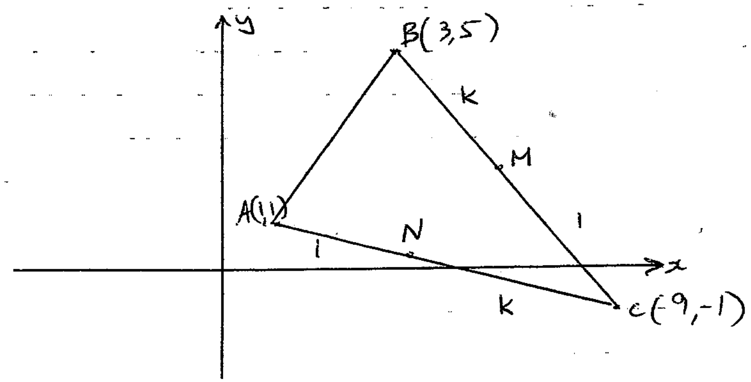
∴ slope AB = slope BC

$$\frac{-6}{6} = -1 = \frac{-3-2}{4-p}$$

(2) ∴ $p - 4 = -5$

$p = -1$

c)



(i) $M(x,y)$ $\quad x = \frac{x_1 + kx_2}{1+k}$ $\quad y = \frac{y_1+ky_2}{1+k}$

(2) $\quad = \frac{3+9k}{1+k}$ $\quad = \frac{5-k}{1+k}.$

$N(x^*, y^*)$ $\quad x^* = \frac{9+k}{1+k}$ $\quad y^* = \frac{-1+k}{1+k}.$

(ii) slope $MN = \dfrac{\frac{5-k}{1+k} - \frac{-1+k}{1+k}}{\frac{9k+3}{1+k} - \frac{9+k}{1+k}}$

$$= \frac{5-k+1-k}{9k+3-9-k}$$

(2)

$$= \frac{6-2k}{8k-6}$$

$$= \frac{3-k}{4k-3}$$

(iii) If $MN \perp CA$,

$$\frac{3-k}{4k-3} \times \frac{1+1}{1-9} = -1 \quad (m_1 m_2 = -1 \text{ for perp. lines})$$

$$\frac{3-k}{4k-3} \times -\frac{1}{4} = -1$$

(2)

$$\frac{3-k}{4k-3} = 4.$$

$3 - k = 16k - 12$

$k = \frac{15}{17}.$

Question 4

a) $f(x) = \frac{x}{x^2-1}$

(1) (i) $f(-x) = \frac{-x}{x^2 \ldots}$ $= -f(x)$ ∴ odd

(ii) $f(x) = 0 \rightarrow \dfrac{x}{x^2-1} = 0$

(1) $\therefore x = 0$

(1) (iii) $x = \pm 1$

(iv) as $x \to +\infty$, $f(x) \to 0^+$

(1) as $x \to -\infty$, $f(x) \to 0^-$

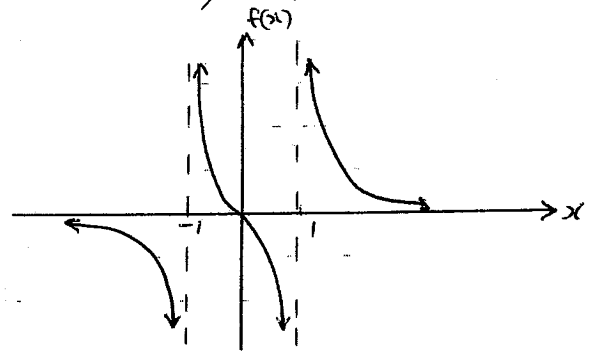
(v) as $x \to -1^+$, $f(x) \to +\infty$

as $x \to -1^-$, $f(x) \to -\infty$

as $x \to 1^+$, $f(x) \to +\infty$

as $x \to 1^-$, $f(x) \to -\infty$

(2)



b) (i) $\dfrac{2x}{x-1} < 1$

$\dfrac{2x}{x-1} \times (x-1)^2 < (x-1)^2$

ie $2x(x-1) < (x-1)^2$

$(x-1)[x-1-2x] > 0$

$(x-1)(-x-1) > 0$

$-(x-1)(x+1) > 0$

$(x-1)(x+1) < 0$

(2) $\therefore -1 < x < 1$

OR "critical value" method ie "hole & test"

ie "intermediate value" theorem

(ii) $\left|\dfrac{2x}{x-1}\right| = 2$

$\dfrac{2x}{x-1} = 2$ or $\dfrac{2x}{x-1} = -2$

(2) $\therefore 2x = 2x - 2$ $\qquad 2x = -2x + 2$

$0 = -2$ $\qquad 4x = 2$

no sol'n. $\qquad \underline{x = \tfrac{1}{2}}$

(26) Question 5

a) $1\times 2 + 2\times 2^2 + 3\times 2^3 + \dots + n\times 2^n = (n-1)2^{n+1} + 2$

Step 1 Show true for $n = 1, 2$

$n=1$: $1\times 2 = (1-1).2^{1+1} + 2$ ✓

$n=2$: $1\times 2 + 2\times 2^2 = (2-1).2^{2+1} + 2$

$10 = 2^3 + 2$ ✓

Step 2 Let $n=k$ be value for which result holds. ie $S_k = (k-1)2^{k+1} + 2$

Consider $n = k+1$

(4)

$S_{k+1} = S_k + T_{k+1}$

$= (k-1)2^{k+1} + 2 + (k+1)2^{k+1}$

$= 2^{k+1}(k-1+k+1) + 2$

$= 2^{k+1}.2k + 2$

$= 2^{k+1+1}.k + 2$

$= 2^{(k+1)+1}(k+1-1) + 2$

which is of the form $S_n = (n-1)2^{n+1} + 2$ with n replaced by $k+1$.

ie if result holds for $n=k$, it also holds for $n = k+1$.

Step 3 Since result holds for $n = 1, 2$ it also holds for $n = 2+1 = 3$ & hence

b) (i) $\dfrac{25^{3n} \times \left(\frac{1}{5}\right)^{2n-1}}{5^{1-2n}} = \dfrac{(5^2)^{3n} \cdot (5^{-1})^{2n-1}}{5^{1-2n}}$

$= 5^{6n-2n+1-(1-2n)}$ (2)

$= 5^{6n}$

(ii) $\dfrac{3^m+9^m}{9^m+27^m} = \dfrac{3^m+9^m}{(3\times3)^m+(3\times9)^m}$

$= \dfrac{3^m+9^m}{3^m(3^m+9^m)}$ (2)

$= \dfrac{1}{3^m}$

$= 3^{-m}$. $x = -2x+2$

(iii) let $\log_{\frac{1}{2}} 8^{-x} = y$

$\therefore 8^{-x} = \frac{1}{2}^y$

$(2^3)^{-x} = (2^{-1})^y$

$2^{-3x} = 2^{-y}$

$\therefore -y = -3x$ (2)

$y = 3x$

$\therefore \log_{\frac{1}{2}} 8^{-x} = 3x$.

Q6

a) $1 + \dfrac{1-p}{p} + \left(\dfrac{1-p}{p}\right)^2 + \dots$ $p>0$

(i) limiting sum if $|r|<1$

ie $\left|\dfrac{1-p}{p}\right|<1$

$|1-p| < |p|$

ie $|1-p| < p$ since $p>0$

$\therefore -p < 1-p < p$

ie $-p<1-p$ $\cap$ $1-p<p$ (2)

$0<1$ ✓ $\cap$ $1<2p$

$\therefore 2p \geq 1$.

$p > \frac{1}{2}$.

ii) $S = \dfrac{a}{1-r}$

$= \dfrac{1}{1-\left(\frac{1-p}{p}\right)} \times \dfrac{p}{p}$ (1)

$= \dfrac{p}{p-(1-p)}$

$= \dfrac{p}{2p-1}$

b) (i) $x,\ 2x-1,\ 2x+2$

AP $\therefore d = 2x-1-x = 2x+2-(2x-1)$ (2)

ie $x-1 = 3$

$\therefore x = 4$

(ii) x, $2x-3$, $3x-2$

G.P. $\therefore r = \frac{2x-3}{x} = \frac{3x-2}{2x-3}$

$$\therefore (2x-3)^2 = x(3x-2)$$ (2)

$$4x^2-12x+9 = 3x^2-2x$$

$$x^2-10x+9=0$$

$$(x-9)(x-1)=0$$

$$\therefore x=1, 9$$

d) $\sum_{r=1}^{20} (2r+2^r)$

$$= 2+2^1+4+2^2+6+2^3+\dots+40+2^{20}.$$

ie $S = (2+4+\dots+40) + 2+2^2+\dots+2^{20}$.

[ie sum of AP & G.P.]

= Using $S_n = \frac{n}{2}(a+l)$; $S_n = a\frac{(r^n-1)}{r-1}$

gives:

(3) $S = 10(2+40) + \frac{2(2^{20}-1)}{2-1}$

$$= 420 + 2^{21} - 2$$

$$= 418 + 2^{21}$$