St George Girls High School

Year 11

End of Preliminary Course Examination

2005

correction- 5-a

Mathematics Extension 1

Time Allowed: 2 hours
(plus 5 minutes reading time)

Instructions

1. All questions should be attempted.
2. All necessary working must be shown.
3. Begin each question on a **new page**.
4. Marks will be deducted for careless work or poorly presented solutions.

Question 1 (16 marks) – Start a New Page **Marks**

a) Evaluate the following limits

(i) $\lim_{x \to \infty} \frac{3-5x}{2x+1}$ 1

(ii) $\lim_{h \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$ 2

b) In what ratio does $C(-4,-2)$ divide the interval joining $A(1,3)$ to $B(-1,1)$? 3

c) Solve for x: 3

$\frac{x}{x-1} \geq 2$

d) (i) Solve for x: $|x-2| = x+1$ 2

(ii) On the same set of axes sketch $y = |x-2|$ and $y = x+1$ 2

(iii) For which values of x is $|x-2| < x+1$? 1

e) Show that the expression $x^2 - 4x + 9$ is positive for all values of x. 2

Question 2 (16 marks) – Start a New Page **Marks**

a) Find the derivatives of:

(i) $\dfrac{3-2x}{3+2x}$ 2

(ii) $\dfrac{1}{\sqrt{1+x^3}}$ 2

b) (i) Solve for $0^\circ \le x \le 360^\circ$

$$\sin x = \frac{1}{2}$$ 2

(ii) Sketch $y = \sin x$ for $0^\circ \le x \le 360^\circ$ 1

(iii) Hence or otherwise solve for $0^\circ \le x \le 360^\circ$, 2

$$\sin x \le \frac{1}{2}$$

c) Solve for x correct to 3 significant figures: $3^x = 5$ 2

d) Consider the parabola $y^2 + 8y + 4x - 4 = 0$ $y(y+8) + 4(x-1) = 0$

(i) Find the vertex $4x - 4 = 0$ 2

(ii) Find the focus $4(x-1) = 0$ 1

(iii) Find the y intercepts 2

$x - 1 = 0$

$\therefore x = 1$

$x^2 + 8x + 4y - 4 = 0$

Question 3 (16 marks) – Start a New Page **Marks**

a) (i) Prove that $\sec^2 x + \tan x - 7 \equiv \tan^2 x + \tan x - 6$ 1

(ii) Hence solve for $0^\circ \le x \le 360^\circ$ (correct to the nearest minute) 4

$$\sec^2 x + \tan x - 7 = 0$$

b) Evaluate $\sum_{r=1}^{\infty} 32\left(-\frac{1}{2}\right)^r$

c) (i) Factor $2^{m+1} - 2^{m-1}$ 1

(ii) Hence or otherwise solve for x: $2^x = \dfrac{2^{3004} - 2^{3002}}{3}$ 2

d) If one root of $x^2 + mx + n = 0$ is twice the other, show that $2m^2 - 9n = 0$ 3

e) Use Mathematical Induction to prove that $7^n - 5^n$ is an even number for all integers $n \ge 1$. 3

$ax^2 + by + c = 0$

Question 4 (16 marks) – Start a New Page **Marks**

a) (i) Prove that the circle $x^2 + y^2 - 6x - 2y + 9 = 0$ has centre (3, 1) and radius 1. 2

(ii) Prove that $3x - 4y = 0$ is a tangent to the above circle. 2

b) (i) Solve for $x: x^2 - 2x - 8 \le 0$ 2

(ii) Hence find the domain of $f(x) = \log(8 + 2x - x^2)$ 2

c) Solve for θ where $0^\circ \le \theta \le 360^\circ$ $\tan(2\theta + 45^\circ) = 1$ 3

d) The following is a correct solution to a problem. 2

$$y = x^2 + 1$$

$$\therefore \frac{dy}{dx} = 2x$$

at (2, 5) $\frac{dy}{dx} = 2(2)$

$$= 4$$

$\therefore$ Equation is

$$y - 5 = 4(x - 2)$$

$$y - 5 = 4x - 8$$

$$y = 4x - 3$$

ie $4x - y - 3 = 0$

Question: Explain in detail what the student was asked to find.

e) Find the equation of the normal to $y = \frac{3}{x}$ at (3, 1) 3

Question 5 (16 marks) – Start a New Page **Marks**

a) Find the values of a and b if $2x^2 - 15x \equiv a(x-3)^2 + b(x-3)$ 2

$x(2x-15)$

b) A spherical soap bubble is expanding so that its volume is increasing at a constant rate of 10mm³/s. At what rate is the radius increasing when the surface area is 100mm² 3

c) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$.

(i) Prove that the tangent to $x^2 = 4ay$ at P is $px - y - ap^2 = 0$ 3

(ii) Find the point T, the intersection of the tangents at P and Q. 2

(iii) Find the mid-point M of the chord PQ. 1

(iv) Find the locus of M as P and Q vary given that T always lies on the line $y = -2a$ 3

d) If $f(x) = 3^x + 1$ find $f^{-1}(x)$. 2

End of Paper

...S OF PRELIMINARY EXT 1. 2005.

QUESTION 1.

a) i) $\lim_{x\to\infty} \frac{3-5x}{2x+1}$

$= \lim_{x\to\infty} \frac{\frac{3}{x}-5}{2+\frac{1}{x}}$

$= -\frac{5}{2}$ as $\frac{1}{x}\to 0$.

ii) $\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$

$= \lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}$

$= \frac{1}{2\sqrt{x}}$

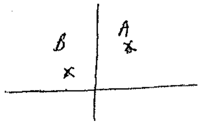
b) A $(1,3)$, C $(-4,-2)$, B $(-1,1)$

$AC = \sqrt{5^2+5^2}$

$= \sqrt{50}$

$BC = \sqrt{3^2+3^2}$

$= \sqrt{18}$



$\therefore AC : BC = 5\sqrt{2} : 3\sqrt{2}$

$= 5 : 3$

(Ext. division in ratio 5 : 3

c) $\frac{x}{x-1} \geq 2$, $x \neq 1$

$x(x-1) \geq 2(x-1)^2$; $[\times(x-1)^2]$

$\therefore 0 \geq (x-1)(2x-2-x)$

$0 \geq (x-1)(x-2)$

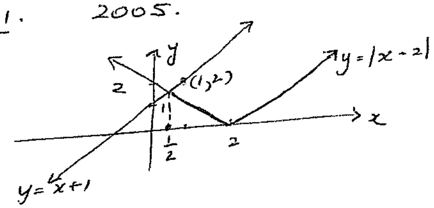
$\therefore 1 < x \leq 2$.

d) $|x-2| = x+1$ $\quad x+1 \geq 0$, $x \geq -1$

$x-2 = x+1$, $-2 = 1$, No solution $\quad$ OR $\quad -x+2 = x+1$, $1 = 2x$, $x = \frac{1}{2}$

$\therefore$ This eqn has one real solution $x = \frac{1}{2}$

ii)



iii) $|x-2| < x+1$

to the right of $x = \frac{1}{2}$

i.e. $x > \frac{1}{2}$

e) $x^2-4x+9 = y$

$a > 0$ $\therefore$ graph is concave up

$\Delta = b^2-4ac$

$= 16 - 4\times 9$

$= -20$ $\therefore$ no real roots.

$\therefore y > 0$ (or positive) $\forall x$.

OR $x^2-4x+9 = (x-2)^2+5$

$(x-2)^2$ has a lowest value of 0 when $x = 2$

$\therefore$ lowest value of quadratic is 5 (when $x = 2$)

i.e. it is always positive.

11.35

QUESTION 2.

a) i) $\frac{d}{dx}\left(\frac{3-2x}{3+2x}\right) = \frac{(3+2x)\cdot(-2)-(3-2x)\cdot 2}{(3+2x)^2}$

$= \frac{-6-6}{(3+2x)^2}$

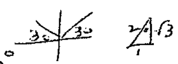
$= \frac{-12}{(3+2x)^2}$

ii) $\frac{d}{dx}(1+x^3)^{-\frac{1}{2}} = -\frac{1}{2}(1+x^3)^{-\frac{3}{2}}\cdot 3x^2$

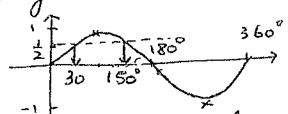
$= \frac{-3x^2}{2(1+x^3)^{3/2}}$.

b) i) $\sin x = \frac{1}{2}$

$x = 30^\circ, 150^\circ$



ii) $y = \sin x$



iii) $y = \sin x$ below $y = \frac{1}{2}$

$0^\circ \leq x \leq 30^\circ$, $150^\circ \leq x \leq 360^\circ$

2. c) $3^x = 5$

$x \log 3 = \log 5$

$x = \frac{\log 5}{\log 3}$

$= 1\cdot 46$ (c 3 sig fig)

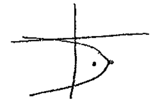
d) $y^2+8y+4x-4 = 0$

i) $y^2+8y+16 = 4-4x+16$

$(y+4)^2 = -4(x-5)$

Vertex is $(5,-4)$

ii) focal length: $a = 1$



Focus is $(4,-4)$

iii) For y intercepts: $x = 0$

$y^2+8y-4 = 0$

$y = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$= \frac{-8\pm\sqrt{64-4\times 1\times -4}}{2}$

$= \frac{-8\pm\sqrt{80}}{2}$

$= \frac{-8\pm 4\sqrt{5}}{2}$

$= -4\pm 2\sqrt{5}$.

$\therefore$ y-int. are $-4+2\sqrt{5}, -4-2\sqrt{5}$.

QUESTION 3

a) i) $\sec^2 x+\tan x-7 = \tan^2 x+\tan x-6$

LS. $= (1+\tan^2 x)+\tan x-7$

$= \tan^2 x+\tan x-6$

$=$ RS.

ii) $\sec^2 x+\tan x-7 = 0$

$\tan^2 x+\tan x-6 = 0$.

$(\tan x+3)(\tan x-2) = 0$

$\tan x = -3$ or 2

$x = 108^\circ 26', 288^\circ 26',$

$63^\circ 26', 243^\circ 26'$

b) $\sum_{1}^{\infty} 32(-\frac{1}{2})^r = -16+8-4+2-\ldots$

This is geom. $a = -16$, $r = -\frac{1}{2}$

$S_\infty = \frac{a}{1-r}$

$= \frac{-16}{\ldots}$

c) i) $2^{m+1}-2^{m-1}$

$= 2^{m-1}(2^2-1)$

$= 3\times 2^{m-1}$

ii) $2^x = \frac{2^{3004}-2^{3002}}{3}$

$2^x = \frac{\cancel{3}\times 2^{3002}}{\cancel{3}}$

$\therefore x = 3002$.

d) $x^2+mx+n = 0$

Let roots be $\alpha, 2\alpha$

$\alpha+2\alpha = -m$ ①

$2\alpha^2 = n$ ②

From ① $3\alpha = -m$, $\alpha = -\frac{m}{3}$

sub into ② $2\cdot\frac{m^2}{9} = n$

$2m^2 = 9n$

$\therefore 2m^2-9n = 0$.

e) 7^n-5^n, n an integer

If $n = 1$ $\quad$ Exp $= 7^1-5^1 = 2$ which is even

$\therefore$ true if $n = 1$.

Assume $7^k-5^k = 2N$ where k, N is an integer. $(7^k = 2N+5^k)$

Examine $n = k+1$

$7^{k+1}-5^{k+1} = 7\times(2N+5^k)-5^{k+1}$

$= 14N+7\times 5^k-5\times 5^k$

$= 14N+2\times 5^k$

$= 2(7N+5^k)$

which is even as $7N+5^k$ is an integer.

$\therefore$ If true for $n = k$, also true for $n = k+1$. We see it is true for $n = 1$, so by induction it is true for $n = 1+1 = 2$, $n = 2+1 = 3$ and so on for all integral values of $n \geq 1$.

QUESTION 4

a) i) $x^2+y^2-6x-2y+9=0$

$x^2-6x+9+y^2-2y+1=-9+9+1$

$(x-3)^2+(y-1)^2=1$

∴ centre $(3,1)$ radius 1.

ii) For a tangent, ⊥ distance from line to centre equals radius.

$D=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$ $\quad 3x-4y=0$, $(3,1)$

$=\frac{|9-4+0|}{\sqrt{3^2+4^2}}$

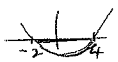
$=\frac{|5|}{5}$

$=1$ (radius)

∴ $3x-4y=0$ is a tangent.

b) i) $x^2-2x-8\le 0$

$(x-4)(x+2)\le 0$



$-2\le x\le 4$

ii) $f(x)=\log(8+2x-x^2)$

D: $8+2x-x^2>0$

$x^2-2x-8<0$

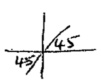
$-2<x<4$.

c) $\tan(2\theta+45^\circ)=1$

$0^\circ\le\theta\le 360^\circ$

$45^\circ\le 2\theta+45\le 785^\circ$

$\left[2\theta+45^\circ=45^\circ\right]$ acute



∴ $2\theta+45^\circ=45^\circ, 225^\circ, 405^\circ, 585^\circ, 765^\circ$

$2\theta=0^\circ, 180^\circ, 360^\circ, 540^\circ, 720^\circ$

$\theta=0^\circ, 90^\circ, 180^\circ, 270^\circ, 360^\circ$

d) Asked to find the equation of the tangent to the curve $y=x^2+1$ at the point $(2,5)$. Answer in general form.

x) $y=\frac{3}{x}$

$\frac{dy}{dx}=-\frac{3}{x^2}$

at $(3,1)$: $\frac{dy}{dx}=-\frac{1}{3}$.

Equ^n of tangent is

$y-1=-\frac{1}{3}(x-3)$

$3y-3=-x+3$

i.e. $x+3y-6=0$.

QUESTION 5

a) $2x^2-15x\equiv a(x-3)^2+b(x-3)$

$=ax^2-6ax\ldots$

∴ $a=2$

if $x=0$: $0=9a-3b$

$3b=18$

∴ $b=6$.

b) $V=\frac{4}{3}\pi r^3$

$\frac{dV}{dr}=4\pi r^2$

$\frac{dV}{dt}=\frac{dV}{dr}\cdot\frac{dr}{dt}$

$10=4\pi r^2\cdot\frac{dr}{dt}$

$\frac{dr}{dt}=\frac{10}{4\pi r^2}$

When SA = 100 mm

$4\pi r^2=100$

∴ $\frac{dr}{dt}=\frac{10}{100}$

$=0\cdot 1$

∴ rate of increase of radius is $0\cdot 1$ mm/s.

c) i) $x^2=4ay$

$y=\frac{x^2}{4a}$

$\frac{dy}{dx}=\frac{x}{2a}$

at $(2ap, ap^2)$: $\frac{dy}{dx}=\frac{2ap}{2a}$

$=p$

∴ Equ^n of tangent is

$y-ap^2=p(x-2ap)$

$y-ap^2=px-2ap^2$

or $px-y-ap^2=0$ — ①

q.e.d.

ii) ∴ Tangent at Q. is

$qx-y-aq^2=0$ — ②

Solving sim. to find T

①−② $(p-q)x=ap^2-aq^2$

$x=a(p+q)$

$y=px-ap^2$

$=p\cdot a(p+q)-ap^2$

$=apq$.

∴ T is $(a(p+q), apq)$

iii) M is midpt of PQ

$M\equiv\left(\frac{2ap+2aq}{2}, \frac{ap^2+aq^2}{2}\right)$

$\equiv\left(a(p+q), \frac{ap^2+aq^2}{2}\right)$

iv) $x=a(p+q)\Rightarrow p+q=\frac{x}{a}$

$y=\frac{a(p^2+2pq+q^2)-2apq}{2}$.

$=\frac{a(p+q)^2}{2}-apq$.

But $apq=-2a$ as T lies on $y=-2a$.

∴ $y=\frac{a\cdot\frac{x^2}{a^2}}{2}-(-2a)$

$=\frac{x^2}{2a}+2a$.

d) $f(x)=3^x+1$

$y=3^x+1$

Inverse is

$x=3^y+1$

$3^y=x-1$

$y=\log_3(x-1)$

∴ $f^{-1}(x)=\log_3(x-1)$