St. George Girls High School

Year 11

End of Preliminary Course Examination

2002

Mathematics Extension 1

Time Allowed: 2 hours
(plus 5 minutes reading time)

Instructions

1. All questions should be attempted.
2. All necessary working must be shown.
3. Begin each question on a **new page**.
4. Marks will be deducted for careless work or poorly presented solutions.

Question 1 (16 marks) – Start a New Page **Marks**

a) Simplify: $\dfrac{25^{2n} \times 5^{n-1}}{5^{2n+1}}$ 1

b) Make x the subject of: $y = m.10^{ax}$ 2

c) For the parabola $y^2 + 4y = 2x - 6$ write down the coordinates of the vertex and focus, the focal length and the equation of the directrix. 3

d) Write down the equation of the vertical asymptote for the graph

$$y = \frac{4x}{x+1}$$ 1

e) Evaluate: $\displaystyle\lim_{x \to \infty} \frac{x^2+4}{1-2x^2}$ 2

f) Sketch $y = |2x|$ and $y = |x-1|$ on the same set of axes. Shade the region where $y \le |2x|$ and $y \ge |x-1|$ 3

g) Prove by Mathematical Induction that:

$$\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \ldots.. + \frac{1}{x^n} = \frac{1}{x-1} - \frac{1}{x^n(x-1)}$$ 4

for all positive integral n where $x \ne 0, 1$.

Question 2 (16 marks) – Start a New Page **Marks**

a) Differentiate:

(i) $y = \sqrt{9 - 2x^2}$ 2

(ii) $y = \dfrac{1}{4x^3}$ 2

b) Sketch the graph: $f(x) = \begin{cases} 1 - x^2 & \text{for } x \le 1 \\ x - 1 & \text{for } x > 1 \end{cases}$

(i) prove this graph is continuous at $x = 1$. 1

(ii) determine whether this graph is differentiable at $x = 1$. 2

c) A point $A(3, 0)$ lies on the parabola $y = 3x - x^2$.

(i) find the equation of the tangent at A. This tangent crosses the y axis at B. State the coordinates of B. 2

(ii) find the equation of the normal at A. This normal crosses the y axis at C. State the coordinates of C. 2

(iii) find the area of triangle ABC. 1

(iv) find the angle that the normal at A makes with the positive direction of the x axis. 2

Question 3 (16 marks) – Start a New Page **Marks**

a) If roots of the quadratic $3x^2 - 8x - 1 = 0$ are α and β evaluate the following:

(i) $\alpha + \beta$ 1

(ii) $\alpha\beta$ 1

(iii) $\dfrac{\alpha}{2\beta} + \dfrac{\beta}{2\alpha}$ 2

(iv) $\alpha^3 + \beta^3$ 2

b) The equation $2x^2 + px + q = 0$ has one root three times the other. Show that $3p^2 = 32q$. 3

c) Solve: $(x^2 + 5x)^2 - 84 = 8(x^2 + 5x)$ 3

d) Show that the roots of the equation $mx^2 - (m+n)x + n = 0$ are rational for all rational values of m and n. 2

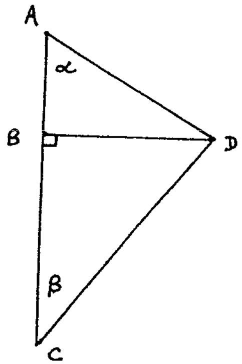
e) For what values of k is $2x^2 - 5x + 4k$ positive definite. 2

Question 4 (16 marks) – Start a New Page **Marks**

a) Prove $\dfrac{\sin^3\theta}{\cos\theta}+\sin\theta\cos\theta\equiv\tan\theta$ 2

b) Solve: $\cos^3\theta-2\cos^2\theta+\cos\theta=0$ for $0°\le\theta\le360°$ 3

c)



$AC=x$ units

Prove: $BD=\dfrac{x\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$ 3

d) The point $P(2ap, ap^2)$ is a point on the parabola $x^2=4ay$.

(i) Prove that the equation of the normal at P is given by $x+py=2ap+ap^3$. 3

(ii) This normal meets the y axis at Q. Find the coordinates of Q. 1

(iii) Find the point $M(X, Y)$ which divides PQ externally in the ratio 2:1. 2

(iv) Hence, find the Cartesian equation of the locus of M. 2

Question 5 (16 marks) – Start a New Page **Marks**

a) Solve: $\dfrac{3x}{x-2}\ge1$ 4

b) A hollow cone with a vertical angle of 60° is held with its axis vertical and vertex pointing downwards. Water is poured into the cone at $10\text{cm}^3/\text{s}$.

(i) Prove that $r=\dfrac{h}{\sqrt{3}}$ where h is the perpendicular height of the cone. 1

(ii) Hence, find the rate at which the surface of the water is rising when the depth is 6cm. 3

c) During a large bush fire a helicopter was used to scoop water from the ocean and then drop it on the fire. The first drop was 500 metres from the ocean pick up point. The next drop was 100 metres further inland. Each successive drop was a further 100 metres inland than the previous one. When a helicopter makes a drop, it flies from the ocean pick up point to the drop area and returns to the ocean pick up point again.

(i) How far did the helicopter fly from the ocean pick up point to the drop area and return to the ocean pick up point on its 10^{th} drop? 2

(ii) Calculate the total distance the helicopter flew from the first pick up to returning to the pick up point after the 10^{th} drop. 2

(iii) Show the total number of kilometres (K) flown in making "d" drops is given by $K=0.9d+0.1d^2$. 2

(iv) The helicopter flew 22km. How many drops did it make? 2

Question 1

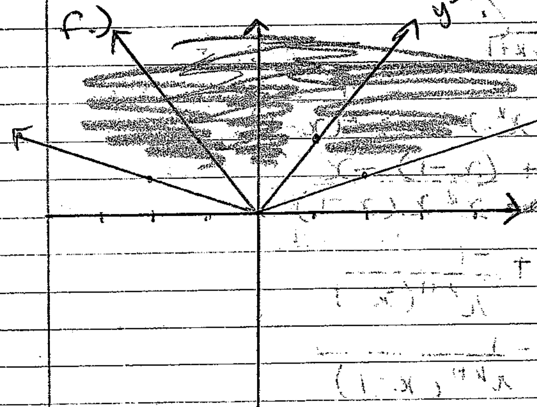
a) $\frac{5^{4n} \times 5^{n-1}}{5^{2n+1}} = 5^{5n-1} \div 5^{2n+1}$
$= 5^{3n-2}$ ✓

b) $y = m.10^{ax}$
$\frac{y}{m} = 10^{ax}$
$\log_{10}\frac{y}{m} = \log_{10} 10^{ax}$
$ax = \log_{10}\frac{y}{m}$ ✓
$x = \frac{\log_{10}\frac{y}{m}}{a}$ ✓

c) $y^2 + 4y = 2x - 6$ or $y^2+4y+4 = 2x-2$
$2x = y^2+4y+6$ $\quad (y+2)^2 = 2x-2$ ✓
$x = \frac{y^2}{2} + 2y + 3$ $\quad (y+2)^2 = 2(x-1)$
$(y-k)^2 = 4a(x-h)$
$4a = 2$; $a = \frac{1}{2}$
focal length $= \frac{x(x-1)}{2}$
directrix $x = -(x-1)$
Focus $(1\frac{1}{2}, -2)$ $\quad$ focus $((x-1), 0)$
Directrix: ( ) $\quad$ Vertex $(1, -2)$ ✓
$x = \frac{1}{2}$

d) $x = -1$ ✓

e) $\lim_{x\to\infty} \frac{\frac{x^2}{x^2} + \frac{4}{x^2}}{\frac{1}{x^2} - \frac{2x^2}{x^2}}$ ✓
$= \frac{1}{0-2}$
$= -\frac{1}{2}$ ✓

f) 

1. Prove true for $n=1$
LHS $\frac{1}{x}$ $\quad$ RHS $= \frac{1}{x-1} - \frac{1}{x^2(x-1)}$
$= \frac{1}{x}$ ✓ $\quad = \frac{x^2-1}{x^2(x-1)} = \frac{(x-1)(x+1)}{x^2(x-1)}$

RHS $= \frac{1}{x-1} - \frac{1}{x^1(x-1)}$
$= \frac{1}{x(x-1)}$

RHS $= \frac{1}{x-1} - \frac{1}{x(x-1)}$
$= \frac{x-1}{x(x-1)}$ ✓
$= \frac{1}{x}$

LHS=RHS
true for $n=1$

2. Assume true for $n=k$
$\frac{1}{x} + \frac{1}{x^2} + \dots + \frac{1}{x^k} = \frac{1}{x-1} - \frac{1}{x^k(x-1)}$

3. Prove true for $n=k+1$
$\frac{1}{x} + \frac{1}{x^2} + \dots + \frac{1}{x^k} + \frac{1}{x^{k+1}} = \frac{1}{x-1} - \frac{1}{x^{k+1}(x-1)}$

LHS $= \frac{1}{(x-1)} - \frac{1}{x^k(x-1)} + \frac{1}{x^{k+1}}$ ✓
$= \frac{1}{(x-1)} - \frac{1}{x^k(x-1)} + \frac{1}{x^k.x}$
$= \frac{x^k - x + (x-1)}{x^k x.(x-1)}$
$= \frac{x^k+1}{x^{k+2}+x^{k+1}}$
$= \frac{1}{x-1} + \frac{1}{x^k.x} - \frac{1}{x^k(x-1)}$
$= \frac{1}{x-1} + \frac{(x-1)-x}{x^k.x.(x-1)}$ ✓
$= \frac{1}{x+1} + \frac{-1}{x^{k+1}(x-1)}$
$= \frac{1}{x+1} - \frac{1}{x^{k+1}(x-1)}$ ✓

∴ Since true for $n=1$ & by assertion true for $n=k$ & $n=k$ and by mathematical induction true for all

2.

$y = \sqrt{9-2x^2}$

$y = (9-2x^2)^{\frac{1}{2}}$

$= \frac{1}{2}(9-2x^2)^{-\frac{1}{2}} \times -4x$

$= \cancel{2x(9-2x^2)}$ ✗ $\quad \frac{-2x}{\sqrt{9-2x^2}}$

(ii) $y = \frac{1}{4x^3} = \frac{1}{4}x^{-3}$

$y' = \frac{1}{12x^2}$ ✗ $\quad y' = \frac{1}{4} \cdot -3x^{-4} = -\frac{3}{4x^4}$

b.) $f(x) = 1-x^2$

$f(1) = 1-1$

$= 0$

$\lim_{x\to 1^-} f(x) = 1-1$

$\lim_{x\to 1^+} f(x) = x-1$

$f(1) = 1-1$

$= 0$.

∴ graph continous at $x = 1$ ✓

(ii) $f'(x) = 1-2x$.

$f'(1) = 1-2$

$= -1$ ✓

$f'(x) = 1-1$

$= 0$ ✓

∴ This graph is not differentable at $x = 1$.

c.) $y = 3x - x^2$

$y' = 3 - 2x$.

at $x = 3$

$y' = 3-6$ ✓

$y' = -3$

$y - 0 = -3(x-3)$

$y = -3x + 9$ ✓

When $x = 0$

$y = 9$ ✓

$B(0, 9)$ ✓

(ii) normal to A

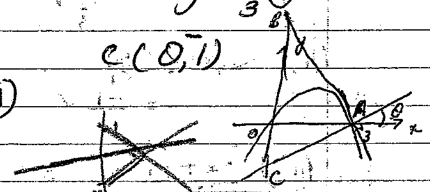
$m_2 = \frac{1}{3}$ ✓

$y = \frac{1}{3}(x-3)$

$y = \frac{x}{3} - 1$

$C(0, -1)$

(iii)



$AC = \sqrt{(3-0)^2 + (0-1)^2}$

$= \sqrt{10}$

Easier to use $\frac{1}{2} \times BC \times OA$

$AB = \sqrt{(3-0)^2 + (0-9)^2}$

$= \sqrt{9+81}$

$= \sqrt{90}$

$\frac{1}{2} \times 10 \times 3$

$= 15$

$\frac{1}{2} \times \sqrt{10} \times 3\sqrt{10} = 15$.

(IV) ??

$\tan\theta = m_{AC}$

$= \frac{1}{3}$

$\theta = 18°26'$

3.

$3x^2 - 8x - 1 = 0$

$\alpha + \beta = -\frac{b}{a}$ ✓

$= -\frac{-8}{3}$

$= \frac{8}{3}$ ✓

(ii) $\alpha\beta = \frac{c}{a}$ ✓

$= \frac{-1}{3}$.

(iii) $\frac{\alpha}{2\beta} + \frac{\beta}{2\alpha} = \frac{\alpha^2 + \beta^2}{2\alpha\beta} = \frac{\alpha^2+\beta^2}{2\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{2\alpha\beta}$

$\frac{2\alpha^2 + 2\beta^2}{2\beta + 2\alpha} = \frac{2(\alpha^2+\beta^2)}{2(\beta+\alpha)}$

$= \frac{(\frac{8}{3})^2 + \frac{2}{3}}{2(-\frac{1}{3})} = \frac{-35}{3}$

$= -11\frac{2}{3}$

$(\alpha+\beta)^2 = (\frac{8}{3})^2 - 2 \times -\frac{1}{3}$

$= 7\frac{7}{9}$

$(2 \times 7\frac{7}{9}) \div (2 \times -\frac{1}{3}) = \frac{140}{9} \div -\frac{2}{3}$

$= -23\frac{1}{3}$.

(iv) $\alpha^3 + \beta^3 = (\alpha+\beta)(\alpha^2 - 2\alpha\beta + \beta^2)$

Please ask me for a "better" method.

$= (\frac{8}{3})(7\frac{7}{9} - 2 \times -\frac{1}{3})$

$= 22\frac{14}{27}$

b) $\alpha + 3\alpha = \frac{-b}{a}$

$4\alpha = \frac{-p}{2}$

$8\alpha = -p$

$\cancel{\alpha = \frac{-p}{8}}$

$p = -8\alpha$

$p^2 = 64\alpha$.

$3 \times 64\alpha = 192\alpha^2$ ✓

∴ $3p^2 = 32q$ ✓

$3\alpha^2 = \frac{q}{2}$.

$\cancel{\alpha^2 = \frac{q}{6}}$

$q = 6\alpha^2$

$6\alpha^2 \times 32 = 192\alpha^2$

c) Let u be (x^2+5x)

$u^2-8u-84=0$

$(u-14)(u+6)=0$

$u=14,-6$

$x^2+5x+6=0$

$(x+3)(x+2)=0$

$x=-2,-3$

$x^2+5x-14=0$

$(x+7)(x-2)=0$

$x=2,-7$

b) Let $\cos\theta$ be u.

$u^3-2u^2+u=0$

$u(u^2-2u+1)=0$

$u(u-1)^2=0$

$u=0$ or 1

$\cos\theta=0$

$\theta=90°,270°$

$\cos\theta=1$

$\theta=360°,0°$

d) $\Delta=b^2-4ac$

$\Delta=(m+n)^2-4\times m\times n$

$=m^2+2mn+n^2-4mn$

$=m^2-2mn+n^2$

$=(m-n)^2$

∴ roots are rational.

c) $\dfrac{AC}{\sin(180-(\alpha+\beta))}=\dfrac{x}{\sin(\alpha+\beta)}=\dfrac{AD}{\sin\beta}$

$\sin\alpha=\dfrac{BD}{AD}$

$AD=\dfrac{BD}{\sin\alpha}$

$AD=\dfrac{x\sin\beta}{\sin(\alpha+\beta)}$

$BD=\dfrac{x\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$

e) positive definite.

$\Delta<0$

$\Delta=25-4\times2\times4k$

$25-32k<0$

$25<32k$

$k>\dfrac{25}{32}$

Question 4

a) $LHS=\dfrac{\sin^3\theta}{\cos\theta}+\sin\theta\cos\theta$

$=\dfrac{\sin^3\theta+\sin\theta\cos^2\theta}{\cos\theta}$

$=\dfrac{\sin\theta(\sin^2\theta+\cos^2\theta)}{\cos\theta}$

$=\dfrac{\sin\theta}{\cos\theta}$

$=\tan\theta$

$=RHS.$

7mm /4

5)

$\dfrac{3x}{x-2}\geq 1$

$3x(x-2)\geq(x-2)^2$ ... Try again.

$3x^2-2\geq x-2$

$3x^2-x\geq 0$

$x(3x-1)\geq 0$

$x=0,\frac{1}{3}$

$x\geq 0$

$x\geq\frac{1}{3}$

(IV) $22=0.9d+0.1d^2$

$220=9d+d^2$

$d^2+9d-220=0$

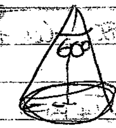
$(d+20)(d-11)=0$

$d=11,-20$

$d\neq-20$

∴ It made 11 drops.

b)

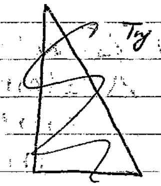


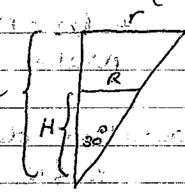
$\dfrac{dV}{dt}=\dfrac{dV}{dr}\times\dfrac{dr}{dt}$

$V=\frac{1}{3}\pi r^2h$

$\dfrac{dV}{dt}=\frac{2}{3}\pi rh\times\dfrac{dr}{dt}$

Try again





b) $d=100$

$a=500$

$T_n=500+(n-1)100$

$=500+100n-100$

$=400+100n$

$T_{10}=400+1000$

$=1400$ m.

but to and fro distance $=2\times1400m=2800m$.

(ii) $S_n=2\times\frac{1}{2}\times10(500+1400)$

$=9500\times2=19000$ m.

(iii) $S_n=\frac{1}{2}\times d(1000+(d-1)100)$

$=\frac{d}{2}(1000+100d-100)$

$=~~9000+100d~~$

$S_n=2\times\frac{1}{2}\times\frac{d}{1000}\left[1000+(d-1)\frac{100}{1000}\right]$

$K=[d+d(d-1)\,0.1]$

$=d+0.1d^2-0.1d$

$=0.9d+0.1d^2$.

Sub. $22=0.9d+0.1d^2$

$\therefore 0.1d^2+0.9d-22=0$

$d^2+9d-220=0$

$(d+20)(d-11)=0$ ∴ $d=11$ drops >0.