St George Girls High School

Year 11

End of Preliminary Course Examination

2006

Mathematics Extension 1

Time Allowed: 2 hours
(plus 5 minutes reading time)

Instructions

1. Attempt all 6 questions.
2. All necessary working must be shown.
3. Begin each question on a **new page**.
4. Marks will be deducted for careless work or poorly presented solutions.

Question 1 (14 marks) – Start a New Page **Marks**

a) Find the values of a, b and c if $2x^2+x+4 \equiv a(x+1)^2+b(x+1)+c$ 3

b) Find the co-ordinates of the point C which divides the join of $A(-3,-2)$ to $B(1,4)$ externally in the ratio 1:3.

In what ratio does B divide CA? 2

c) Find the domain of $f(x)=\sqrt{\dfrac{3-x}{x}}$ 3

d) Find $\lim\limits_{h \to 0} \dfrac{\sqrt{x+h}-\sqrt{x-h}}{2h}$ 3

e) Find $\dfrac{dy}{dx}$ in simplest factored form if $y=(2x-1)^3(3x+1)^2$

Question 2 (14 marks) – Start a New Page **Marks**

a) Solve for x: **3**

$\log_2 x + \log_2 (x-2) = 3$

b) Solve for θ where $0^\circ \le \theta \le 360^\circ$ **6**

(i) $\cos\left(\theta + 60^\circ\right) = \frac{1}{2}$

(ii) $2\sin^2\theta - \sin\theta - 1 = 0$

c) The rate at which the world's pollution, P, is changing over time t (years) was reported by three scientists.

Their results were

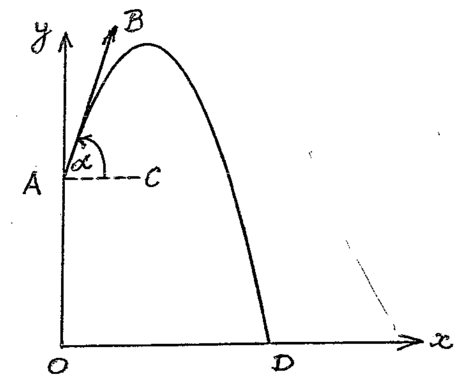
1. $\frac{dP}{dt} = 0.6$
2. $\frac{dP}{dt} = \frac{1}{t}$ $(t > 0)$
3. $\frac{dP}{dt} = -0.1$

Discuss the meaning of <u>each</u> of the above. **3**

d) If $y = \frac{x}{\sqrt{1-x^2}}$ find $\frac{dy}{dx}$ in the form $\left(1-x^2\right)^n$ where n is a rational number. **2**

Question 3 (14 marks) – Start a New Page **Marks**

a)



A ball is thrown from A at the top of a cliff. This ball follows the path of the parabola $y = 60 + 20x - 5x^2$ (as shown above) to the ground at D. The line AB is tangential to the parabola at A and makes an angle α with the horizontal line AC. [<u>Note</u>: All measurements are in metres].

(i) How high is the cliff? **1**

(ii) Find the value of $\frac{dy}{dx}$ at A. **2**

(iii) Find α correct to the nearest degree. **1**

(iv) Find the maximum height reached by the ball above ground level. **2**

(v) Find the distance OD where O is the origin and D is on the same horizontal level as O. **2**

b) Consider the series $20 + 10 + 5 + \ldots$

(i) Find the sum of the first n terms, S_n, in simplest form. **2**

(ii) Find the limiting sum, S, of the series. **1**

(iii) Find the least integer n for which $S - S_n < 0.001$ **3**

Question 4 (14 marks) – Start a New Page **Marks**

a) Prove by Mathematical Induction that $3^{2n}-1$ is divisible by 8 for all integers $n \geq 1$ 4

b) Consider the function $f(x)=\dfrac{4-x^2}{1-x^2}$

(i) Evaluate $f(0)$ 1

(ii) Solve for x: $f(x)=0$ 1

(iii) State the equations of the vertical asymptotes. 1

(iv) Find the equation of the horizontal asymptote. 1

(v) Determine whether $f(x)$ is an even function, an odd function or neither. 1

(vi) Sketch $y=f(x)$ clearly showing all of the above features. 2

(vii) Evaluate $f'(2)$ 2

(viii) Find the equation of the tangent to $y=f(x)$ at $(2,0)$ 1

Question 5 (14 marks) – Start a New Page **Marks**

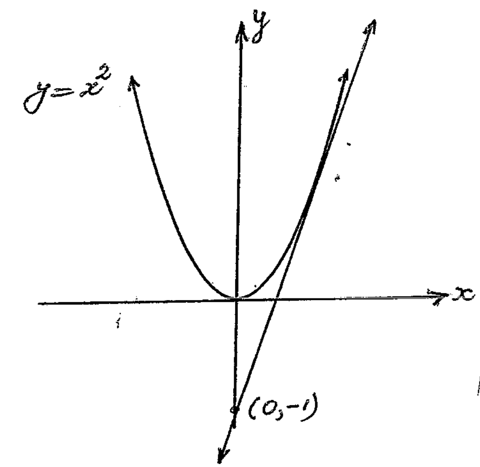
a) Show that the roots of $x^2+(k-1)x-1=0$ are real and distinct for all values of k where k is a real number. 2

b) Consider the function $f(x)=\begin{cases} 2x & \text{for } 0 \leq x \leq 2 \\ ax^2+b & \text{for } x>2 \end{cases}$ 4

$f'(x)=2$

Find the values of 'a' and 'b' if $f(x)$ is both continuous and differentiable at $x=2$

c)



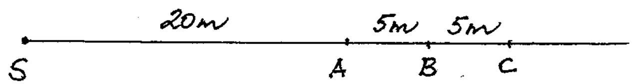
A line is drawn through the point $(0,-1)$ with gradient m.

(i) Show that the equation of this line is $mx-y-1=0$ 1

(ii) If this line is tangential to $y=x^2$ as above, find the value of m given that m is positive. 3

Question 5 (cont'd) **Marks**

d)



A load of sand at S is to be distributed equally, along a straight line, to twenty positions A, B, C, D... which are separated by 5m. A is 20m from S.

The worker starts at S and returns to S after delivering to <u>each</u> position. Find:

(i) the distance travelled from S to the 20th position and then returning to S. 2

(ii) the total distance travelled completing this task. 2

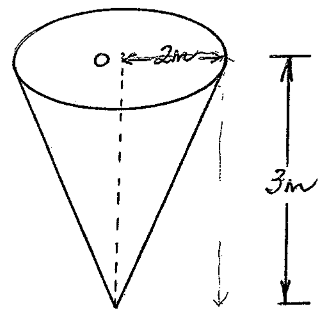
<u>Question 6</u> (14 marks) – Start a New Page **Marks**

a) Consider the function $f(x) = (x-2)^2$ for $x \geq 2$

(i) State the range of $f(x)$ 1

(ii) Find $f^{-1}(x)$ clearly indicating the domain and range. 3

b) 4



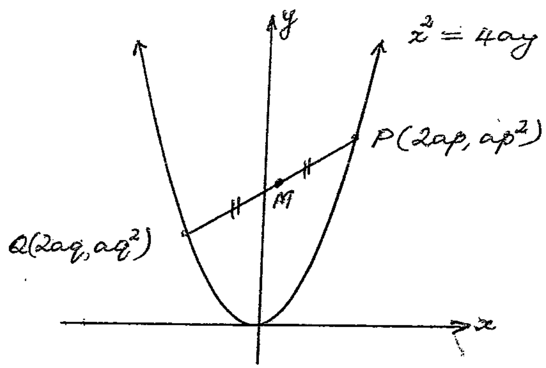
Water is entering this container at the rate of $\frac{\pi}{10}$ metres^3/hour.

When the depth of the water is hm the volume of water is $V = \frac{4}{3}\pi h^3$ metres^3.

At what rate is the depth increasing (correct to 2 significant figures) when the depth is 0.4m?

Question 6 (cont'd) **Marks**

c)



The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two variable points on the parabola $x^2 = 4ay$. M is the mid-point of PQ.

(i) Find the gradient of PQ. 1

(ii) Show that the equation of PQ is $y - \frac{1}{2}(p+q)x + apq = 0$ 1

(iii) If PQ always passes through $(0, 2a)$

a. Show that $pq = -2$ 1

b. Show that the locus of M as P and Q vary is the parabola $x^2 = 2a(y - 2a)$ 3

End of Paper

(e) $y = \underbrace{(2x-1)^3}_{u}\underbrace{(3x+1)^2}_{v}$

$\frac{dy}{dx} = vu' + uv'$

$= (3x+1)^2 . 3(2x-1)^2 . 2 + (2x-1)^3 . 2(3x+1) . 3$

$= 6(3x+1)^2(2x-1)^2 + 6(2x-1)^3(3x+1)$

$= 6(3x+1)(2x-1)^2[3x+1+2x-1]$

$= 30x(3x+1)(2x-1)^2$

~~1~~ ~~2~~ 3 4 ~~5~~ ~~6~~

QUESTION 2:

(a) $\log_2 x + \log_2(x-2) = 3$

$\Rightarrow \log_2[x(x-2)] = 3$

$\Rightarrow x^2 - 2x = 8$

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$x = 4, -2$

BUT $x > 0$ & $x - 2 >$

$\Rightarrow x > 2$

$\therefore x = 4$

(b) (i) $\cos(\theta + 60^\circ) = \frac{1}{2}$

$0^\circ \le \theta \le 360^\circ$

$60^\circ \le \theta + 60^\circ \le 420^\circ$

$\therefore \theta + 60^\circ = 60^\circ, 300^\circ, 420^\circ$

$\therefore \theta = 0^\circ, 240^\circ, 360^\circ$

$(\theta + 60^\circ)_{acute} = 60^\circ$

(ii) $2\sin^2\theta - \sin\theta - 1 = 0$

$(\sin\theta - 1)(2\sin\theta + 1) = 0$

$\Rightarrow \sin\theta = -\frac{1}{2}, 1$

$\sin\theta = 1$

$\Rightarrow \theta = 90^\circ$

$\sin\theta = -\frac{1}{2}$

$\therefore \theta = 210^\circ, 330^\circ$

$\theta_{acute} =$

$\therefore \theta = 90^\circ, 210^\circ, 330^\circ$

(c) (i) 1. Pollution is increasing at a constant ra

2. Pollution is increasing but at a decreasing

3. Pollution is decreasing at a constant rat

(d) $y = \frac{x}{(1-x^2)^{\frac{1}{2}}}$

$\frac{dy}{dx} = \frac{(1-x^2)^{\frac{1}{2}}.1 - x.\frac{1}{2}(1-x^2)^{-\frac{1}{2}}.(-2x)}{1-x^2}$

$= \frac{(1-x^2)^{\frac{1}{2}} + \frac{x^2}{(1-x^2)^{\frac{1}{2}}}}{}$

$= \frac{1}{(1-x^2)^{3/2}}$

$= (1-x^2)^{-\frac{3}{2}}$

QUESTION 3:

(a) (i) $y = 60 + 20x - 5x^2$

at $x = 0$, $y = 60$

$\therefore$ Cliff is 60 m high

(ii) $\frac{dy}{dx} = 20 - 10x$

at $x = 0$, $\frac{dy}{dx} = 20$

(iii) $\tan \alpha = 20$

$\therefore \alpha = 87 \cdot 13^\circ$

$= 87^\circ$ (correct to nearest degree)

(iv) $y = 60 + 20x - 5x^2$

$= -5(x^2 - 4x - 12)$

$= -5[(x^2 - 4x + 4) - 16]$

$= 80 - 5(x-2)^2$

$\therefore$ Maximum height is 80 m

(v) when $y = 0$, $80 - 5(x-2)^2 = 0$

$80 = 5(x-2)^2$

$16 = (x-2)^2$

$x - 2 = 4, -4$

$\therefore x = 6, -2$

$\therefore x = 6 \quad (x > 0)$

$\therefore$ OD is 6 m

(b) $\underbrace{20 + 10 + 5 + \cdots}_{\text{Geometric series}}$

$a = 20$, $r = \frac{1}{2}$

(i) $S_n = \frac{a(1-r^n)}{1-r}$

$= \frac{20[1 - (\frac{1}{2})^n]}{1 - \frac{1}{2}}$

$= 40[1 - (\frac{1}{2})^n]$

(ii) $S = \frac{a}{1-r}$

$= \frac{20}{1-\frac{1}{2}}$

$= 40$

(iii) $S - S_n < 0 \cdot 001$

$\Rightarrow 40 - 40[1 - (\frac{1}{2})^n] < 0 \cdot 001$

$\therefore 40(\frac{1}{2})^n < 0 \cdot 001$

$(\frac{1}{2})^n < \frac{0 \cdot 001}{40}$

$\therefore \log_{10}(\frac{1}{2})^n < \log_{10}(\frac{0 \cdot 001}{40})$

$n \log_{10}(\frac{1}{2}) < \log_{10}(\frac{0 \cdot 001}{40})$

$\therefore n > \frac{\log_{10}(\frac{0 \cdot 001}{40})}{\log_{10} \frac{1}{2}}$ since $\log_{10}(\frac{1}{2}) < 0$

ie $n > 15.28 \ldots$

$\therefore n = 16$

QUESTION 4:

(a) Let $S(n)$ be the assertion that $3^{2n}-1$ is divisible by 8

Then $S(1)$: $3^2-1 = 8$ which is divisible by 8

$\therefore S(1)$ is true

Assume $S(k)$ is true for some integer $k > 1$

ie $3^{2k}-1 = 8N$ $\quad N$ integer ——— ①

We now aim to prove that $S(k+1)$ is true

ie $3^{2(k+1)}-1$ is divisible by 8

now
$$\begin{aligned} 3^{2(k+1)}-1 &= 3^{2k+2}-1 \\ &= 3^2 . 3^{2k}-1 \\ &= 3^2(8N+1)-1 \quad \text{from ①} \\ &= 72N+8 \\ &= 8(9N+1) \\ &= 8M \quad M \text{ integer} \end{aligned}$$

$\therefore$ If $S(k)$ is true then $S(k+1)$ is true

But $S(1)$ is true, hence $S(2)$ is true and by the principle of mathematical induction $S(n)$ is true for all integers $n \geq 1$

(b) $$f(x) = \frac{4-x^2}{1-x^2}$$

(i) $f(0) = 4$

(ii) $f(x) = 0 \Rightarrow \dfrac{4-x^2}{1-x^2} = 0$

$\therefore 4-x^2 = 0$

$\therefore x = \pm 2$

(iii) Vertical asymptotes are $x = 1$, $x = -1$

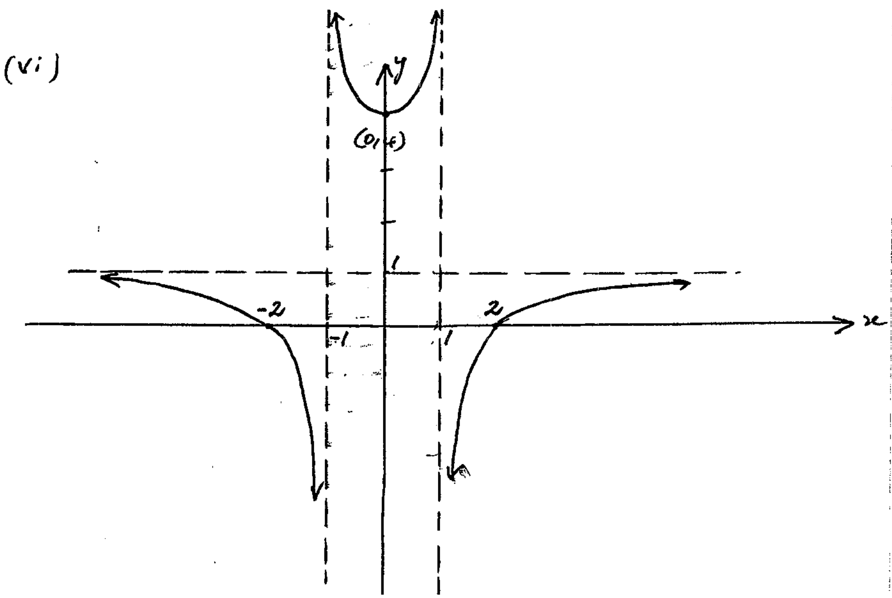
(iv) $$\lim_{x \to \pm\infty} \frac{4-x^2}{1-x^2} = \lim_{x\to\pm\infty} \frac{\frac{4}{x^2}-1}{\frac{1}{x^2}-1} = 1$$

$\therefore y = 1$ is horizontal asymptote

(v) $$\begin{aligned} f(-x) &= \frac{4-(-x)^2}{1-(-x)^2} \\ &= \frac{4-x^2}{1-x^2} \\ &= f(x) \end{aligned}$$

$\therefore f(x)$ is an EVEN function.

(vi)



(vii) $$f(x) = \frac{4-x^2}{1-x^2}$$

$$f'(x) = \frac{(1-x^2)(-2x) - (4-x^2)(-2x)}{(1-x^2)^2}$$

$$= \frac{-2x + 2x^3 + 8x - 2x^3}{(1-x^2)^2}$$

$$= \frac{6x}{(1-x^2)^2}$$

$$\therefore f'(2) = \frac{12}{9}$$

$$= \frac{4}{3}$$

(viii) Tangent at $(2,0)$ is

$$y - 0 = \frac{4}{3}(x-2)$$

$$3y = 4x - 8$$

ie $\underline{4x - 3y - 8 = 0}$

QUESTION 5:

(a) $$x^2 + (k-1)x - 1 = 0$$

$$\Delta = b^2 - 4ac$$

$$= (k-1)^2 - 4(1)(-1)$$

$$= (k-1)^2 + 4$$

since Δ is sum of 2 squares then $\Delta > 0$ for all k.

ie Roots must be real and distinct

(b) CONTINUOUS at $x=2$

(i) $f(2)$ exists and equals 4

(ii) $$\lim_{x\to 2^-} f(x) = 4$$

$$\lim_{x\to 2^+} f(x) = 4a + b$$

$$\therefore 4a + b = 4 \quad \text{———} \; (1)$$

DIFFERENTIABLE at $x=2$

(i) $f(2)$ exists and equals 4

(ii) $$\lim_{x\to 2^-} f'(x) = \lim_{x\to 2^-} 2$$

$$= 2$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 2ax$$

$$= 4a$$

$$\therefore 4a = 2 \quad \text{———} \; (2)$$

$$\therefore a = \tfrac{1}{2} \text{ sub in } (1)$$

$$\Rightarrow \quad 2 + b = 4$$

$$\therefore b = 2$$

$$\therefore \underline{a = \tfrac{1}{2},\ b = 2}$$

(c) (i) Equation is $y + 1 = m(x - 0)$

$= mx$

ie $mx - y - 1 = 0$

(ii) $mx - y - 1 = 0$ —— (1)

$y = x^2$ —— (2)

sub $y = x^2$ in (1)

$\Rightarrow mx - x^2 - 1 = 0$

ie $x^2 - mx + 1 = 0$

$\Delta = b^2 - 4ac$

$= (-m)^2 - 4(1)(1)$

$= m^2 - 4$

Tangent $\Rightarrow$ equal roots

$\Rightarrow \Delta = 0$

ie $m^2 - 4 = 0$

$m^2 = 4$

$m = \pm 2$

$\therefore m = 2$ ($m > 0$, given)

(d) Series is $\underbrace{40 + 50 + 60 + \cdots}$

Arithmetic series

$a = 40$, $d = 10$

(i) $T_{20} = a + 19d$

$= 40 + 19(10)$

$= 230$

$\therefore$ Travel 230 m

(ii) $S_{20} = \frac{20}{2}(a + l)$

$= 10(40 + 230)$

$= 2700$

QUESTION 6:

(a) $f(x) = (x-2)^2$ $x \geq 2$

(i) Range is $f(x) \geq 0$

(ii) $f: y = (x-2)^2$ $\begin{cases} x \geq 2 \\ y \geq 0 \end{cases}$

$f^{-1}: x = (y-2)^2$

$\therefore y - 2 = \pm\sqrt{x}$

$y = 2 \pm \sqrt{x}$ $\begin{cases} x \geq 0 \\ y \geq 2 \end{cases}$

since $y \geq 2$ then

$y = 2 + \sqrt{x}$

ie $f^{-1}(x) = 2 + \sqrt{x}$ Domain: $x \geq 0$

Range: $y \geq 2$

(b) $\frac{dV}{dt} = \frac{\pi}{10}$ metres^3/hour

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dr}{dt} = \frac{dr}{dV} \cdot \frac{dV}{dt}$

$= \frac{1}{4\pi r^2} \cdot \frac{\pi}{10}$ m/hr

$= \frac{1}{40r^2}$ m/hr

$r = \frac{2}{5} \Rightarrow \frac{dr}{dt} = \frac{1}{40 \cdot \left(\frac{4}{25}\right)}$

$= 0.15625$

$= 0.16$ (correct to 2 sig. figs)

$\therefore$ Increasing at 0.16 m/hr

(c) (i) $m_{PQ} = \dfrac{ap^2 - aq^2}{2ap - 2aq}$

$= \dfrac{a(p+q)(p-q)}{2a(p-q)}$

$= \dfrac{p+q}{2} \qquad p \neq q$

(ii) Equation of PQ:

$y - ap^2 = \left(\dfrac{p+q}{2}\right)(x - 2ap)$

ie $2y - 2ap^2 = (p+q)x - 2ap^2 - 2apq$

ie $2y = (p+q)x - 2apq$

$\Rightarrow y - (p+q)x + apq = 0$

(iii)(a) $(0, 2a) \Rightarrow 2a - 0 + apq = 0$

$\therefore apq = -2a$

$pq = -2$

(b) $M \equiv \left(a(p+q), \frac{a}{2}(p^2+q^2)\right)$

ie $x = a(p+q)$ ——— (1)

$y = \frac{a}{2}(p^2+q^2)$

$= \frac{a}{2}\left[(p+q)^2 - 2pq\right]$

$= \frac{a}{2}\left[\left(\frac{x}{a}\right)^2 + 4\right]$ since $pq = -2$ and from (1)

$= \frac{a}{2}\left(\frac{x^2}{a^2} + 4\right)$

$= \dfrac{x^2}{2a} + 2a$

ie $2ay = x^2 + 4a^2$

ie $x^2 = 2ay - 4a^2$

$= 2a(y - 2a)$