St George Girls' High School

Year 11

Common Test 2

June 2003

Mathematics Extension 1

Time Allowed: 75 minutes

Instructions

1. All questions are of equal value.
2. Attempt all questions.
3. All questions are to be completed in the answer booklet.
4. Start each question on a new page.

Question 1 – 12 marks – (Start a new page) **Marks**

a) If $f(x) = x^2 - x$ show that $f(1-x) = f(x)$ **2**

b) (i) Find the points of intersection of $y = x^2$ and $x^2 + y^2 = 2$ **5**

(ii) Sketch the graphs of both curves in (i) on the same diagram.

(iii) Shade the region which is the intersection of $x^2 + y^2 \le 2$ and $y \le x^2$

c) If $0 < a < 1$, show that $1 + a < \dfrac{1}{1-a}$ **2**

d) For the function $f(x) = \dfrac{9+x^2}{9-x^2}$, find the equations of the horizontal and vertical asymptotes. **3**

Marks

Question 2 – 12 marks – (Start a new page)

a) Simplify $1-\dfrac{\sin A\cos A}{\tan A}$ **2**

b) If $\tan\theta = 2$ and $-180^\circ \le \theta \le 0^\circ$ find the exact value of $\cos\theta$. **2**

c) Leana walks 5.1km due East, then turns and walks 3.7km on a bearing of 205°. **5**

(i) Draw a diagram to show the above information.

(ii) Calculate how far Leana is from her starting point (correct to 1 decimal place).

(iii) What is the bearing of Leana's starting point from her final position? (to the nearest degree).

d) Show that $(\operatorname{cosec}\alpha + \cot\alpha)^2 = \dfrac{1+\cos\alpha}{1-\cos\alpha}$ **3**

Question 3 – 12 marks – (Start a new page) **Marks**

a) (i) Find the centre and radius of the circle **5**

$$x^2 - 6x + y^2 + 4y + 8 = 0$$

(ii) Two tangents to this circle have the form

$$x - 2y + k = 0$$

Using the formula for perpendicular distance of a point from a line, find the equations of these two tangents.

b) Find the equation of the line through the intersection of $x - 2y - 3 = 0$ and $3x - y - 5 = 0$, which has a gradient of $\frac{1}{3}$. **4**

c) (i) Show the graphs $y = \frac{3}{x}$ and $y = x + 2$ intersect at $(-3, -1)$ and $(1, 3)$. **3**

(ii) By sketching or otherwise solve $\frac{3}{x} < x + 2$.

Marks

Question 4 – 12 marks – (Start a new page)

a) The n^{th} term of a series is given by $T_n = 3 - 4n$ **6**

(i) Show that the series is arithmetic.

(ii) Find the sum of 20 terms of the series.

(iii) Find the first term less than -500.

b) For the series $\sqrt{3} - 3 + 3\sqrt{3} - \ldots$ **3**

(i) Show it is geometric.

(ii) Find the sum of 6 terms of the series in simplest form.

c) Find the first term and the common difference in the arithmetic series whose 3^{rd} term is 12 and 10^{th} term is -9. **3**

Question 5 – 12 marks – (Start a new page) **Marks**

a) If $\log_5 2 = m$ and $\log_5 3 = n$ find an expression in terms of m and n for: **3**

(i) $\log_5 18$

(ii) $\log_5 1.5$

(iii) $\log_5 1.2$

b) Prove by the method of mathematical induction for all integers $n \geq 1$: **5**

$$1 \times 3 + 2 \times 4 + 3 \times 5 + \ldots + n(n+2) = \frac{1}{6}n(n+1)(2n+7)$$

c) Solve for x **4**

(i) $5^{2x} = \dfrac{1}{125\sqrt{5}}$

(ii) $2^{x+1} - 2^{x-1} = 9$ (answer correct to 1 decimal place)

Marks

Question 6 – 12 marks – (Start a new page)

a) Solve for $0^\circ \le x^\circ \le 360^\circ$ **4**

(i) $2\sin x + \cos x = 0$

(ii) $2\cos^2 x = \cos x$

b) Simplify $\dfrac{1-x^{-2}}{1-x^{-1}}$ **2**

c) A tree 10m tall grows 6m during the next year and each year after that its height increases by $\frac{2}{3}$ the previous years growth. What will be the ultimate height of the tree? (assuming it does not die or get chopped down). **3**

d) The sum of the first and third terms of a geometric sequence is 13 and the sum of the second and fourth terms is 19½. **3**

Find the first term and the common ratio.

Question 6.

a) (i) $2\sin x + \cos x = 0$

$2\sin x = -\cos x$

$\tan x = -\frac{1}{2}$

(2) $\therefore x = 180° - 26°34',\ 360° - 26°34'$

$= 153°26',\ 333°26'$

(ii) $2\cos^2 x = \cos x$

$2\cos^2 x - \cos x = 0$

$\cos x(2\cos x - 1) = 0$

(2) $\cos x = 0$ or $\frac{1}{2}$

$\therefore x = 90°, 270°, 60°, 300°$

b) $\dfrac{1 - \frac{1}{x^2}}{1 - \frac{1}{x}} \times \dfrac{x^2}{x^2} = \dfrac{x^2 - 1}{x^2 - x}$

(2) $= \dfrac{(x+1)(x-1)}{x(x-1)}$

$= \dfrac{x+1}{x}$

c) height of tree $= 10 + 6 + 6 \times \frac{2}{3} + 6 \times \left(\frac{2}{3}\right)^2 + \ldots$

$= 10 + 6 + 4 + 4 \times \frac{2}{3} + \ldots$

infinite geometric series with $r = \frac{2}{3}$ $(|r| < 1)$

(3)

ie ultimate height $= 10 + \dfrac{a}{1-r}$

$= 10 + \dfrac{6}{1 - \frac{2}{3}}$

$= 10 + 18 = 28$

ie ultimate height is 28 m.

d) $a + ar^2 = 13 \Rightarrow a(1 + r^2) = 13$ ①

$ar + ar^3 = 19\frac{1}{2} \Rightarrow ar(1 + r^2) = \frac{39}{2}$ ②

② ÷ ① gives

$r = \frac{39}{2} \times \frac{1}{13}$

$= \frac{3}{2}$

(3) $\therefore$ ① $\Rightarrow a\left(1 + \frac{9}{4}\right) = 13$

$a \times \frac{13}{4} = 13$

$\therefore a = 4$

ie $T_1 = 4,\ r = \frac{3}{2}$

Question 5

a) (i) $\log_5 18 = \log_5 2\times 3^2$

$= \log_5 2 + 2\log_5 3$

$= m + 2n$ (1)

(ii) $\log_5 1\cdot 5 = \log_5\left(\frac{3}{2}\right)$

$= \log_5 3 - \log_5 2$ (1)

$= n - m$

(iii) $\log_5 1\cdot 2 = \log_5 6/5$

$= \log_5\left(\frac{3\times 2}{5}\right)$ (1)

$= \log_5 3 + \log_5 2 - \log_5 5$

$= n + m - 1$

b) $1\times 3 + 2\times 4 + 3\times 5 + \ldots + n(n+2) = \frac{1}{6}\cdot n(n+1)(2n$

Step 1 Show true for $n = 1, 2$.

$n = 1$ $1\times 3 = \frac{1}{6}\times 1\times 2\times 9$ ✓

$n = 2$ $1\times 3 + 2\times 4 = \frac{1}{6}\times 2\times 3\times 11$

$11 = 11$ ✓

$\therefore$ true for $n = 1, 2$

Step 2 Let $n = k$ be a value for which result holds

ie $S_k = \frac{1}{6} k(k+1)(2k+7)$

Consider now $n = k+1$.

$S_{k+1} = S_k + T_{k+1}$

$= \frac{1}{7} k(k+1)(2k+7) + (k+1)(k+3)$

$= \frac{1}{6}(k+1)[k(2k+7) + 6(k+3)]$

$= \frac{1}{6}(k+1)(2k^2 + 13k + 18)$

$= \frac{1}{6}(k+1)(k+2)(2k+9)$ (5

$= \frac{1}{6}(k+1)[(k+1)+1][2(k+1)+7]$

which is of the form $S_n = \frac{1}{6}n(n+1)(2n$ with $n = k+1$.

ie if result holds for $n = k$, it also holds for $n = k+1$

Step 3 Since result holds for $n = 1, 2$ it must hold for $n = 2+1 = 3$, and hence $n = 3+1 = 4$ etc ie for all integral $n \geq 1$.

c) (i) $5^{2x} = \frac{1}{125\sqrt{5}}$

$\therefore 5^{2x} = 5^{-3\frac{1}{2}}$ (2)

$\therefore 2x = -\frac{7}{2}$

$x = -7/4$

(ii) $2^{x+1} - 2^{x-1} = 9$.

$2^{x-1}(2^2 - 1) = 9$

$2^{x-1}\times 3 = 9$

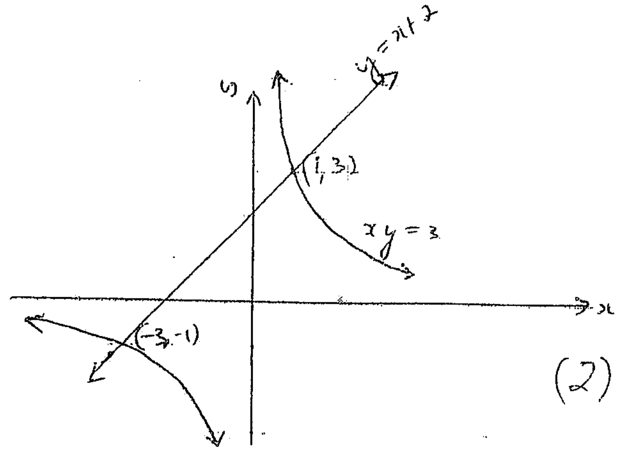
$2^{x-1} = 3$

$(x-1)\log_{10} 2 = \log_{10} 3$ (2)

$x = \frac{\log 3}{\log 2} + 1$

$\doteq 2\cdot 6$ (correct to 1 dec. pl)

(ii)



(2)

$\frac{3}{x} < x+2$ when $-3 < x < 0$, $x >$

Q4

a) $T_n = 3 - 4n$

$\therefore T_1 = 3 - 4 = -1$

$T_2 = 3 - 8 = -5$

$T_3 = 3 - 12 = -7$.

$T_2 - T_1 = -4$

$T_3 - T_2 = -4$

$\therefore$ arithmetic

(2)

or

$T_n - T_{n-1} = 3 - 4n$

$- (3 - 4(n-1))$

$= 3 - 4n - 3 + 4n$

$= -4$.

$\therefore$ A.P with d=

(ii) $S_n = \frac{n}{2}(2a + (n-1)d)$

$n = 20$

$a = -1$

$d = -4$

$S_{20} = 10(-2 + 19 \times -4)$

$= 10(-2 - 76)$

$= -780$

(2)

(iii) $3 - 4n < -500$

$-4n < -503$

$n > 125\frac{3}{4}$ ie $n = 126$

(2)

ie $T_{126} = -501$ is first term < -500

b) (i) $\sqrt{3} - 3 + 3\sqrt{3} \ldots$

$\frac{T_2}{T_1} = \frac{-3}{\sqrt{3}} = -\sqrt{3}$

$\frac{T_3}{T_2} = \frac{3\sqrt{3}}{-3}$

$= -\sqrt{3}$

$\therefore$ G.P with $r = -\sqrt{3}$

(1)

(ii) $S_n = \frac{a(r^n - 1)}{r-1}$

$\therefore S_6 = \frac{\sqrt{3}((-\sqrt{3})^6 - 1)}{-\sqrt{3} - 1}$

$= \frac{\sqrt{3}(27 - 1)}{-\sqrt{3} - 1}$

$= \frac{-26\sqrt{3}}{\sqrt{3} + 1} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$

(2)

$= \frac{-26\sqrt{3}(\sqrt{3} - 1)}{2}$

$= -13(3 - \sqrt{3})$

c) ~~$T_3 = ar^2 = 12$~~

~~$T_{10} = ar^9 = -9$~~

~~$\therefore r^7 = \ldots$~~

~~$\therefore r = 3$~~

~~$\therefore a \times 3^2 = 3$~~

~~ie $a = \frac{1}{3}$.~~

~~ie $T_1 = \frac{1}{3}$, $r = 3$~~

~~3, 6561, 2187~~

(3)

A.P.

$T_3 = 12 \Rightarrow a + 2d = 12$ ①

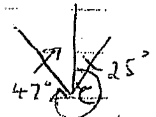
$a + 9d = -9$ ②

$T_{10} = -9$ ② − ① ⇒ $7d = -21$ ie

$\therefore a = 12 + 6 = 18$

$\therefore \angle C \doteq 72^\circ$ to nearest degree.

(2)



$\therefore$ Bearing of start from final position $\doteq 360^\circ - 47^\circ$
$\doteq 313^\circ$.

d) $(\operatorname{cosec}\alpha + \cot\alpha)^2 = \operatorname{cosec}^2\alpha + 2\operatorname{cosec}\alpha\cot\alpha + \cot^2$

$$= \frac{1}{\sin^2\alpha} + 2\cdot\frac{1}{\sin\alpha}\cdot\frac{\cos\alpha}{\sin\alpha} + \frac{\cos^2}{\sin^2}$$

$$= \frac{1 + 2\cos\alpha + \cos^2\alpha}{\sin^2\alpha}$$

$$= \frac{(1+\cos\alpha)^2}{1-\cos^2\alpha}$$

$$= \frac{(1+\cos\alpha)^2}{(1+\cos\alpha)(1-\cos\alpha)} \qquad (3)$$

$$= \frac{1+\cos\alpha}{1-\cos\alpha}$$

Q3

a) (i) $x^2 - 6x + y^2 + 4y = -8$.

$x^2 - 6x + 9 + y^2 + 4y + 4 = -8 + 13$ (2)

$(x-3)^2 + (y+2)^2 = 5$

$\therefore$ Centre is $(3, -2)$ radius $= \sqrt{5}$ u

(ii) $d = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$ $ax + by + c \equiv x - 2y + k$ and $(x_1, y_1) = (3, -2)$, $d = \sqrt{5}$

$\therefore \sqrt{5} = \dfrac{|3 + 4 + k|}{\sqrt{1^2 + (-2)^2}}$ (3)

$\therefore 5 = |7 + k|$

$\therefore k + 7 = 5$ or $k + 7 = -5$

ie $k = -2, -12$

$\therefore$ 2 tangents are $x - 2y - 2 = 0$ or $x - 2y - 12 = 0$

b) Line has equation in the form

$x - 2y - 3 + k(3x - y - 5) = 0$

ie $(1 + 3k)x + y(-2 - k) = 5k + 3$

ie $y = \dfrac{3k+1}{k+2}x - \dfrac{5k+3}{k+2}$

ie $\dfrac{3k+1}{k+2} = \dfrac{1}{3}$

$\Rightarrow 9k + 3 = k + 2$. (4)

$8k = -1$

$k = -\frac{1}{8}$.

$\therefore$ line is

$x - 2y - 3 - \frac{1}{8}(3x - y - 5) = 0$

ie $8x - 16y - 24 - 3x + y + 5 = 0$

ie $5x - 15y - 19 = 0$.

c) (i) $y = \dfrac{3}{x}$ ① , $y = x + 2$ ② } solve simult'ly

Sub ② into ① $\Rightarrow$

$x + 2 = \dfrac{3}{x}$

ie $x^2 + 2x - 3 = 0$.

$(x+3)(x-1) = 0$ (1)

$x = 1, -3$.

when $x = 1$, $y = 3$

$x = -3$, $y = -1$

$\therefore (-3, -1)$, $(1, 3)$ are pts of $\times$'n

(OR sub both points into both equations)

d) $f(x) = \dfrac{9+x^2}{9-x^2}$

Vertical asymptotes occur at $x = \pm 3$ (

As $x \to \pm\infty$, $f(x) \to -1$

ie $y = f(x) = -1$ is horizontal asymptote.

Q2

a) $1 - \dfrac{\sin A \cos A}{\tan A}$

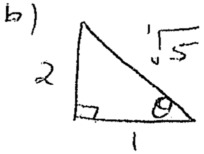
$= 1 - \sin A \cos A \div \dfrac{\sin A}{\cos A}$

$= 1 - \sin A \cos A \times \dfrac{\cos A}{\sin A}$

$= 1 - \cos^2 A$ (2).

$= \sin^2 A$.

b)

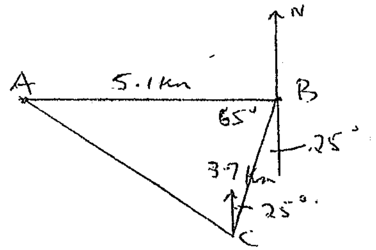


θ in Quad 3 from conditions given. $\therefore \cos\theta < 0$

$\cos\theta = -\dfrac{1}{\sqrt{5}}$. (2)

c)

(i)



(1)

(ii) $b^2 = a^2 + c^2 - 2ac \cos B$

$= 3.7^2 + 5.1^2 - 2 \times 3.7 \times 5.1 \cos 65$.

$= 23.75$

$\therefore b = 4.9$ km (to 1 dec. pl) (2

ie Leana ~ 4.9 km from start.

(iii) We find $\angle C$

– either sine or cosine Rule

Sine Rule:

$\dfrac{\sin C}{c} = \dfrac{\sin B}{b} \Rightarrow \dfrac{\sin C}{5.1} \doteq \dfrac{\sin 65°}{4.8734}$

$\sin C \doteq \dfrac{5.1 \sin 65}{4.8734}$

SOLUTIONS

YEAR 11 EXT 1

COMMON TEST 2

2003

Q1 a) $f(x) = x^2 - x$

$f(1-x) = (1-x)^2 - (1-x)$

$= 1 - 2x + x^2 - 1 + x$

$= x^2 - x$

$= f(x)$. (2)

b) (i) $y = x^2$ ①

$x^2 + y^2 = 2$ ②

Sub ① into ② $\Rightarrow x^2 + x^4 = 2$

let $u = x^2$

$\therefore u^2 + u - 2 = 0$

$(u+2)(u-1) = 0$

$\therefore u = 1, -2$

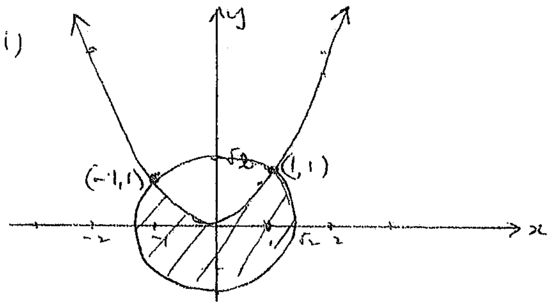
$\therefore x^2 = 1, \ -2$ n/a

ie $x = \pm 1$

when $x = \pm 1$, $y = 1$

$\therefore (1,1)$ $(-1,1)$ are pts of i.

(ii) (iii)



(5)

(iv) a) $a^2 > 0$ for all $a > 0$

$-a^2 < 0$

$1 - a^2 < 1$ (2)

$\therefore (1+a)(1-a) < 1$

$\therefore 1 + a < \frac{1}{1-a}$ Since $1 - a > 0$ for a