St George Girls High School

Year 11

Common Test 2

June 2004

Mathematics Extension 1

Time Allowed: 75 minutes

Instructions

1. All questions are of equal value.
2. Attempt all questions.
3. All questions are to be completed in the answer booklet.
4. Start each question on a new page.

Question 1 – 15 marks – (Start a new page) **Marks**

a) Solve for $0^\circ \le x \le 360^\circ$:

(i) $\sin 2x = \dfrac{-\sqrt{3}}{2}$ **2**

(ii) $\sin x + \cos x = 0$ **2**

b) Find the equation of the line through $(-1,\ 2)$ and making an angle of 150° with the positive x-axis **2**

c) Solve this equation:

$\log_2 x = \log_2 16 + \log_4 32$ **2**

d) (i) Determine what type of sequence is: $\log 3,\ \log 9,\ \log 27,\ \log 81$..... (show all working) **1**

(ii) Find an expression for T_n, the *nth* term of the above sequence. Hence find: **1**

(iii) the eighth term. **1**

(iv) the eighth partial sum of the progression. **1**

e) The length of a newly hatched python is 24cm. It increases in length by 9cm during its first year. In each succeeding year, its increase in length is $\dfrac{2}{3}$ of that in the previous year. What is the greatest length that this python could expect to reach? **3**

Question 2 – 15 marks – (Start a new page) **Marks**

a) Evaluate $\sum_{r=1}^{10} 3\left(\frac{1}{2}\right)^r$ correct to 3 decimal places (without calculating the individual terms). **3**

b) Insert three geometric means between 16 and $\frac{81}{16}$. **3**

c) Simplify $\frac{3^{2-n} \times 6^{2n}}{12^{n-2}}$ **2**

(d) (i) Write down: **1**

(α) the centre

and (β) the radius of the circle $(x-2)^2+(y+1)^2=25$

(ii) Find the distance from the line $y=2x+3$ to the centre of the above circle. (Give exact answer). **3**

(iii) Hence, find the length of the chord cut off from the line by the circle. (Give answer correct to 2 significant figures). **3**

Question 3 – 15 marks – (Start a new page) **Marks**

a) The sides of a triangle are in the ratio $2:4:5$. Find the largest angle of the triangle, correct to the nearest minute. **2**

b) Find the equation of the line (in general form) passing through the point of intersection of $x-2y+1=0$ and $2x+3y-1=0$, having a slope of 2. (Do not find the point of intersection of the two given lines). **4**

c) A is the point $(-2, 1)$ and B is the point (x, y). The point $P(13, -9)$ divides AB externally in the ratio $5:3$. Find the values of x and y. **3**

d) The sum of the first three terms of a geometric series is 4, and the sum of the next three terms is 32. Find the first term, a, and the common ratio, r, of the progression. **3**

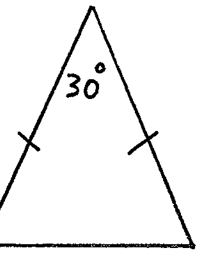
e) Solve for θ $\left(0^\circ \le \theta \le 360^\circ\right)$: **3**

$$6\cos^2\theta + \sin\theta - 4 = 0$$

(Give answers correct to the nearest minute).

Question 4 – 15 marks – (Start a new page) **Marks**

a) The vertical angle of an isosceles triangle is 30°, and its area is 40cm^2. Find the length of the equal sides. (Give the exact answer, in simplified form). **2**



b) Prove the following identity: **3**

$$\frac{1+\cos A}{1-\cos A} \equiv (\operatorname{cosec} A + \cot A)^2$$

c) Two roads diverge from an intersection, X, at an angle of 132°. At the same time, two cars leave the intersection – one along each road. The cars are travelling at 30km/h and 40km/h. How far apart are they after 2 hours? (Give the answer to the nearest kilometre). **3**

d) If $x = \log_m 2$, $y = \log_m 3$ and $z = \log_m 5$, express $\log_m \dfrac{25}{54m}$ in terms of x, y and z. **2**

e) Using the principle of mathematical induction, prove (for all positive integral values of n) that $5^n - 1$ is divisible by 4. **5**

12 × 20

Stephanie Pham

Preliminary Solutions Common Test 2 Extension 1

2004

Question 1

(a) (i) $\sin 2x = -\frac{\sqrt{3}}{2}$ 1 for ref ∠ = 60°

$2x = 240°, 300°, 600°, 660°$ 1

$x = 120°, 150°, 300°, 330°$ 1

$-\frac{1}{2}$ per   (2)

(ii) $\sin x + \cos x = 0$

$\sin x = -\cos x$

$\frac{\sin x}{\cos x} = -1$ (2)

$\tan x = -1$ 1

for $\tan x = 1$ ∴ $x = 45$ ✓

$x = 135°, 315°$ 1

(b) 1 for $\tan x = 1$ $x = 45°, 135°, 225°$

$m = \tan\theta$

$m = \tan 150$ } $\frac{1}{2}$

$m = -\frac{1}{\sqrt{3}}$ $(-1, 2)$ $\frac{1}{2}$

$y - y_1 = m(x - x_1)$ $(\frac{1}{2})$

$y - 2 = -\frac{1}{\sqrt{3}}(x + 1)$ (2)

$-\frac{1}{2}$ if say $\tan 150 = -0.577..$

$\sqrt{3}y - 2\sqrt{3} = -x - 1$

$y = -0.58x + 1.4x$

$x + \sqrt{3}y - 2\sqrt{3} + 1 = 0$ $\frac{1}{2}$

(c) $\log_2 x = \log_2 16 + \log_4 32$

$\log_2 x = \log_2 2^4 + \frac{\log_2 32}{\log_2 4}$ (2)

$\log_2 x = 4\log_2 2 + \frac{5\log_2 2}{2\log_2 2}$

$\log_2 x = 4 + \frac{5}{2}$ $\frac{1}{2}$, $\frac{1}{2}$ $6\frac{1}{2}$ $-\frac{1}{2}$ per

$\log_2 x = \frac{13}{2}$

$x = 2^{\frac{13}{2}}$ $\frac{1}{2}$ $= 64\sqrt{2}$ or $\sqrt{2^{13}}$

(d) $\log 3 + 2\log 3 + 3\log 3 + 4\log 3$

(i) AP ‖ $a = \log 3$
$d = \log 3$

(ii) $T_n = a + (n-1)d$ } $\frac{1}{2}$

$= \log 3 + (n-1)\log 3$

If say GP, don't penalise subsequent errors

$= \log 3 + n\log 3 - \log 3$ (1)

$= n\log 3$

$T_n = \log 3^n$ } $\frac{1}{2}$

(iii) $T_8 = \log 3^8$ or $8\log 3$ $= \log 6561$ 1

(iv) $S_8 = \frac{8}{2}(\log 3 + \log 3^8)$ if say GP give $\frac{1}{2}$ for GP formula

$= 4(\log 3 + \log 3^8)$

$= 4\log 3 + 4\log 3^8$

$= 4\log 3 + 32\log 3$

$= 36\log 3$ 1

or $4\log 3^9$ $= 4\log 19683$

(e) $24 + 9 + \frac{2}{3}\cdot 9 + \frac{2}{3}\cdot\frac{2}{3}\cdot 9 + \ldots$ $(\frac{1}{2})$

$(\frac{1}{2})$ 1

$24 + S_\infty$

$24 + \frac{a}{1-r}$

$24 + \frac{9}{1-\frac{2}{3}}$ } 1   (3)

$24 + 27$

total length $= 51$ cm.   1

-1 if just 27

2 for 12's $24 + \frac{a}{1-r}$

Question 2.

(a) $\sum_{r=1}^{10} 3\left(\frac{1}{2}\right)^n$

$a = 3 \cdot \frac{1}{2} = \frac{3}{2}$

$r = \frac{1}{2}$

$n = 10$

$$S_{10} = \frac{a(1-r^n)}{1-r}$$

$$= \frac{\frac{3}{2}\left(1-\left(\frac{1}{2}\right)^{10}\right)}{1-\frac{1}{2}}$$

$$= 3\left(1-\left(\frac{1}{2}\right)^{10}\right)$$

$$= 3 \cdot \frac{1023}{1024}$$

$$= \frac{3069}{1024}$$

(b) $16 \quad x \quad y \quad z \quad \frac{81}{16}$

$a = 16$

$ar^4 = \frac{81}{16} \quad \therefore \quad 16r^4 = \frac{81}{16}$

$$r^4 = \frac{81}{256}$$

$$r^4 = \frac{3^4}{2^8}$$

$$\therefore r = \left(\frac{3^4}{2^8}\right)^{\frac{1}{4}}$$

$$r = \pm\frac{3}{4}$$

(c) $$\frac{3^{2-n} \times 6^{2n}}{12^{n-2}}$$

$$= \frac{3^{2-n} \times 2^{2n} \cdot 3^{2n}}{(2^2 \cdot 3)^{n-2}}$$

$$= \frac{3^2 \cdot 3^{-n} \cdot 2^{2n} \cdot 3^{2n}}{2^{2n-4} \cdot 3^{n-2}}$$

$$= 3^{2-n+2n-n+2} \cdot 2^{2n-2n+4}$$

$$= 3^4 \cdot 2^4$$

$$= 6^4$$

$$= 1296$$

(d) (i) $(x-2)^2 + (y+1)^2 = 25$

α. centre $= (2,-1)$

β. radius $= 5$

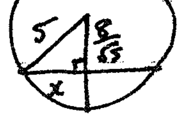
(ii) $2x - y + 3 = 0 \qquad (2,-1)$

$$d = \frac{|2x_1 - y_1 + 3|}{\sqrt{2^2 + (-1)^2}}$$

$$= \frac{|4 + 1 + 3|}{\sqrt{5}}$$

$$= \frac{8}{\sqrt{5}} \qquad (3.58)$$

(iii)



$$x^2 + \frac{64}{5} = 25$$

$$x^2 = \frac{61}{5}$$

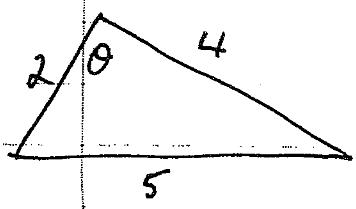
$$x = \sqrt{\frac{61}{5}}$$

$$\therefore \text{chord} = \frac{2\sqrt{61}}{\sqrt{5}}$$

$$= \frac{2\sqrt{305}}{5} \qquad (\approx 7)$$

Question 3.

(a)



$$\cos\theta = \frac{2^2+4^2-5^2}{2.2.4}$$

$$= -\frac{5}{16}$$

$$\theta = 108^\circ 13'$$

(b) $x-2y+1+k(2x+3y-1)=0$

$x-2y+1+2kx+3ky-k=0$

$(2-3k)y = (1+2k)x + 1-k$

$$y = \frac{1+2k}{2-3k}x + \frac{1-k}{2-3k}$$

$$m = \frac{1+2k}{2-3k}$$

$$\therefore \frac{1+2k}{2-3k} = 2$$

$1+2k = 4-6k$

$8k = 3$

$$k = \frac{3}{8}$$

$x-2y+1+\frac{3}{8}(2x+3y-1)=0$

$8x-16y+8+6x+9y-3=0$

$14x-7y+5=0$

(c) $-5:3$

$A(-2,1)$ $B(x,y)$ $P(13,-9)$

$$x = \frac{nx_1+mx_2}{m+n} \qquad y = \frac{ny_1+my_2}{m+n}$$

$$13 = \frac{-6-5x_2}{-2} \qquad -9 = \frac{3-5y_2}{-2}$$

$-26 = -6-5x_2$ $\qquad 18 = 3-5y_2$

$-20 = -5x_2$ $\qquad 15 = -5y_2$

$4 = x_2$ $\qquad -3 = y_2$

$\therefore B = (4,-3)$

(d) $T_1+T_2+T_3 = 4$

$T_4+T_5+T_6 = 32$

$$S_6 = 36 = \frac{a(r^6-1)}{r-1}$$

$$S_3 = 4 = \frac{a(r^3-1)}{r-1}$$

$$\frac{S_6}{S_3} = \frac{36}{4} = \frac{a(r^6-1)}{\cancel{(r-1)}}\cdot\frac{\cancel{(r-1)}}{a(r^3-1)}$$

$$\therefore \frac{r^6-1}{r^3-1} = 9$$

$r^6-1 = 9r^3-9$

$r^6-9r^3+8=0$

let $m=r^3$

$m^2-9m+8=0$

$(m-1)(m-8)=0$

$\therefore r^3=1 \qquad r^3=8$

$r=1 \qquad r=2$

$a(2^3-1)=4$

$7a=4$

$a=4$

$$\therefore \frac{4}{7}+\frac{8}{7}+\frac{16}{7}+\frac{32}{7}+\frac{64}{7}+\frac{128}{7}$$

(e) $6\cos^2\theta + \sin\theta - 4 = 0$

$6(1-\sin^2\theta) + \sin\theta - 4 = 0$

$6 - 6\sin^2\theta + \sin\theta - 4 = 0$

$-6\sin^2\theta + \sin\theta + 2 = 0$

$6\sin^2\theta - \sin\theta - 2 = 0$

$(3\sin\theta - 2)(2\sin\theta + 1) = 0.$

$\sin\theta = \frac{2}{3}, \ -\frac{1}{2}$

$\sin\theta = \frac{2}{3}$

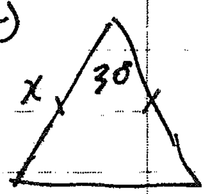
$\theta = 41^\circ 49' \quad 138^\circ 11'$

$\sin\theta = -\frac{1}{2}$

$\theta = 210^\circ, \ 330^\circ$

Question 4

(a)



$A = 40$

$\text{Area} = \frac{1}{2} \cdot x \cdot x \cdot \sin 30$

$40 = \frac{x^2}{2} \cdot \sin 30$

$40 = \frac{x^2}{2} \cdot \frac{1}{2}$

$160 = x^2$

$x = \sqrt{160}$

$x = 4\sqrt{10}$

(b) $\frac{1+\cos A}{1-\cos A} = (\operatorname{cosec} A + \cot A)^2$

$\text{LHS} = \frac{1+\cos A}{1-\cos A}$

$= \frac{\frac{1}{\sin A} + \frac{\cos A}{\sin A}}{\frac{1}{\sin A} - \frac{\cos A}{\sin A}}$

$= \frac{\operatorname{cosec} A + \cot A}{\operatorname{cosec} A - \cot A}$

$= \frac{\operatorname{cosec} A + \cot A}{\operatorname{cosec} A - \cot A} \cdot \frac{\operatorname{cosec} A + \cot A}{\operatorname{cosec} A + \cot A}$

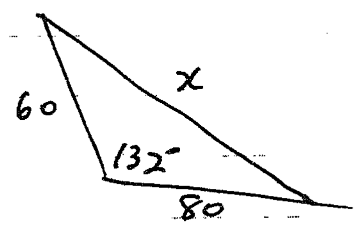
$= \frac{(\operatorname{cosec} A + \cot A)^2}{\operatorname{cosec}^2 A - \cot^2 A}$

$= \frac{(\operatorname{cosec} A + \cot A)^2}{1}$

$= (\operatorname{cosec} A + \cot A)^2$

$=$ RHS.

(c)



$x^2 = 60^2 + 80^2 - 2 \cdot 60 \cdot 80 \cdot \cos 132$

$= 3600 + 6400 - 960 \cos 132$

$= 16423.65$

$x = \sqrt{16423.65}$

$= 128.15$

≈ 128 km.

(d) $x = \log_m 2 \quad y = \log_m 3 \quad z = \log_m 5$

$$\log \frac{25}{54m} = \log 25 - \log 54m$$

$$= \log 5^2 - (\log 27 + \log 2 + \log m)$$

$$= 2\log 5 - (3\log 3 + \log 2 + 1)$$

$$= 2z - (3y + x + 1)$$

$$= 2z - 3y - x - 1$$

(e) test $n = 1$

$$5^1 - 1 = 5 - 1$$
$$= 4$$

$\therefore$ true for $n = 1$

let statement be true for $n = k$

$\therefore 5^k - 1 = 4M$

need to show $5^{k+1} - 1$ also divisible by 4.

$$5^{k+1} - 1 = 5^k . 5 - 1$$
$$= 5 . 5^k - 5 + 4$$
$$= 5(5^k - 1) + 4$$
$$= 5 . 4M + 4$$
$$= 20M + 4$$
$$= 4(5M + 1)$$

$\therefore$ divisible by 4.

$\therefore$ true for $n = k+1$

Since true for $n = 1$

~~$\therefore$ true for $k = 1$~~

shown true for $k+1$ $\therefore$ true for 2 and so on

$\therefore$ true for all n.