St George Girls High School

Year 12

Mid-HSC Course Examination

2009

Mathematics Extension 2

General Instructions

- Working time: 1½ hours
- Reading time: 5 minutes
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- A separate section with graphs is supplied for Question 2 responses.

Total marks – 72

- Attempt Questions 1 – 3
- All questions are of equal value
- Begin each question in a new booklet
- All necessary working should be shown in every question.

Question 1 – (24 marks) – (Start a new booklet) Marks

a) Find the equation of the ellipse which has eccentricity $\frac{3}{4}$ and foci at $S'(-3,\ 0)$ and $S(3,\ 0)$ 2

b) Consider the hyperbola H with equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$

(i) Find the eccentricity, e 1

(ii) Find the coordinates of the foci, S and S' 1

(iii) Find the equation of each directrix 1

(iv) Find the equation of each of the asymptotes 1

(v) Sketch the hyperbola H, showing all the main features. 2

(vi) Given $P(4\sec\phi,\ 3\tan\phi)$ is any arbitrary point on H

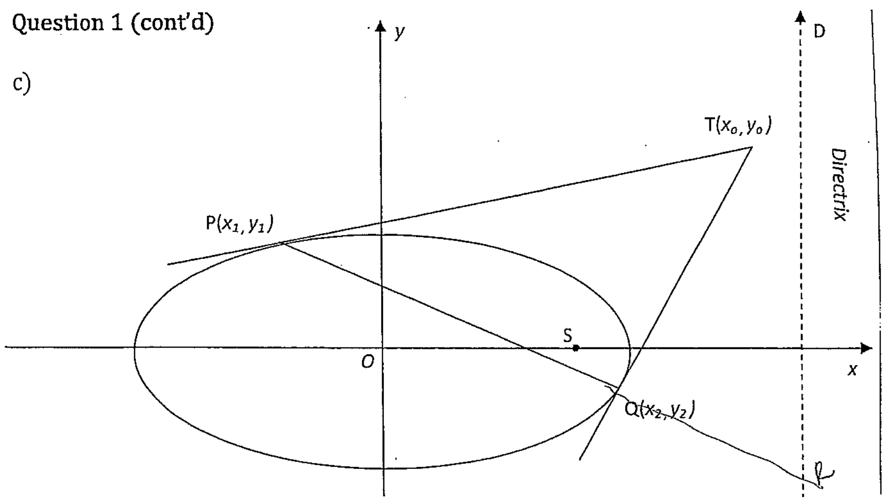
(α) Show that the equation of the tangent at P is $\frac{x\sec\phi}{4} - \frac{y\tan\phi}{3} = 1$ 3

(β) The tangent at P meets the asymptote of the hyperbola H at the points Q and Q'. Find the coordinates of Q and Q' 2

(γ) Hence, or otherwise, show that $PQ = PQ'$ 2

Question 1 (cont'd) **Marks**

c)



The ellipse E has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with focus S and directrix D as shown on the diagram. The point $T(x_0, y_0)$ lies outside the ellipse and is not on the x-axis.

The chord of contact PQ from T intersects the directrix at R.

(i) Show that the equation of the tangent to ellipse at the point $P(x_1, y_1)$ is **2**

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1$$

(ii) Show that the equation of the normal to the ellipse at the point $P(x_1, y_1)$ is **2**

$$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$$

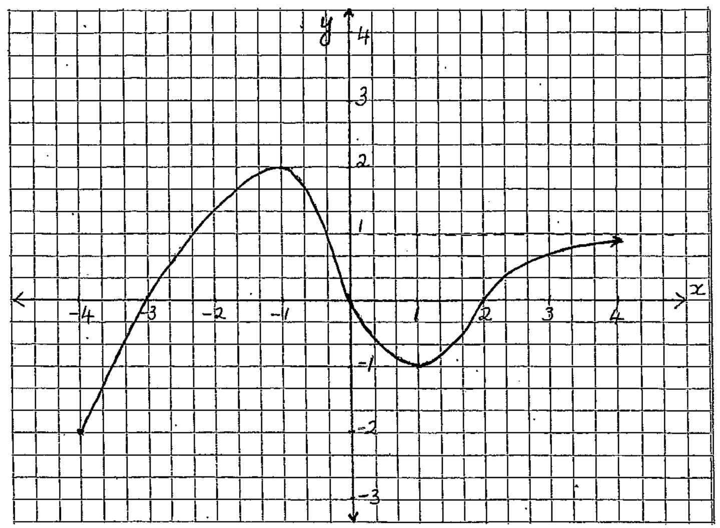
(iii) Given the equation of the chord of contact from T is $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$, show that $R(x, y)$ has ordinate **2**

$$y = \frac{b^2}{y_0}\left(1 - \frac{x_0}{ae}\right)$$

(iv) Show that TS is perpendicular to SR. **3**

Question 2 – (24 marks) – (Start a new booklet) **Marks**

a) The graph of $y = f(x)$ for $x \geq -4$ is shown in the following diagram.



On the separate diagrams provided, neatly sketch the following graphs.

(i) $y = f(x+1)$ **2**

(ii) $y = [f(x)]^2$ **2**

(iii) $y = \sqrt{f(x)}$ **2**

(iv) $y^2 = f(x)$ **2**

(v) $y = f(|x|)$ **2**

(vi) $y = \frac{1}{f(x)}$ **2**

(vii) $y = f'(x)$ **2**

Question 2 (cont'd) Marks

b) For the curve $2x^2 - xy + y^2 = 14$

(i) Use implicit differentiation to find $\frac{dy}{dx}$ 3

(ii) Find the gradient of the tangent at the point $(1, -3)$ 1

(iii) Show that any stationary points must lie on the line $y = 4x$ 1

(iv) Hence find the coordinates of the stationary points 3

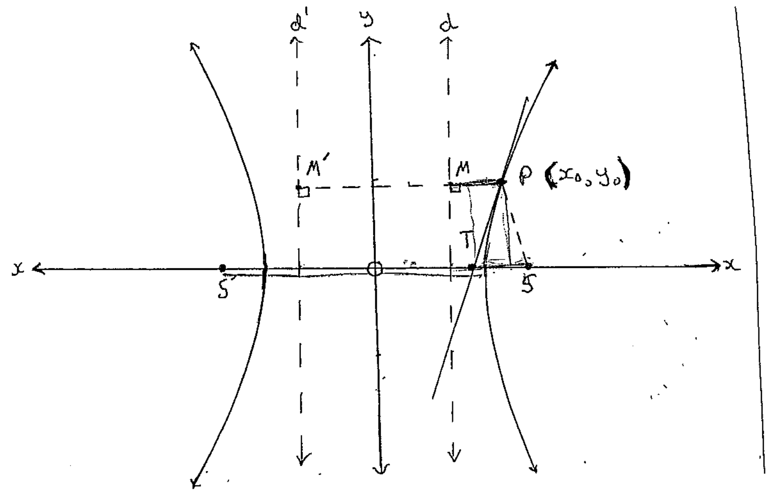
c) Sketch the graph of $\cos(x + y) = 0 \quad -3\pi \le x \le 3\pi$ 2

Question 3 – (24 marks) – (Start a new booklet) Marks

a) (i) On separate diagrams draw neat sketches of $y = \cos x$ for $-2\pi \le x \le 2\pi$ and $y = 3^x$ for $-1 \le x \le 1$ 2

(ii) Using part (i) or otherwise draw a neat sketch of $y = 3^{\cos x}$ $-2\pi \le x \le 2\pi$ 2

b)



The point $P(x_0, y_0)$ lies on the hyperbola $\frac{x^2}{9} - \frac{y^2}{7} = 1$

The tangent to the hyperbola at P cuts the x-axis at T, and has equation $\frac{x_0 x}{9} - \frac{y_0 y}{7} = 1$. The two foci of the hyperbola are S and S', and the two directrices are d and d'.

The points M and M' are the feet of the perpendicular from P to d and d'.

The eccentricity is $\frac{4}{3}$, the foci $S(4, 0)$ and $S'(-4, 0)$, the directrices are $x = \pm\frac{9}{4}$

(i) Show that T has coordinates $(\frac{9}{x_0}, 0)$ 1

(ii) Using the focus directrix definition, or otherwise, show that $\frac{PS}{PS'} = \frac{TS}{TS'}$ 3

(iii) Prove that $|PS' - PS| = 6$ 1

Question 3 (cont'd) Marks

c) Show that, if $y = mx + k$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ then $m^2a^2 - b^2 = k^2$ 3

d) (i) Draw a neat sketch of the parabola $y = x^2 - 2x - 3$. Your sketch should clearly show the intercepts with the coordinate axes and the coordinates of the vertex. 3

(ii) Hence, or otherwise, sketch the graph of $y = 4 - |x^2 - 2x - 3|$ showing all important features. 2

(iii) For what values of m does the equation $mx = 4 - |x^2 - 2x - 3|$ have 4 distinct real solutions? 3

e) P is any point on the ellipse $E: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and S is a focus of E.

Prove that the line through S perpendicular to the tangent at P meets the line OP produced at a point on the directrix. 4

You may use the fact that the equation of the tangent through P is

$$y = \frac{-b\cos\theta}{a\sin\theta}\left(x - \frac{a}{\cos\theta}\right)$$

End of Paper

Year 12 Mid-Course Assessment Ext 2 Maths

Question 1.

a) $e=\frac{3}{4}$, $S(3,0)$ $\therefore$ $ae=3$

$$a\times\frac{3}{4}=3$$
$$\therefore a=4$$

Now $b^2=a^2(1-e^2)$

$$b^2=16\left(1-\frac{9}{16}\right)$$
$$=16-9$$
$$=7$$

$\therefore$ Equation of ellipse is $\frac{x^2}{16}+\frac{y^2}{7}=1$

b) (i) Hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ has $a=4$, $b=3$

Since $b^2=a^2(e^2-1)$

$$9=16(e^2-1)$$
$$\frac{9}{16}+1=e^2$$
$$\frac{25}{16}=e^2$$
$$\therefore e=\frac{5}{4},\ e>0$$

(ii) Foci are $S(ae,0)$ ie $S\left(4\times\frac{5}{4},0\right)$; $S(5,0)$
$S'(-5,0)$

(iii) Directrices are $x=\pm\frac{a}{e}$

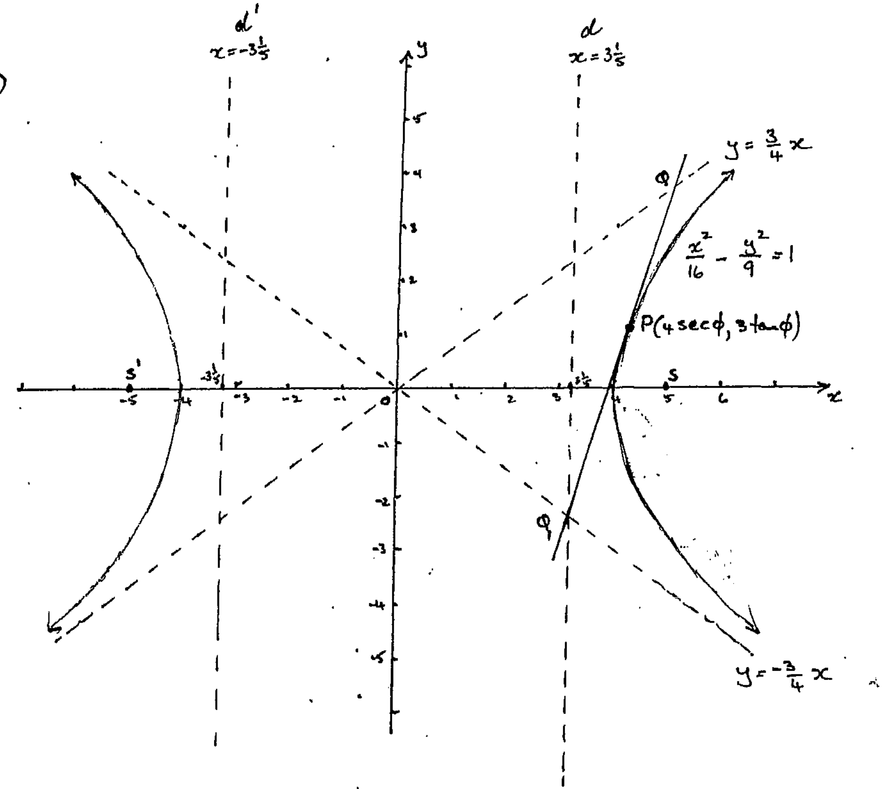
ie $x=\pm\frac{4}{5/4}$

ie $x=\frac{16}{5}$ or $x=-3\frac{1}{5}$

(iv) Asymptotes are $y=\pm\frac{b}{a}x$

ie $y=\frac{3}{4}x$ and $y=-\frac{3}{4}x$

(v)



(vi) $P(4\sec\phi, 3\tan\phi)$

(α) $\frac{x^2}{16}-\frac{y^2}{9}=1$

$$\frac{2x}{16}-\frac{2y}{9}\frac{dy}{dx}=0$$
$$\frac{dy}{dx}=-\frac{2x}{16}\cdot\frac{-9}{2y}$$
$$=\frac{9x}{16y}=\frac{9\times4\sec\phi}{16\times3\tan\phi},\text{ when at P.}$$

$\therefore$ Equation of tangent is

$$y-3\tan\phi=\frac{3\sec\phi}{4\tan\phi}(x-4\sec\phi)$$
$$4y\tan\phi-12\tan^2\phi=3x\sec\phi-12\sec^2\phi$$
$$3x\sec\phi-4y\tan\phi=12(\sec^2\phi-\tan^2\phi)$$
$$3x\sec\phi-4y\tan\phi=12\quad,\ \tan^2\phi+1=\sec^2\phi$$

$\div 12$, $\therefore\ \frac{x\sec\phi}{4}-\frac{y\tan\phi}{3}=1$

β) tangent at P meets asymptotes at Q & Q'

ie $y = \pm \frac{3}{4}x$ —(1)

$\frac{x\sec\phi}{4} - \frac{y\tan\phi}{3} = 1$ —(2)

Sub (1) into 2: $\frac{x\sec\phi}{4} - \frac{\pm 3x\tan\phi}{4\times 3} = 1$

$$\frac{x}{4}\left(\sec\phi \mp \tan\phi\right) = 1$$

ie $\begin{cases} x = \dfrac{4}{\sec\phi \mp \tan\phi} \\ y = \pm\dfrac{3}{4} \times \dfrac{4}{\sec\phi \mp \tan\phi} \end{cases}$

$\therefore Q\left(\frac{4}{\sec\phi - \tan\phi}, \frac{3}{\sec\phi - \tan\phi}\right)$ & $Q'\left(\frac{4}{\sec\phi+\tan\phi}, \frac{-3}{\sec\phi+\tan\phi}\right)$

γ) Now $PQ^2 = \left(4\sec\phi - \frac{4}{\sec\phi - \tan\phi}\right)^2 + \left(3\tan\phi - \frac{3}{\sec\phi - \tan\phi}\right)^2$

$= 16\left(\frac{\sec^2\phi - \sec\phi\tan\phi - 1}{\sec\phi - \tan\phi}\right)^2 + 9\left(\frac{\sec\phi\tan\phi - \tan^2\phi - 1}{\sec\phi - \tan\phi}\right)^2$

$= 16\left(\frac{\tan^2\phi - \sec\phi\tan\phi}{\sec\phi - \tan\phi}\right)^2 + 9\left(\frac{\sec\phi\tan\phi - \sec^2\phi}{\sec\phi - \tan\phi}\right)^2$, $\tan^2\phi + 1 = \sec^2\phi$

$= 16\tan^2\phi\left(\frac{\tan\phi - \sec\phi}{\sec\phi - \tan\phi}\right)^2 + 9\sec^2\phi\left(\frac{\tan\phi - \sec\phi}{\sec\phi - \tan\phi}\right)^2$

$= 16\tan^2\phi + 9\sec^2\phi$

Also, $[PQ']^2 = \left(4\sec\phi - \frac{4}{\sec\phi + \tan\phi}\right)^2 + \left(3\tan\phi - \frac{-3}{\sec\phi + \tan\phi}\right)^2$

$= 16\left(\frac{\sec^2\phi + \sec\phi\tan\phi - 1}{\sec\phi + \tan\phi}\right)^2 + 9\left(\frac{\sec\phi\tan\phi + \tan^2\phi + 1}{\sec\phi + \tan\phi}\right)^2$

$= 16\left(\frac{\tan^2\phi + \sec\phi\tan\phi}{\sec\phi + \tan\phi}\right)^2 + 9\left(\frac{\sec\phi\tan\phi + \sec^2\phi}{\sec\phi + \tan\phi}\right)^2$, $\tan^2\phi + 1 = \sec^2\phi$

$= 16\tan^2\phi\left(\frac{\tan\phi + \sec\phi}{\sec\phi + \tan\phi}\right)^2 + 9\sec^2\phi\left(\frac{\tan\phi + \sec\phi}{\sec\phi + \tan\phi}\right)^2$

$= 16\tan^2\phi + 9\sec^2\phi$

$\therefore$ Since $[PQ]^2 = [PQ']^2$

then,

$PQ = PQ'$

OR ALTERNATIVELY and more SIMPLY!

(γ) Midpoint of QQ' is:

$$M\left[\frac{1}{2}\left(\frac{4}{\sec\phi - \tan\phi} + \frac{4}{\sec\phi + \tan\phi}\right), \frac{1}{2}\left(\frac{3}{\sec\phi - \tan\phi} + \frac{-3}{\sec\phi + \tan\phi}\right)\right]$$

$$\text{ie } M\left[\frac{4}{2}\left(\frac{\sec\phi + \tan\phi + \sec\phi - \tan\phi}{\sec^2\phi - \tan^2\phi}\right), \frac{3}{2}\left(\frac{\sec\phi + \tan\phi - \sec\phi + \tan\phi}{\sec^2\phi - \tan^2\phi}\right)\right]$$

$$\text{ie } M\left[2\left(\frac{2\sec\phi}{1}\right), \frac{3}{2}\left(\frac{2\tan\phi}{1}\right)\right]$$

$\because \sin^2\phi + \cos^2\phi = 1$
$\tan^2\phi + 1 = \sec^2\phi$

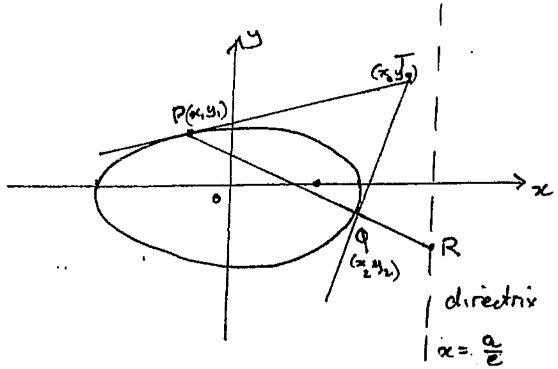
Thus midpoint of QQ' is

$(4\sec\phi, 3\tan\phi)$

But this is point P

Thus, $PQ = PQ'$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



(c) (i)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{2x}{a^2}\cdot\frac{b^2}{2y}$$

$$=\frac{-b^2x}{a^2y} = \frac{-b^2x_1}{a^2y_1} \text{ when at } P(x_1,y_1)$$

$\therefore$ Equation of tangent at P is

$$y-y_1=\frac{-b^2x_1}{a^2y_1}(x-x_1)$$

$$a^2y_1y-a^2y_1^2=-b^2x_1x+b^2x_1^2$$

$$b^2x_1x+a^2y_1y=a^2y_1^2+b^2x_1^2$$

$\div a^2b^2$;
$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$$

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1 \quad \text{since } (x_1,y_1) \text{ lies on } E.$$

(ii) Gradient of normal is $\frac{a^2y_1}{b^2x_1}$

$\therefore$ Equation of normal is

$$y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1)$$

$$b^2x_1y-b^2x_1y_1=a^2y_1x-a^2x_1y_1$$

$$a^2y_1x-b^2x_1y=a^2x_1y_1-b^2x_1y_1$$

$\div x_1y_1$:
$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$$

(iii) Equation of chord from T is

$$\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1$$

Ordinate of R when $x=\frac{a}{e}$

ie
$$\frac{x_0\times a}{a^2e}+\frac{y_0y}{b^2}=1$$

$$\frac{x_0}{ae}+\frac{y_0y}{b^2}=1$$

$$\frac{y_0y}{b^2}=1-\frac{x_0}{ae}$$

$$y_0y=b^2\left(1-\frac{x_0}{ae}\right)$$

$$y=\frac{b^2}{y_0}\left(1-\frac{x_0}{ae}\right)$$

(iv) Now,

$$m_{TS}=\frac{y_0-0}{x_0-ae}=\frac{y_0}{x_0-ae}$$

$$m_{SR}=\frac{\frac{b^2}{y_0}\left(1-\frac{x_0}{ae}\right)-0}{\frac{a}{e}-ae}$$

$$=\frac{b^2\left(1-\frac{x_0}{ae}\right)}{ay_0\left(\frac{1}{e}-e\right)} \begin{matrix}\times ae\\ \times ae\end{matrix}$$

$$=\frac{b^2(ae-x_0)}{a^2y_0(1-e^2)}$$

$$=\frac{b^2(ae-x_0)}{b^2y_0} \qquad b^2=a^2(1-e^2)$$

$$=\frac{ae-x_0}{y_0}$$

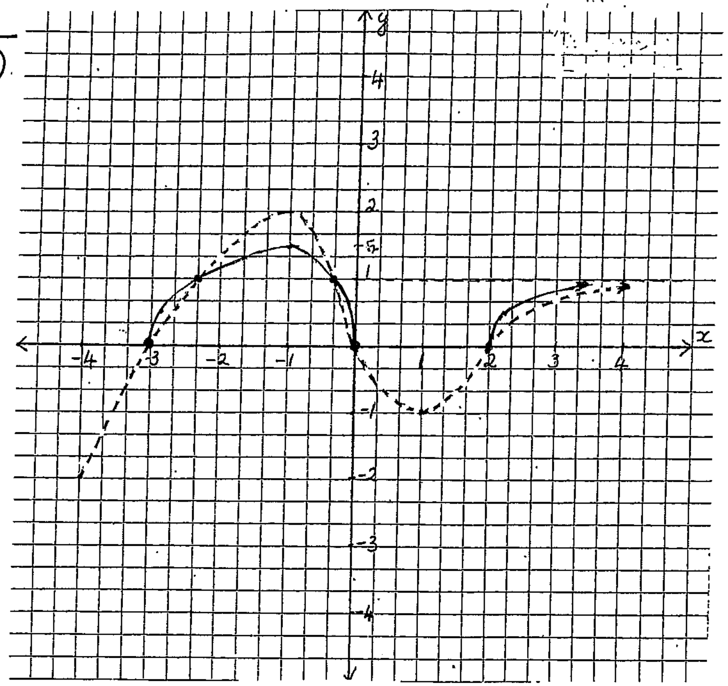
Since $m_{TS}\cdot m_{SR}=\frac{y_0}{\cancel{x_0-ae}}\cdot\frac{\cancel{ae-x_0}^{-1}}{y_0}$

$=-1$ then, $TS\perp SR$

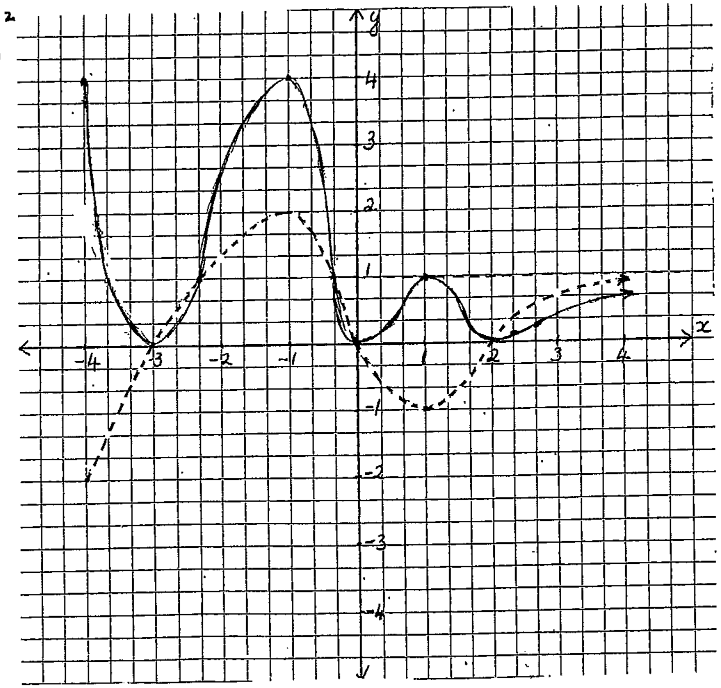
Question 2 (a)

(i) $y = f(x+1)$

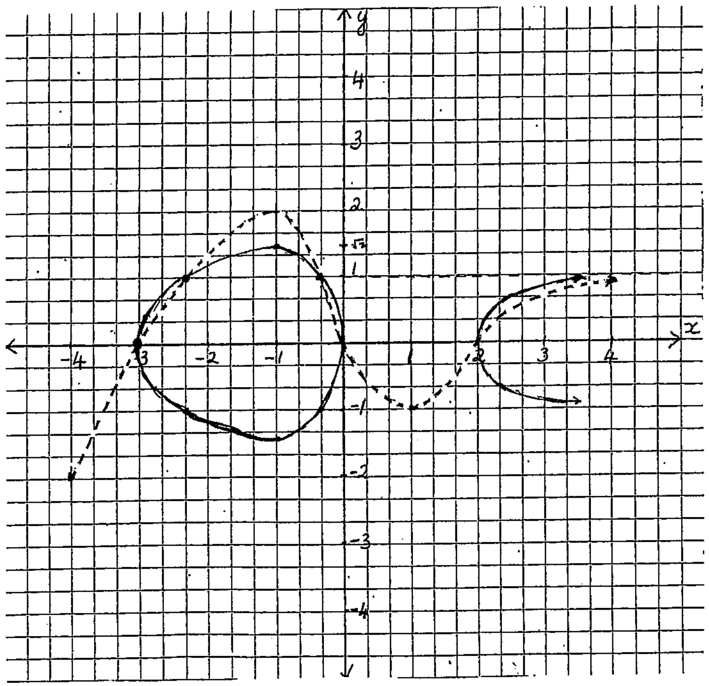
(iii) $y = \sqrt{f(x)}$



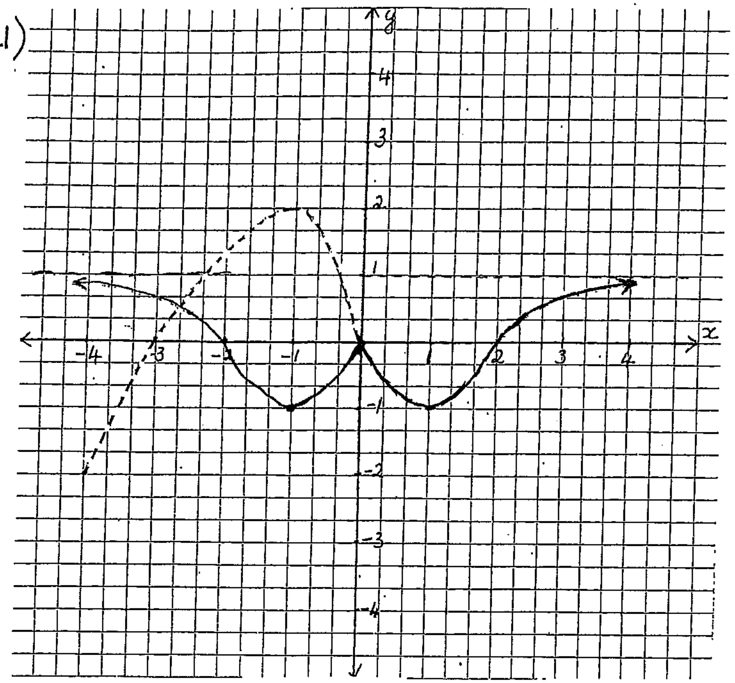
(ii) $y = [f(x)]^2$



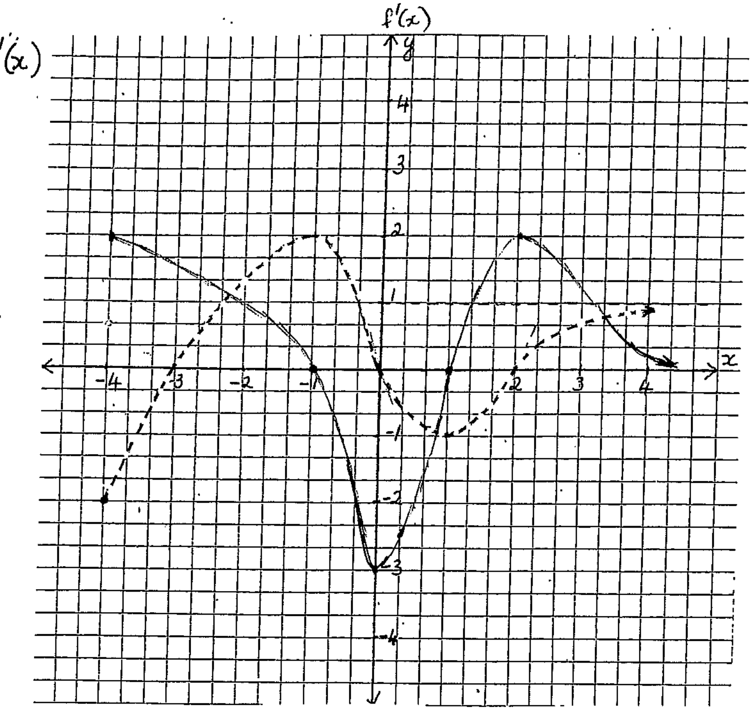
(iv) $y^2 = f(x)$



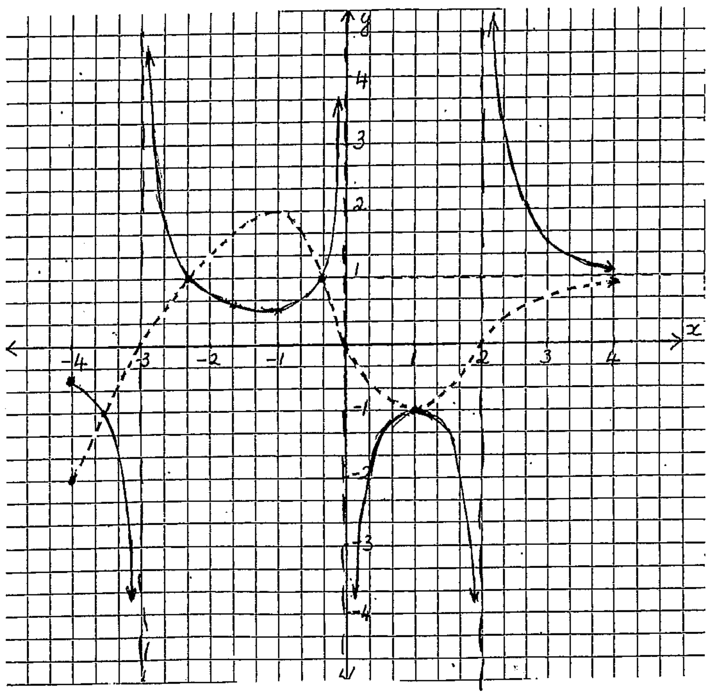
(v) $y = f(|x|)$



(vii) $y = f'(x)$



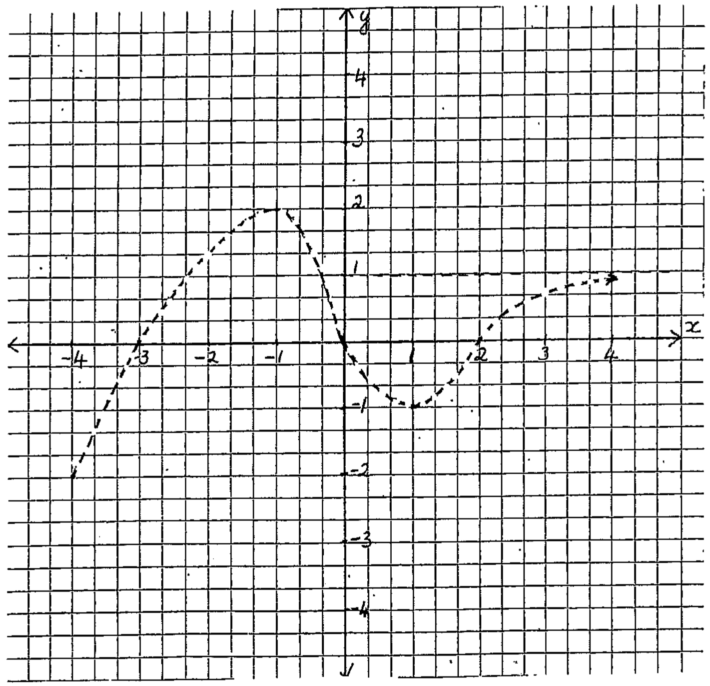
(vi) $y = \frac{1}{f(x)}$



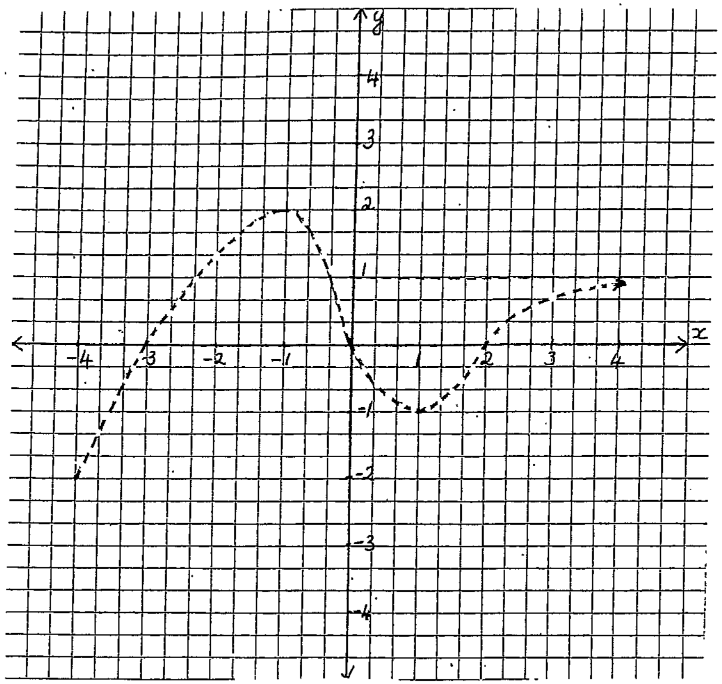
Spare

part:

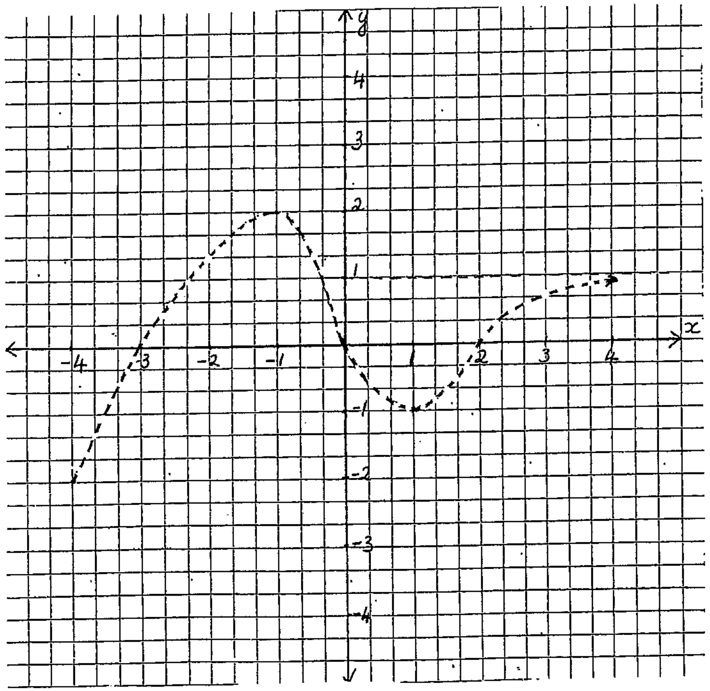
equation:



part:
equation:



part:
equation:



Question 2 (b)

(i)
$$2x^2 - xy + y^2 = 14$$
$$4x - \left(y.1 + x.\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$$
$$4x - y + \frac{dy}{dx}(2y - x) = 0$$
$$\frac{dy}{dx} = \frac{y - 4x}{2y - x}$$

(ii) when $x = 1$, $y = -3$
$$\frac{dy}{dx} = \frac{-3 - 4\times 1}{2\times -3 - 1} = \frac{-3-4}{-6-1} = \frac{-7}{-7} = 1$$

(iii) Stationary Points occur when $\frac{dy}{dx} = 0$

ie. $\frac{y - 4x}{2y - x} = 0$, $2y \neq x$

ie $y - 4x = 0$

ie all lie on $y = 4x$

(iv) Stationary points must occur when $y = 4x$

ie
$$2x^2 - x(4x) + (4x)^2 = 14$$
$$2x^2 - 4x^2 + 16x^2 = 14$$
$$14x^2 = 14$$
$$x^2 = 1$$
$$x = \pm 1$$

ie Stat Points are $(-1, -4)$ and $(1, 4)$

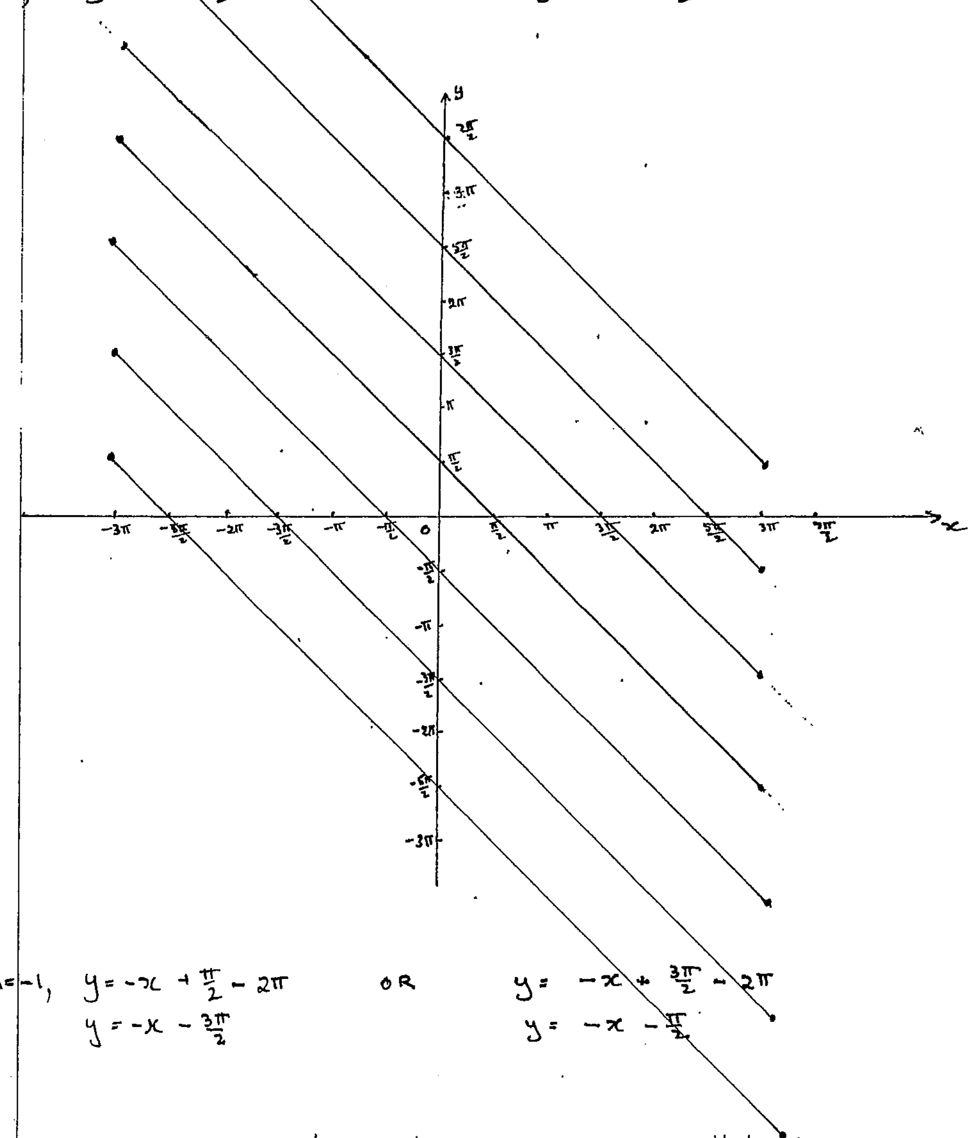
(d) $\cos(x+y) = 0$ for $-3\pi \le x \le 3\pi$

$x+y = \frac{\pi}{2} + 2\pi n$ **OR** $x+y = \frac{3\pi}{2} + 2n\pi$

$\therefore y = -x + \frac{\pi}{2} + 2n\pi$ OR $y = -x + \frac{3\pi}{2} + 2n\pi$

$n=0, \quad y = -x + \frac{\pi}{2}$ $\quad y = -x + \frac{3\pi}{2}$

$n=1, \quad y = -x + \frac{\pi}{2} + 2\pi$ $\quad y = -x + \frac{3\pi}{2} + 2\pi$



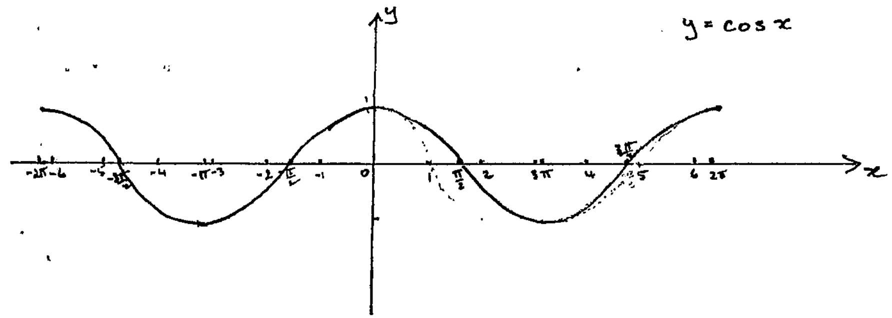
$n=-1, \quad y = -x + \frac{\pi}{2} - 2\pi$ OR $y = -x + \frac{3\pi}{2} - 2\pi$

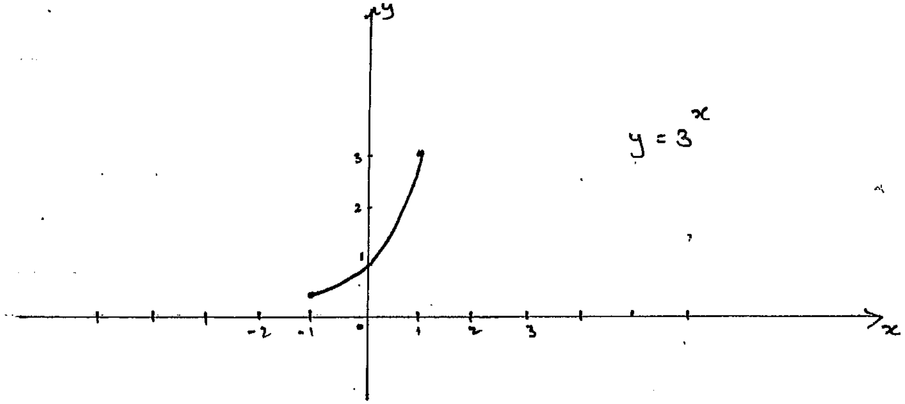
$y = -x - \frac{3\pi}{2}$ $\quad y = -x - \frac{\pi}{2}$

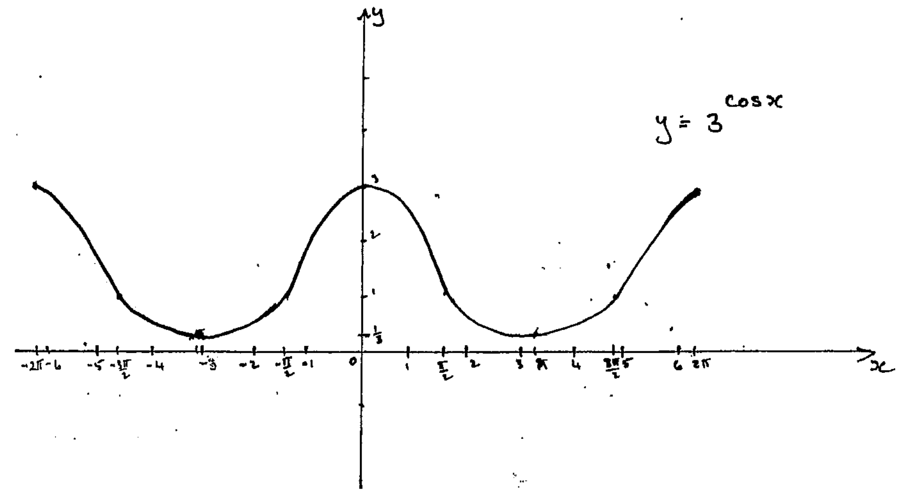
Graph continues as parallel lines separated by π units.

<u>Question 3.</u>

a) (i)







b) (i) $$\frac{x^2}{9} - \frac{y^2}{7} = 1$$

Equation of tangent is

$$\frac{x_0 x}{9} - \frac{y_0 y}{7} = 1$$

Tangent crosses x-axis when $y=0$

ie $\dfrac{x_0 x}{9} = 1 \Rightarrow x = \dfrac{9}{x_0}$

ie $T\left(\dfrac{9}{x_0}, 0\right)$

(ii) From the definition of Hyperbola,

$\dfrac{PS}{PM} = e$ and $\dfrac{PS'}{PM'} = e$

$\therefore PS = ePM$, $PS' = ePM'$

$$\therefore \frac{PS}{PS'} = \frac{ePM}{ePM'} = \frac{x_0 - \frac{3}{e}}{x_0 - \frac{-3}{e}}$$ where M is on $x = \frac{3}{e}$ and M' is on $x = \frac{-3}{e}$

$$= \frac{ex_0 - 3}{ex_0 + 3}$$

Now also, $$\frac{TS}{TS'} = \frac{3e - \frac{9}{x_0}}{\frac{9}{x_0} - -3e} \begin{matrix} \times x_0 \\ \times x_0 \end{matrix}$$

$$= \frac{3ex_0 - 9}{9 + 3ex_0}$$

$$= \frac{3(ex_0 - 3)}{3(3 + ex_0)}$$

$$= \frac{ex_0 - 3}{3 + ex_0}$$

Thus, $$\frac{TS}{TS'} = \frac{PS}{PS'}$$

(iii) $$|PS' - PS| = |ePM' - ePM|$$
$$= e|PM' - PM|$$
$$= e\left|\left(x_0 - \frac{3}{e}\right) - \left(x_0 - \frac{-3}{e}\right)\right|$$
$$= e\left|x_0 - \frac{3}{e} - x_0 - \frac{3}{e}\right|$$
$$= e\left|-2 \times \frac{3}{e}\right|$$
$$= |-6|$$
$$= 6$$

ie $|PS' - PS| = 6$

c) $y = mx + k$ is a tangent to $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ when Δ of quadratic formed on solving is ZERO.

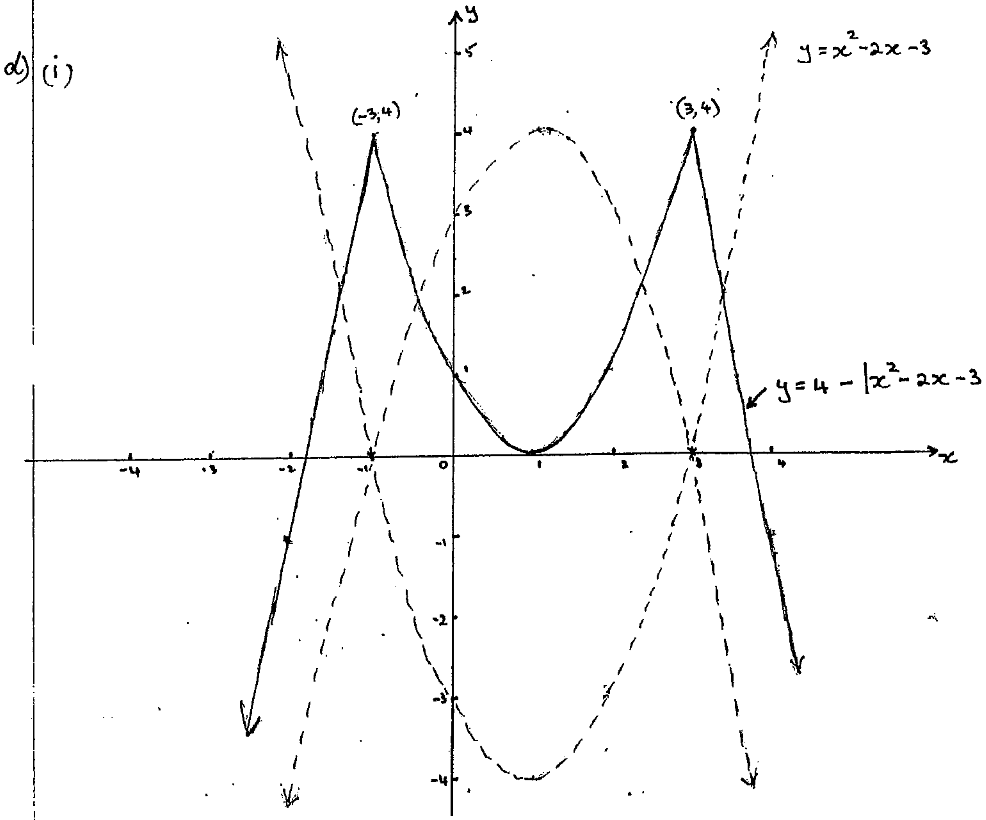
ie $$\frac{x^2}{a^2} - \frac{(mx+k)^2}{b^2} = 1$$

$\times a^2b^2$: $$b^2x^2 - a^2(mx+k)^2 = a^2b^2$$
$$b^2x^2 - a^2m^2x^2 - 2a^2mkx - a^2k^2 = a^2b^2$$
$$x^2(b^2 - a^2m^2) - 2a^2mkx - a^2(k^2+b^2) = 0$$
$$\therefore \Delta = (-2a^2mk)^2 - 4 \times (b^2 - a^2m^2) \times -a^2(k^2+b^2) = 0$$
ie $4a^2[a^2m^2k^2 + b^2k^2 + b^4 - a^2m^2k^2 - a^2b^2m^2$

ie $b^2(k^2 + b^2 - a^2m^2) = 0$

$b^2 \neq 0$, $\quad m^2a^2 - b^2 = k^2$

d) (i)



$y = (x-3)(x+1)$

$x = 1,\ y = -2 \times 2$
$= -4$

$(1,-4)$ vertex

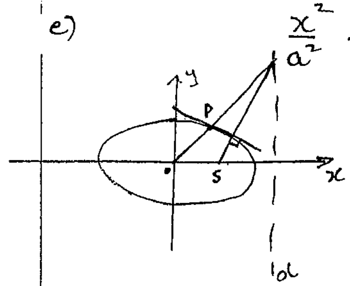
$x = -2,\ y = -(x^2 - 2x - 3)$
$= -(4+4-3)$
$= -5$

(ii) $mx = 4 - |x^2 - 2x - 3|$

Now, $y = mx$ passes through origin
m is the gradient
Thus for 4 distinct solutions

$$0 < m < \frac{4}{3}$$

e) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



Given the equation of tangent at P is

$$y = \frac{-b\cos\theta}{a\sin\theta}\left(x - \frac{a}{\cos\theta}\right)$$

then, gradient of line perp. to tangent at P and through S is: $\dfrac{a\sin\theta}{b\cos\theta}$

$\therefore$ Equation of this line is:

$$y - 0 = \frac{a\sin\theta}{b\cos\theta}(x - ae)$$

$$y = \frac{a\sin\theta}{b\cos\theta}(x - ae) \quad \text{——— (1)}$$

Equation of line OP is $y = mx$

ie $$y = \frac{b\sin\theta - 0}{a\cos\theta - 0}x$$

$$y = \frac{b\sin\theta}{a\cos\theta}x \quad \text{——— (2)}$$

Point of intersection of lines (1) + (2) is:

$$\frac{a\sin\theta}{b\cos\theta}(x - ae) = \frac{b\sin\theta}{a\cos\theta}x$$

$$\frac{a}{b}(x - ae) = \frac{b}{a}x$$

$\times ab$: $a^2(x - ae) = b^2 x$

$$a^2 x - b^2 x = a^3 e$$

$$x = \frac{a^3 e}{a^2 - b^2}$$

But $b^2 = a^2(1 - e^2)$

$$\therefore x = \frac{a^3 e}{a^2 - a^2(1-e^2)}$$

$$= \frac{ae}{1 - 1 + e^2}$$

$$x = \frac{a}{e}$$

ie. lines meet on the directrix $x = \frac{a}{e}$