St George Girls High School

Year 12

Common Test 3

May 2008

Mathematics Extension 2

Time Allowed: 75 Minutes

Instructions to Candidates

1. All 3 questions may be attempted.
2. All necessary working must be shown.
3. Start each question on a new page.

Question 1 - (20 marks) – Start a New Page **Marks**

a) Find these integrals: 8

(i) $\int \cot x \ \mathrm{cosec}^2 x \ dx$

(ii) $\int \frac{dx}{\sqrt{4x^2-9}}$

(iii) $\int \sec^6 x \ dx$

(iv) $\int \frac{4x-1}{x^2-4} \ dx$

b) (i) Evaluate $\int_0^1 u\, e^u \ du$ 4

and (ii) hence, using a suitable substitution, evaluate $\int_0^1 x^5 \ e^{x^3} \ dx$

c) Find the value of $\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\cos x}$, using the 't' method. 4

d) (i) Express $\frac{7t}{(3-t)(3t-2)}$ as the sum of its partial fractions. 4

and (ii) hence, find $\int \frac{7t}{-3t^2+11t-6} \ dt$

Question 2 – (20 marks) Polynomials – Start a New Page **Marks**

a) (i) If α, β and γ are the roots of the cubic equation $x^3 + mx + n = 0$, find in terms of m and n, the values of:

1. $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **3**

2. $\alpha^2 + \beta^2 + \gamma^2$ **2**

(ii) Determine the cubic equation whose roots are α^2, β^2 and γ^2 **3**

b) The polynomial $P(x)$ has equation $P(x) = x^4 - 2x^3 - x^2 + 2x + 10$ **4**

Given that $x - 2 + i$ is a factor of $P(x)$, express $P(x)$ as a product of two real quadratic functions.

c) Given that $Q(x) = x^4 - 5x^3 + 4x^2 + 3x + 9$ has a zero of multiplicity 2, solve the equation $Q(x) = 0$ over the complex field. **4**

~~P(x) > 0~~ or < 0

d) Prove that $2x + \frac{x^2}{2} - \frac{2x^3}{3} - \frac{x^4}{4} + e = 0$ has no real roots if $e < \frac{-19}{12}$ **4**

How many real roots are there if $e \leq \frac{-19}{12}$?

Question 3 – (20 marks) – Start a New Page **Marks**

a) Using suitable substitutions, or otherwise, find: **4**

(i) $\int \frac{e^{3x}}{\sqrt{4-e^{6x}}}\, dx$ (ii) $\int \sec^9 x \tan x \, dx$

b) (i) By considering the expansion of $(\cos\theta + i\sin\theta)^3$ and by using De Moivre's Theorem show that $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ and $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$ **2**

(ii) Show that $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$ **1**

(iii) Find the general solution of the equation $\tan 3\theta = \sqrt{3}$ **2**

(iv) Use the substitution $x = \tan\theta$ to express the equation $\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta} = \sqrt{3}$ as a polynomial in terms of x **1**

(v) Hence, show that $\tan\frac{\pi}{9} + \tan\frac{4\pi}{9} + \tan\frac{7\pi}{9} = 3\sqrt{3}$ **2**

(vi) Find the polynomial of least degree that has zeros **2**

$\left(\cot\frac{\pi}{9}\right)^2, \left(\cot\frac{4\pi}{9}\right)^2, \left(\cot\frac{7\pi}{9}\right)^2$

c) (i) Show that $\int \sec x \, dx = \ln(\sec x + \tan x) + c$ **4**

(ii) Use the method of integration by parts to find: $\int \sec^3 x \, dx$

d) Find $\int \sqrt{\frac{2+x}{3-x}}\, dx$ **2**

End of Paper

SOLUTIONS.

QUESTION 1

(a) (i) $\int \cot x \operatorname{cosec}^2 x \, dx$ $\quad u = \cot x$, $du = -\operatorname{cosec}^2 x \, dx$

$= \int u.(-1)\, du$

$= -\frac{u^2}{2} + c$

$= -\frac{1}{2}\cot^2 x + c$

(ii) $\int \frac{dx}{\sqrt{4x^2-9}}$

$= \int \frac{dx}{2\sqrt{x^2-\frac{9}{4}}}$

$= \frac{1}{2}\log\left|x + \sqrt{x^2-\frac{9}{4}}\right| + c$

(iii) $\int \sec^6 x \, dx$

$= \int \sec^2 x . \sec^4 x \, dx$

$= \int \sec^2 x . (\tan^2 x + 1)^2 \, dx$ $\quad$ let $u = \tan x$, $du = \sec^2 x \, dx$

$= \int (u^2+1)^2 . \, du$

$= \int (u^4 + 2u^2 + 1)\, du$

$= \frac{u^5}{5} + \frac{2u^3}{3} + u + c$

$= \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + c$

(iv) $\int \frac{4x-1}{(x+2)(x-2)}\, dx$

$= \int \left(\frac{\frac{9}{4}}{x+2} + \frac{\frac{7}{4}}{x-2}\right) dx$

$= \frac{9}{4}\ln|x+2| + \frac{7}{4}\ln|x-2| + c$

let $\frac{4x-1}{(x+2)(x-2)} = \frac{a}{x+2} + \frac{b}{x-2}$

$\Longrightarrow 4x-1 = a(x-2) + b(x+$

$x=2 \Rightarrow 7 = 4b$

$b = \frac{7}{4}$

$x=-2 \Rightarrow -9 = -4a$

$a = \frac{9}{4}$

(b) (i) $\int_0^1 \underbrace{u}_{p}\underbrace{e^u}_{\frac{dq}{du}}\, du$

$= pq - \int q \frac{dp}{du}\, du$

$= \left[ue^u\right]_0^1 - \int_0^1 e^u \, du$

$= \left[ue^u - e^u\right]_0^1$

$= (e-e) - (0-1)$

$= 1$

(ii) $\int_0^1 x^5 e^{x^3}\, dx$

$= \frac{1}{3}\int_0^1 3x^2 . x^3 . e^{x^3}\, dx$ $\quad$ let $u = x^3$, $du = 3x^2\, dx$

$= \frac{1}{3}\int_0^1 ue^u \, du$

$= \frac{1}{3} . 1$

$= \frac{1}{3}$

(c) $\int_0^{\frac{\pi}{2}} \frac{dx}{5 + 4\cos x}$ $\quad$ $t = \tan\frac{x}{2}$, $dt = \frac{1}{2}(1+t^2).\, dx$, $dx = \frac{2dt}{1+t^2}$

$= \int_0^1 \frac{\frac{2dt}{1+t^2}}{5 + 4.\left(\frac{1-t^2}{1+t^2}\right)}$

$= \int_0^1 \frac{2dt}{5(1+t^2) + 4 - 4t^2}$

$= \int_0^1 \frac{2dt}{9 + t^2}$

$= \frac{2}{3}\tan^{-1}\left(\frac{t}{3}\right)\Big]_0^1$

(d) (i) let $\frac{7t}{(3-t)(3t-2)} = \frac{a}{3-t} + \frac{b}{3t-2}$

$$\Rightarrow 7t = a(3t-2) + b(3-t)$$

$$t=3 \Rightarrow 21 = 7a$$
$$a = 3$$

constant term $\Rightarrow 0 = -2a + 3b$
$$= -6 + 3b$$
$$b = 2$$

$$\therefore \frac{7t}{(3-t)(3t-2)} = \frac{3}{3-t} + \frac{2}{3t-2}$$

(ii) $\int \frac{7t}{(3-t)(3t-2)}\,dt = \int \frac{3}{3-t}\,dt + \int \frac{2}{3t-2}\,dt$

$$= -3\ln|3-t| + \frac{2}{3}\ln|3t-2| + c$$

QUESTION 2:

(a) (i) 1. $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$$= \frac{m}{-n}$$
$$= -\frac{m}{n}$$

2. $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$
$$= 0 - 2(m)$$
$$= -2m$$

(ii) $y = x^2 \Rightarrow x = \pm\sqrt{y}$

$\therefore P(\sqrt{x}) = 0$ has roots $\alpha^2, \beta^2, \gamma^2$

ie $(\sqrt{x})^3 + m\sqrt{x} = -n$

$x\sqrt{x} + m\sqrt{x} = -n$

$\sqrt{x}(x+m) = -n$

$\Rightarrow x(x^2 + 2mx + m^2) = n^2$

ie $x^3 + 2mx^2 + m^2x - n^2 = 0$

(b) $P(x) = x^4 - 2x^3 - x^2 + 2x + 10$

$x = 2-i$ is a zero $\Rightarrow$ $x = 2+i$ is a zero as all co-efficients are real

$\therefore$ One factor is $x^2 - [(2-i)+(2+i)]x + (2-i)(2+i)$

ie $x^2 - 4x + 5$ ——— ①

Hence $P(x) = (x^2-4x+5)(x^2+kx+2)$ for some k

Equating co-efficients of x^3 LHS/RHS of ①

$\Rightarrow -2 = k - 4$

$\therefore k = 2$

$\therefore P(x) = (x^2 - 4x + 5)(x^2 + 2x + 2)$

(c) $Q(x) = x^4 - 5x^3 + 4x^2 + 3x + 9$ $\quad \alpha, \alpha, \beta, \gamma$

$Q'(x) = 4x^3 - 15x^2 + 8x + 3$ $\quad \alpha, \delta, \varepsilon$

$Q'(1) = 4 - 15 + 8 + 3$
$= 0 \Rightarrow x-1$ is a factor of $Q'(x)$

$\therefore Q'(x) = (x-1)(4x^2 + kx - 3)$

co-eff of $x \Rightarrow 8 = -3 - k$
$\therefore k = -11$

$\therefore Q'(x) = (x-1)(4x^2 - 11x - 3)$
$= (x-1)(x-3)(4x+1)$

now $Q(1) = 1 - 5 + 4 + 3 + 9 \neq 0$
$Q(3) = 81 - 135 + 36 + 9 + 9$
$= 0$

Hence $x=3$ is the root of multiplicity 2

Hence $Q(x) = (x-3)^2(x^2 + kx + 1)$

Co-eff of $x \Rightarrow 3 = -6 + 9k$
$k = 1$

$\therefore Q(x) = (x-3)^2(x^2 + x + 1)$

$$Q(x) = (x-3)^2\left[\left(x+\tfrac{1}{2}\right)^2 - \tfrac{3}{4}i^2\right]$$
$$= (x-3)^2\left(x+\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)\left(x+\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i\right)$$

Hence $Q(x) = 0$
$$\Rightarrow \quad x = 3, 3, -\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i, -\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i$$

(d) Consider $y = 2x + \frac{x^2}{2} - \frac{2x^3}{3} - \frac{x^4}{4} + e$ ——— ①

$$\frac{dy}{dx} = 2 + x - 2x^2 - x^3$$

stationary points at $\frac{dy}{dx} = 0$

$$\text{ie}\quad x^3 + 2x^2 - x - 2 = 0$$
$$x^2(x+2) - 1(x+2) = 0$$
$$(x+2)(x+1)(x-1) = 0$$
$$x = -2, -1, 1$$

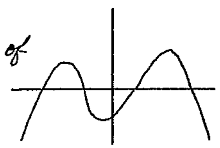
$$\frac{d^2y}{dx^2} = 1 - 4x - 3x^2$$

$x = -2 \Rightarrow \frac{d^2y}{dx^2} = 1 + 8 - 12$
$< 0 \Rightarrow$ rel. max at $x = -2$

$x = -1 \Rightarrow \frac{d^2y}{dx^2} = 1 + 4 - 3$
> 0

$x = 1 \Rightarrow \frac{d^2y}{dx^2} = 1 - 4 - 3$
$< 0 \Rightarrow$ rel. max. at $x = 1$

Since this graph has a basic shape of



we need only check the y-values of the relative maximum turning points

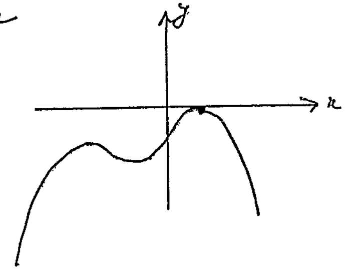
when $x = -2$, $y = -4 + 2 + \frac{16}{3} - 4 + e$
$$= -\tfrac{2}{3} + e$$

$x = 1$, $y = 2 + \frac{1}{2} - \frac{2}{3} - \frac{1}{4} + e$
$$= \tfrac{19}{12} + e$$

There are no real roots if $\frac{19}{12} + e < 0$

ie $e < -\frac{19}{12}$

If $e = -\frac{19}{12}$ graph has the shape



Hence there is a double root at $x = 1$ for $e = -\frac{19}{12}$ and no real roots for $e < -\frac{19}{12}$

QUESTION 7:

(a) (i) $\displaystyle\int \frac{e^{3x}}{\sqrt{4-e^{6x}}}\,dx$ let $u = e^{3x}$, $du = 3e^{3x}$

$$= \frac{1}{3}\int \frac{3e^{3x}}{\sqrt{4-e^{6x}}}\,dx$$
$$= \frac{1}{3}\int \frac{du}{\sqrt{4-u^2}}$$
$$= \frac{1}{3}\sin^{-1}\left(\frac{u}{2}\right) + c$$
$$= \frac{1}{3}\sin^{-1}\left(\frac{e^{3x}}{2}\right) + c$$

(ii) $\int \sec^9 x \tan x \, dx$

$= \int \sec^8 x . \sec x \tan x \, dx$ $\qquad u = \sec x$

$\qquad du = \sec x \tan x \, dx$

$= \int u^8 \, du$

$= \frac{u^9}{9} + c$

$= \frac{1}{9} \sec^9 x + c$

(b) (i) $(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$

$= \cos^3\theta + (3\cos^2\theta\sin\theta)i - 3\cos\theta\sin^2\theta - (\sin^3\theta)i$

$= (\cos^3\theta - 3\cos\theta . \sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$ —— (1)

$= \cos 3\theta + i\sin 3\theta$ —— (2)

by De-Moivres theorem

equating real parts of (1) and (2)

$\Rightarrow \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ —— (3)

equating imaginary parts of (1) and (2)

$\Rightarrow \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$ —— (4)

(ii) Hence $\tan 3\theta = \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta} \div \frac{\cos^3\theta}{\cos^3\theta}$

$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$ —— (5)

(iii) $\tan 3\theta = \sqrt{3}$

$\Rightarrow 3\theta = \frac{\pi}{3} + n\pi$ $\qquad n$ integer

$= \frac{\pi(1+3n)}{3}$

$\therefore \theta = \frac{\pi}{9}(3n+1)$

(iv) let $x = \tan\theta$ in (5)

$\therefore \frac{3x - x^3}{1 - 3x^2} = \sqrt{3}$

ie $3x - x^3 = \sqrt{3} - 3\sqrt{3}x^2$

ie $x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0$ —— (6)

(v) The solutions of $\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \sqrt{3}$

are $\theta = \frac{\pi}{9}(3n+1)$

$= \frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{10\pi}{9}, \frac{13\pi}{9}, \ldots$

Then the solutions of (6) are

$x = \tan\frac{\pi}{9}, \tan\frac{4\pi}{9}, \tan\frac{7\pi}{9}, \tan\frac{\pi}{9}, \tan\frac{4\pi}{9},$

ie $x = \tan\frac{\pi}{9}, \tan\frac{4\pi}{9}$ and $\tan\frac{7\pi}{9}$

Σ roots in (6) $\Rightarrow \tan\frac{\pi}{9} + \tan\frac{4\pi}{9} + \tan\frac{7\pi}{9} = 3\sqrt{3}$

(vi) $y = \frac{1}{x^2} \Rightarrow x = \frac{1}{\sqrt{y}}$

$P(x) = x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0$

Then $P(\frac{1}{\sqrt{x}}) = \frac{1}{x\sqrt{x}} - 3\sqrt{3} . \frac{1}{x} - \frac{3}{\sqrt{x}} + \sqrt{3} = 0$

ie $1 - 3\sqrt{3} . \sqrt{x} - 3x + \sqrt{3}x\sqrt{x} = 0$

$\sqrt{x}(\sqrt{3}x - 3\sqrt{3}) = 3x - 1$

i.e. $x(3x^2 - 18x + 27) = 9x^2 - 6x + 1$

i.e. $3x^3 - 27x^2 + 33x - 1 = 0$

(c) (i) $\int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \, dx$

$= \ln|\sec x + \tan x| + c$

(ii) $I = \int \sec^3 x \, dx$

$= \int \underbrace{\sec x}_{u} \cdot \underbrace{\sec^2 x \, dx}_{dV}$

$= \sec x \tan x - \int \sec x \tan x . \tan x \, dx$

$= \sec x \tan x - \int \sec x(\sec^2 x - 1) \, dx$

$= \sec x \tan x - I + \int \sec x \, dx$

$\therefore 2I = \sec x \tan x + \ln|\sec x + \tan x| + c$

$\therefore I = \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right] + c$

(d) $\int \sqrt{\frac{2+x}{3-x}} \, dx = \int \frac{\sqrt{2+x}}{\sqrt{3-x}} \cdot \frac{\sqrt{2+x}}{\sqrt{2+x}} \, dx$

$= \int \frac{2+x}{\sqrt{6+x-x^2}} \, dx$

$= -\frac{1}{2}\int \frac{1-2x}{\sqrt{6+x-x^2}} \, dx + \frac{5}{2}\int \frac{dx}{\sqrt{\frac{25}{4} - (x-\frac{1}{2})^2}}$

$= -\frac{1}{2}\int \frac{du}{\sqrt{u}} + \frac{5}{2} \sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{5}{2}}\right) + c$

$= -\frac{1}{2} . 2\sqrt{u} + \frac{5}{2} \sin^{-1}\left(\frac{2x-1}{5}\right) + c$

$= -\sqrt{6+x-x^2} + \frac{5}{2} \sin^{-1}\left(\frac{2x-1}{5}\right) + c$