St George Girls High School

Year 12

Mid-HSC Course Examination

2005

Mathematics Extension 2

General Instructions

- Start each question on a new page.
- Attempt questions 1, 2 and 3.
- All questions are of equal value.
- Marks for each question are shown.
- All necessary working should be shown in every question.
- Board-approved calculators may be used.
- A table of standard integrals is provided.

Total marks – 75

- Working time: 1½ hours
- Reading time: 5 minutes
- Write using blue or black pen.
- Do curve sketching questions on the number planes provided.

Question 1 – **25 marks** – (Start a new page) **Marks**

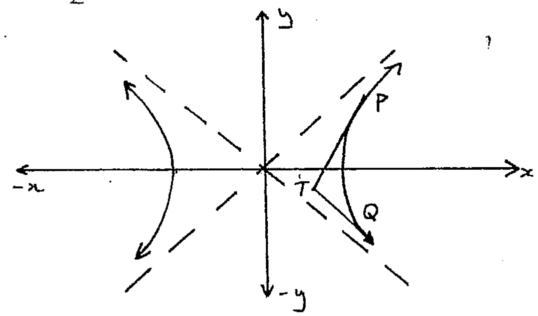
a) The point $P(a\cos\theta,\ b\sin\theta)$ lies on the ellipse $E: \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

(i) Show that the equation of the normal to E at P is $\dfrac{ax}{\cos\theta}-\dfrac{by}{\sin\theta}=a^2-b^2$ 3

(ii) This normal cuts the x-axis at M and the y-axis at N. Find M and N. 2

(iii) Prove that $\dfrac{PM}{PN}=\dfrac{b^2}{a^2}$ 3

b) The points $P(x_1,\ y_1)$ and $Q(x_2,\ y_2)$ lie on the hyperbola $H: \dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$. The point $T(x_0,\ y_0)$ is placed such that tangents can be drawn from T to H with points of contact P and Q as shown.



(i) Prove that the tangents to H at P and Q are 4

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1 \quad \text{and} \quad \frac{xx_2}{a^2}-\frac{yy_2}{b^2}=1 \quad \text{respectively}$$

(ii) Hence show that the equation of the chord PQ is:

$$\frac{xx_0}{a^2}-\frac{yy_0}{b^2}=1$$ 3

(iii) Hence show that if PQ is a focal chord then T must lie on a directrix. 2

Question 1 (cont'd)

Marks

c) The distinct points $P\left(3p, \dfrac{3}{p}\right)$ and $Q\left(3q, \dfrac{3}{q}\right)$ (with both $p > 0$ and $q > 0$) lie on the rectangular hyperbola R: $\left.\begin{array}{l} x = 3t \\ y = \frac{3}{t} \end{array}\right\}$

(i) Show that the gradient of PQ is $-\dfrac{1}{pq}$ 1

(ii) Show that the equation of the chord PQ is $x + pqy = 3(p+q)$ 2

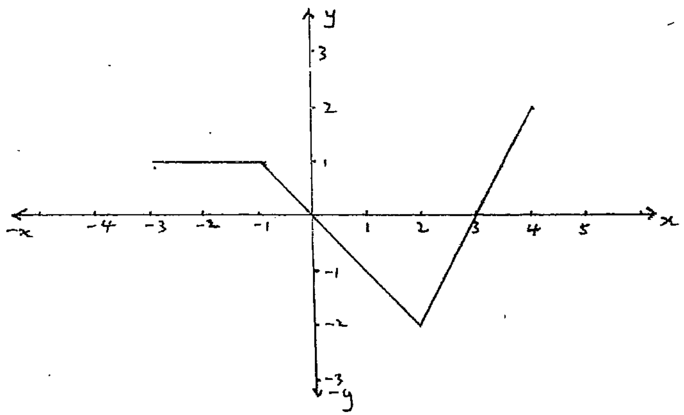
(iii) If PQ always passes through $(0, 4)$ show that the mid-point M of PQ always lies on the line $y = 2$ 3

(iv) Is it possible for M to lie anywhere on the line $y = 2$? Your answer should include any further appropriate conditions. 2

<u>Question 2</u> – **(25 marks)** – (Start a new page) **Marks**

Do all sketches on the number planes provided with the examination paper.

a)



The diagram is a sketch of the function $y = f(x)$, $-3 \le x \le 4$. 10
On separate diagrams (see number planes provided), draw neat sketches of:

(i) $y = |f(x)|$

(ii) $y = -f(x)$

(iii) $y = f(-x)$

(iv) $y = f(|x|)$

(v) $y = \sqrt{f(x)}$

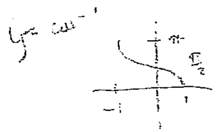
(vi) $y = \dfrac{1}{f(x)}$

(vii) $y = [f(x)]^2$

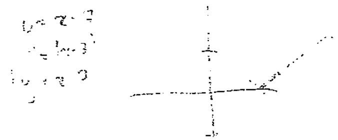
Question 2 (b) and (c) on next page

Question 2 (continued) **Marks**

b) (i) Sketch the curve $y = \cos^{-1}(2x)$ 3



(ii) Hence, sketch $|y| = \cos^{-1}(2x)$ 2



c) Consider the function: $y = \dfrac{1+x^2}{1-x^2}$

(i) What are the equations of all its asymptotes? Give full reasons. 3

(ii) Without using calculus, sketch $y = \dfrac{1+x^2}{1-x^2}$. Show all working. 3

(iii) Hence, sketch:

(α) $y = \sqrt{\dfrac{1+x^2}{1-x^2}}$ For (α), show the relationship of the curve to the original function. 2

(β) $y^2 = \dfrac{1+x^2}{1-x^2}$ 2

Question 3 – (25 marks) – (Start a new page) **Marks**

a) The ellipse E: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ has foci $\left(\pm\sqrt{7}, 0\right)$ and directrices $x = \pm\dfrac{16}{\sqrt{7}}$

(i) Show that the eccentricity e has the value $\dfrac{\sqrt{7}}{4}$ 3

(ii) Show that $a = 4$ and $b = 3$ 2

(iii) Sketch E clearly showing all essential features. 2

b) The point $P(x, y)$ lies on the hyperbola $H: \dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

H has foci S and S'.

(i) Show that $|PS - PS'| = 6$ 1

(ii) By considering the behaviour of H for $|x|$ very large, find the asymptotes of H. 2

(iii) Find the eccentricity e 2

Question 3 c), d) and e) on next page

Question 3 (continued) **Marks**

As with Question 2, for Parts c), d) and e) of Question 3, use the number planes provided with the examination paper.

c) (i) Is the function $y = x^2 \sin x$ odd, even or neither? Give reasons. 1

(ii) Sketch the curve $y = x^2 \sin x$ for $-2\pi \le x \le 2\pi$, showing all important features. 2

d) (i) Sketch the curve $y = |\tan x|$ for $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$ 2

(ii) (α) Is the curve differentiable at the origin? 1

(β) Explain, using calculus, what type of point the origin is. 2

e) On separate number planes, sketch

(i) $u = \sin x$ $(0 \le x \le 2\pi)$ 1

(ii) $y = \ln u$ 1

Hence, sketch

(iii) $y = \ln(\sin x)$ $(0 \le x \le 2\pi)$ 3

End of Paper

Year 12
Mid H.S.C. Ext. 2
2005
SOLUTIONS

Q1

a) $x = a\cos\theta \rightarrow \frac{dx}{d\theta} = -a\sin\theta$

$y = b\sin\theta$

$\frac{dy}{d\theta} = b\cos\theta$

$\therefore \frac{dy}{dx} = -\frac{b\cos\theta}{a\sin\theta}$

$\therefore$ slope of normal $= \frac{a\sin\theta}{b\cos\theta}$

$\therefore$ eqn of normal is

$y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$

$\left(\times \frac{b}{\sin\theta}\right)$

ie $\frac{by}{\sin\theta} - b^2 = \frac{ax}{\cos\theta} - a^2$

$\therefore \frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$ (3)

(ii) at M, $y = 0$

$\therefore \frac{ax}{\cos\theta} = a^2 - b^2$

$x = \frac{(a^2-b^2)\cos\theta}{a}$

$\therefore$ M is $\left(\frac{(a^2-b^2)\cos\theta}{a}, 0\right)$ (2)

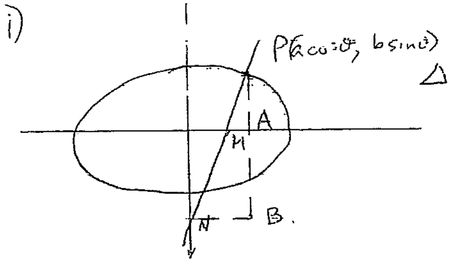
at N, $x = 0$

$\therefore -\frac{by}{\sin\theta} = a^2 - b^2$

$y = \frac{(b^2-a^2)\sin\theta}{b}$

N: $(0, (b^2-a^2) \ldots)$

(iii)



$\Delta PMA \parallel\!\mid \Delta PNB$ (equiangular)

$\therefore \frac{PM}{PN} = \frac{PA}{PB}$

$= \frac{b\sin\theta}{b\sin\theta + \frac{(a^2-b^2)\sin\theta}{b}}$

$= \frac{b}{b + \frac{a^2}{b} - b}$ (3)

$= \frac{b^2}{a^2}$

b) (i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore \frac{d}{dx}\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = 0$

ie $\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0$

$\therefore y' = \frac{b^2}{a^2}\frac{x}{y}$

$\therefore$ at P, $y' = \frac{b^2x_1}{a^2y_1}$

$\therefore$ eqn of tangent at P:

$y - y_1 = \frac{b^2x_1}{a^2y_1}(x - x_1)$

$\left(\times \frac{y_1}{b^2}\right)$

$\frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = \frac{xx_1}{a^2} - \frac{x_1^2}{a^2}$

ie $\frac{xx_1}{\ldots} - \ldots = \frac{x_1^2}{\ldots} - \frac{y_1^2}{\ldots} = 1$

(since (x_1, y_1) lies on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$)

Similarly, tangent at Q is (4)

$$\frac{x x_2}{a^2} - \frac{y y_2}{b^2} = 1.$$

(ii) T lies on tangent at P & at Q

$$\therefore \frac{x_0 x_1}{a^2} - \frac{y_0 y_1}{b^2} = 1 \quad ①$$

and

$$\frac{x_0 x_2}{a^2} - \frac{y_0 y_2}{b^2} = 1 \quad ②$$

By referring to ①, ② we deduce that $P(x_1, y_1)$ and $Q(x_2, y_2)$ both lie on

$$\frac{x x_0}{a^2} - \frac{y y_0}{b^2} = 1 \quad ③ \qquad (3)$$

ie ③ is eqn of PQ (③ is linear)

(iii) $(ae, 0)$ lies on PQ

$$\therefore ③ \Rightarrow \frac{ae\, x_0}{a^2} = 1. \qquad (2)$$

ie $x_0 = \frac{a}{e}$.

ie T must lie on directrix $x = \frac{a}{e}$.

c) (i) gradient $PQ = \dfrac{\frac{3}{q} - \frac{3}{p}}{3q - 3p} \times \dfrac{pq}{pq}$

$$= \frac{3p - 3q}{pq(3q - 3p)}$$

$$= -\frac{1}{pq} \qquad (1)$$

(ii) Eqn of chord PQ is

$$y - \frac{3}{p} = -\frac{1}{pq}(x - 3p)$$

ie $pqy - 3q = -x + 3p$ (2)

ie $x + pqy = 3(p + q)$ ✝

(iii) $(0, 4)$ lies on ✝

$$\therefore 4pq = 3(p + q) \quad ⊕$$

M is $\left(\frac{3p + 3q}{2}, \frac{\frac{3}{p} + \frac{3}{q}}{2}\right)$

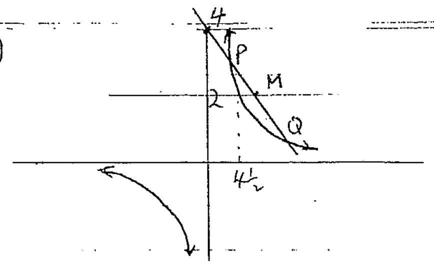
$$= \left(\frac{3}{2}(p+q), \frac{3q + 3p}{2pq}\right) \qquad (3)$$

$$= \left(\frac{3}{2}(p+q), \frac{4pq}{2pq}\right) \quad \text{from } ⊕$$

$$= \left(\tfrac{3}{2}(p+q), 2\right)$$

ie midpoint M of PQ always lies

(iv)



Eqn of hyperbola is $xy = 9$.

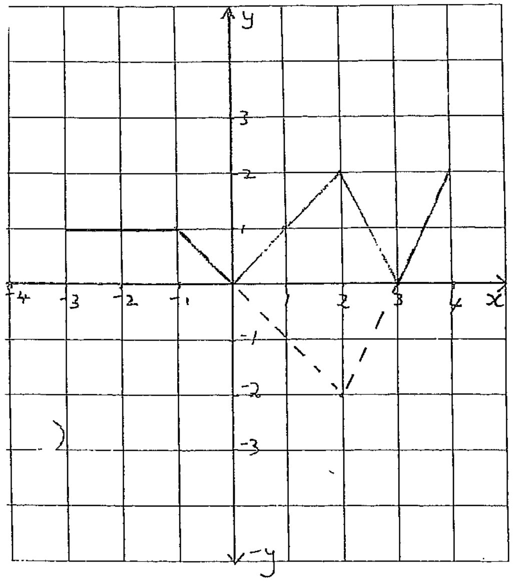
when $y = 2$, $x = 4\frac{1}{2}$

$\therefore$ M lies on $y = 2$, $x \geq 4\frac{1}{2}$ (see diagram)

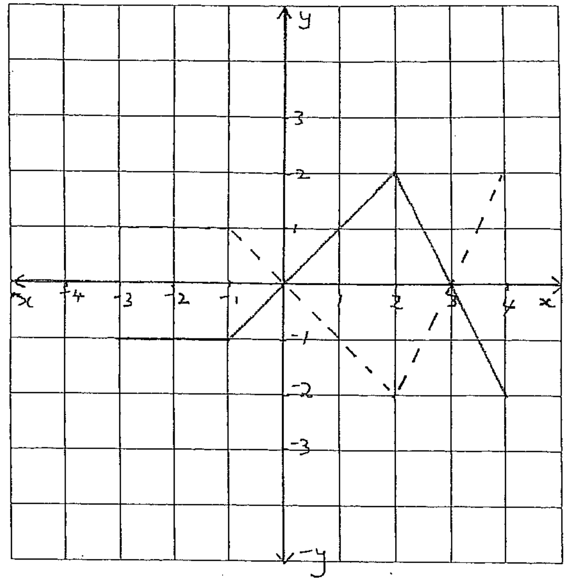
(2)

2 a)

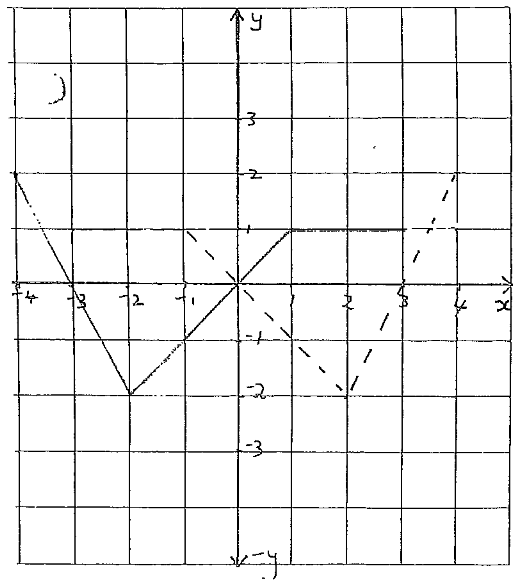
i) $y = |f(x)|$



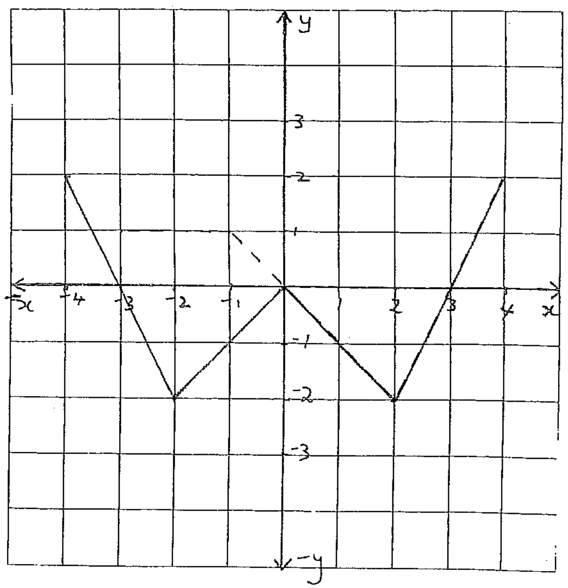
(ii) $y = -f(x)$



iii) $y = f(-x)$

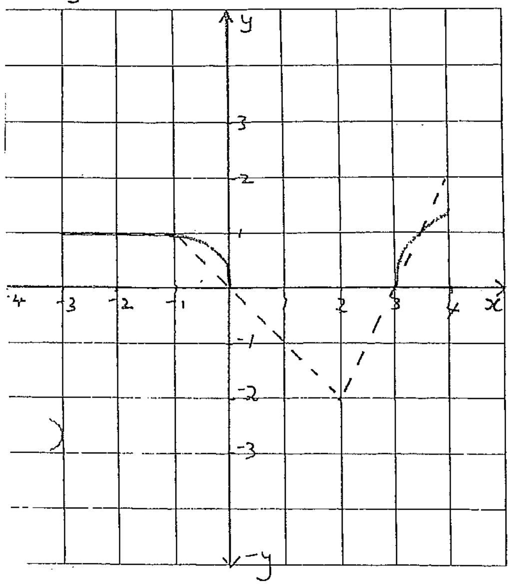


(iv) $y = f(|x|)$

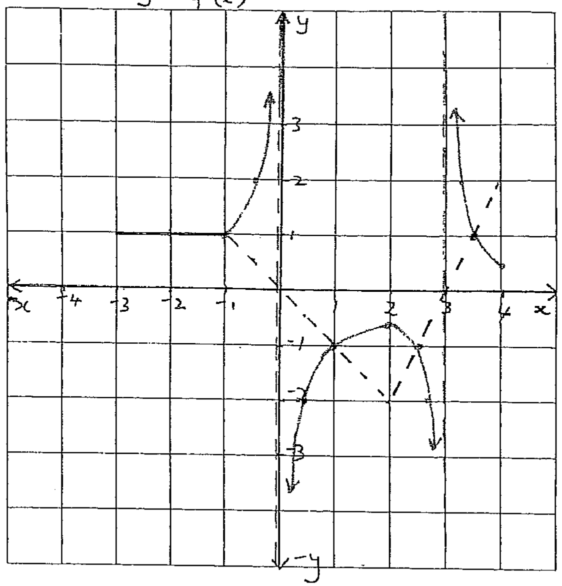


Q2 a)

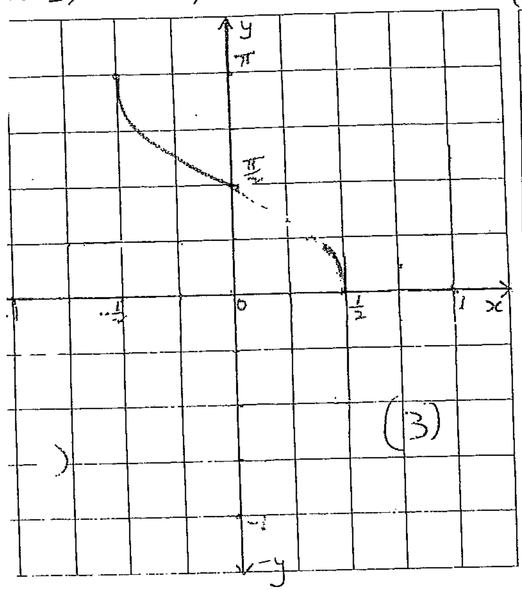
) $y = \sqrt{f(x)}$



(vi) $y = \frac{1}{f(x)}$

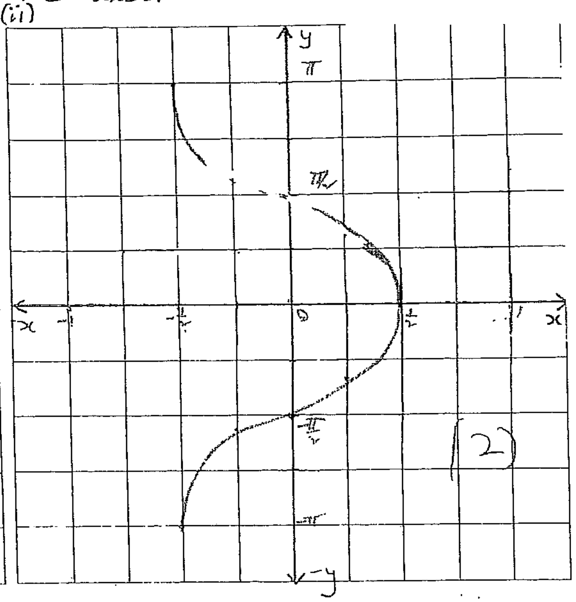


2 b) carefully mark scale used on the axes.



$y = \cos^{-1}(2x)$

(ii)

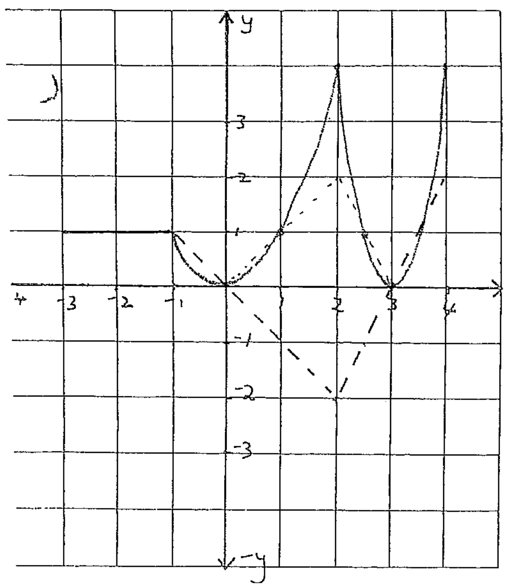


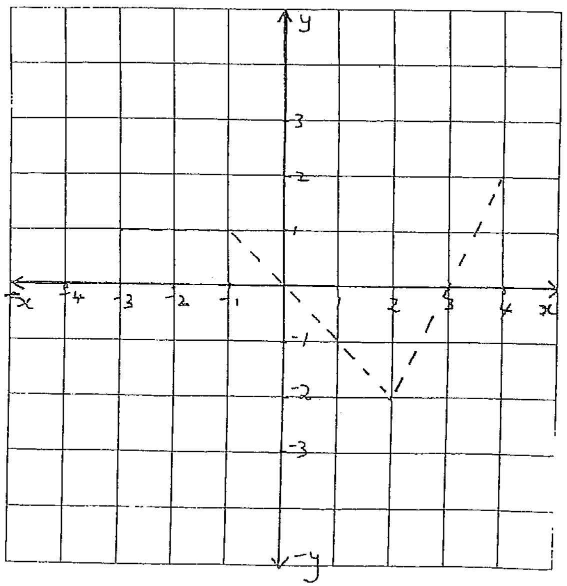
$|y| = \cos^{-1}(2x)$

Q2 c)
(i)

SPARES
Please label if being used.

i) $y = [f(x)]^2$





UESTION 2:

(c) $y = \dfrac{1+x^2}{1-x^2}$

(i) vertical asymptotes: $x=-1,\ x=1$

horizontal asymptotes:

$$\lim_{x\to\infty} \frac{1+x^2}{1-x^2}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x^2}+1}{\frac{1}{x^2}-1}$$

$$= -1$$

(3)

$\therefore y=-1$ is asymptote.

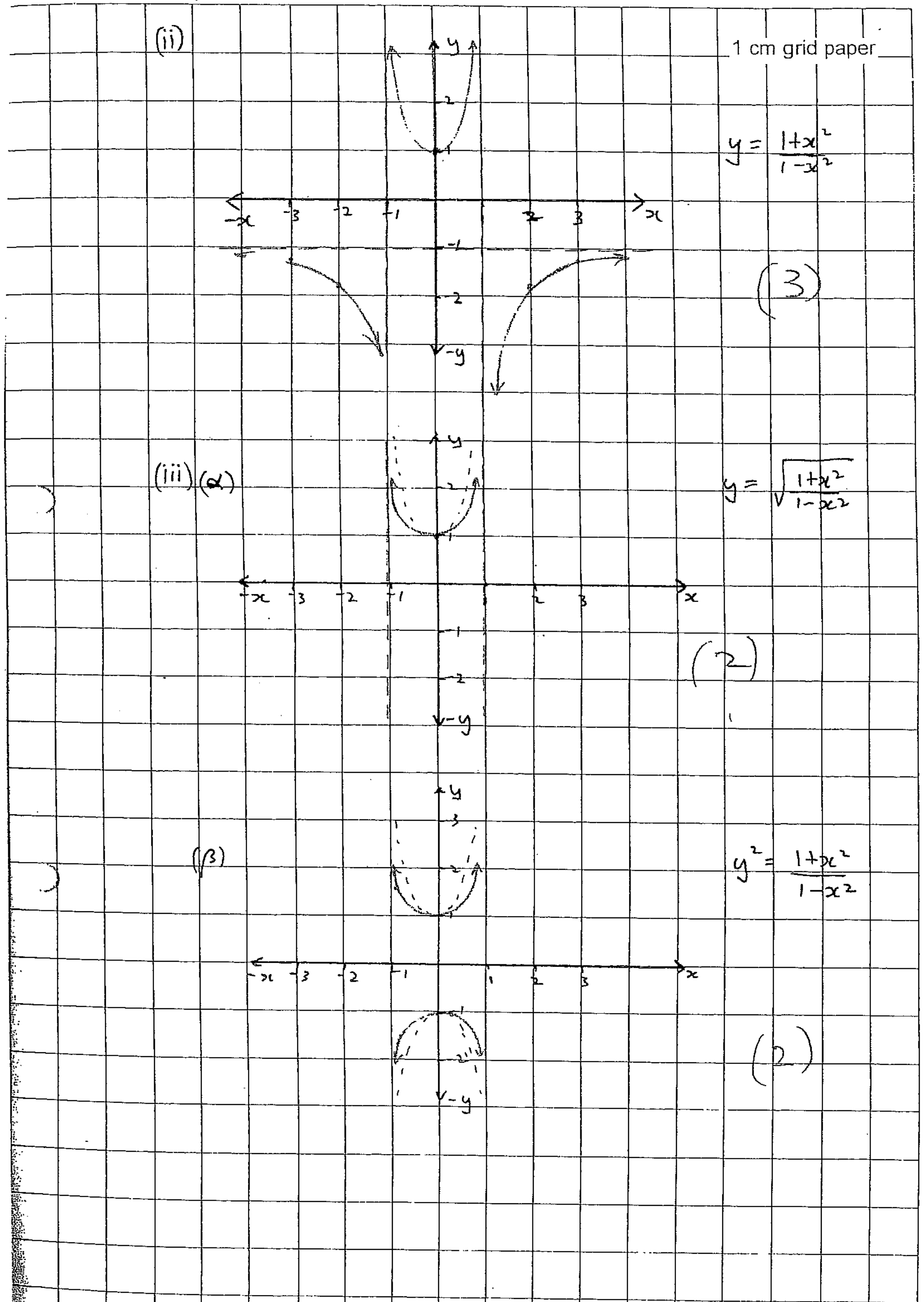
(ii) see graph

- as $x\to 1^+$, $\dfrac{1+x^2}{1-x^2} \to -\infty$
- as $x\to 1^-$, $\dfrac{1+x^2}{1-x^2} \to \infty$
- as $x\to -1^+$, $\dfrac{1+x^2}{1-x^2} \to \infty$
- as $x\to -1^-$, $\dfrac{1+x^2}{1-x^2} \to -\infty$

(iii) (α) see graph

(β) see graph

Q2 c)



x3

a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ foci $(\pm\sqrt{7}, 0)$

directrices $x = \pm \frac{16}{\sqrt{7}}$

$\therefore ae = \sqrt{7}$ ①

$\frac{a}{e} = \frac{16}{\sqrt{7}}$ ②

$\therefore a = \frac{16e}{\sqrt{7}}$

sub into ① $\Rightarrow$ $\frac{16e^2}{\sqrt{7}} = \sqrt{7}$

$\therefore e^2 = \frac{7}{16}$

$e = \frac{\sqrt{7}}{4}$ (3)

(ii) sub $e = \frac{\sqrt{7}}{4}$ into ① $\Rightarrow$ $\therefore a = 4$

$b^2 = a^2(1-e^2)$ for an ellipse

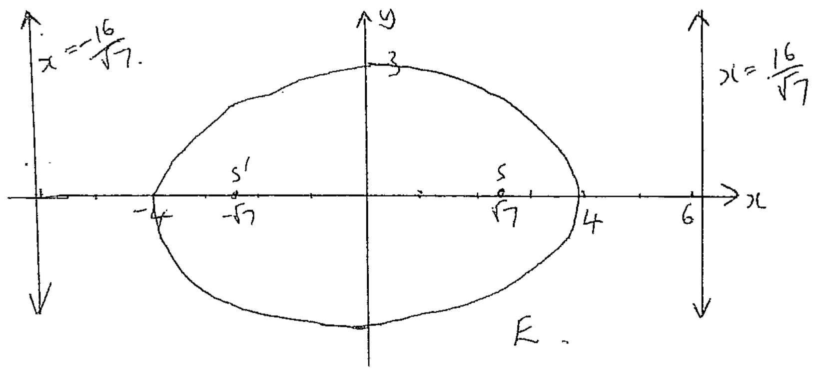
$\therefore b^2 = 16(1-e^2)$

$= 16(1 - \frac{7}{16})$ (2)

$= 9$

$\therefore b = 3$.

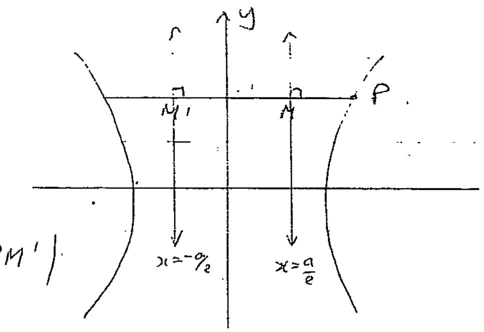
(iii)



(2)

b)(i) $\frac{x^2}{9} - \frac{y^2}{16} = 1$

$\therefore a = 3$, $b = 4$.



$PS = e\,PM$

$PS' = e\,PM'$

$\therefore |PS - PS'| = e|PM - PM'|$

$= e\left|\frac{2a}{e}\right|$

$= 2a$ (1)

$= 6$.

(ii) $\frac{x^2}{9} - \frac{y^2}{16} = 1$

$\therefore \frac{y^2}{16} = \frac{x^2}{9} - 1$

$y^2 = 16\left(\frac{x^2}{9} - 1\right)$

$y = \pm\sqrt{\frac{16x^2}{9} - 16} = \pm\sqrt{\frac{16x^2}{9}\left(1 - \frac{9}{x^2}\right)}$

$= \pm\frac{4x}{3}\sqrt{1 - \frac{9}{x^2}}$ (2)

as $x \to \pm\infty$, $y \to \pm\frac{4x}{3}$

$\therefore$ asymptotes are $y = \pm\frac{4x}{3}$.

(iii) $b^2 = a^2(e^2 - 1)$

$\therefore 16 = 9(e^2 - 1)$ (2)

$e^2 = \frac{25}{9}$

$e = 5/3$

UESTION 3

(c) (i) $f(x) = x^2 \sin x$

$f(-x) = (-x)^2 \sin(-x)$

$= -x^2 \sin x$ (1)

$= -f(x)$

$\therefore f(x)$ is an odd function.

(ii) see graph

(d) (i) see graph

(ii) (α) NO (1)

(β) For $x > 0$, $|\tan x| = \tan x$

If $f(x) = |\tan x| = \tan x$ for $x > 0$

$f'(x) = \sec^2 x$

as $x \to 0^+$, $f'(x) \to 1$ ——— (1)

For $x < 0$, $|\tan x| = -\tan x$

$\therefore f(x) = |\tan x|$

$= -\tan x$ for $x < 0$ (2)

$f'(x) = -\sec^2 x$

as $x \to 0^-$, $f'(x) \to -1$ ——— (2)

Since the two limits are not equal, $f'(x)$ is not differentiable at $x = 0$ and hence the origin is a cuxp.

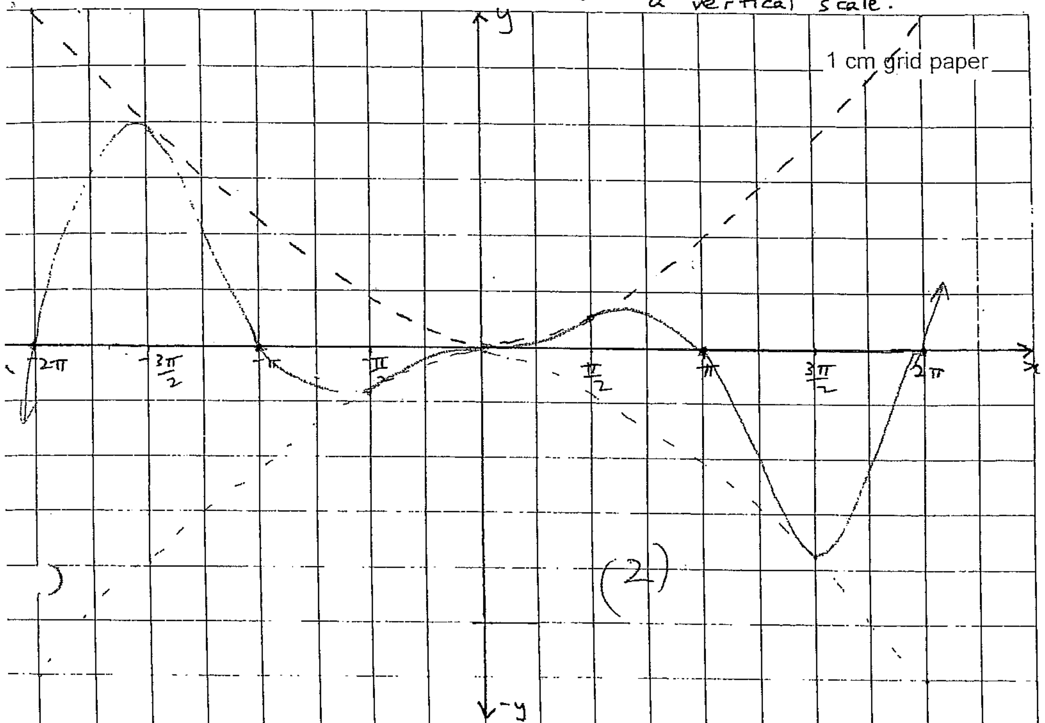
(e) (i) see graph

(ii) see graph

(iii) see graph

Q3 c) (ii) The dotted curve is $y=x$. There is no need to show a vertical scale.

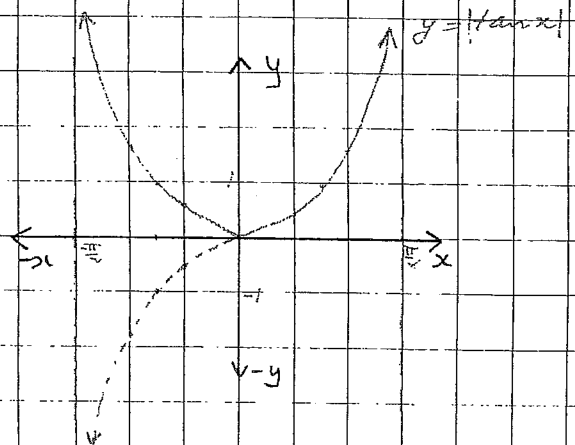
1 cm grid paper



(2)

WORKING SPACE FOR Q3 c)(ii)

Q3 d) (i)

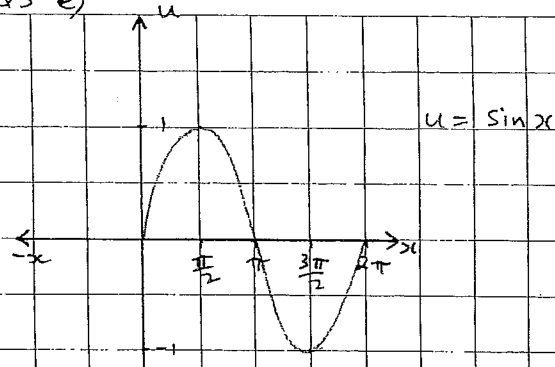


(2)

Q3 e)

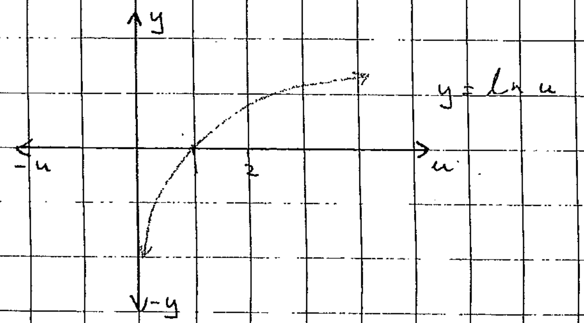
1 cm grid paper

(i)



(1)

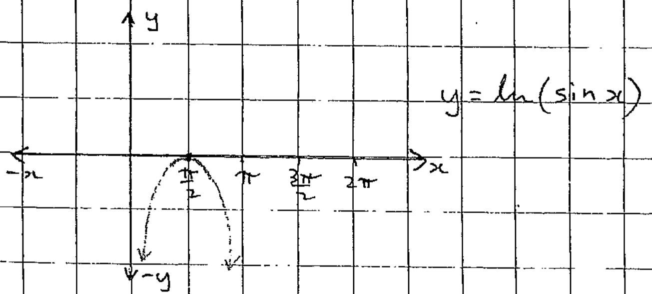
(ii)



note no need to insert vertical scales in parts (ii), (iii)

(1)

(iii)



(3)