St George Girls High School

Year 12

Mid-HSC Course Examination

2008

Mathematics Extension 2

General Instructions

- Working time: 1½ hours
- Reading time: 5 minutes
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.

Total marks – 75

- Attempt Questions 1 – 3
- All questions are of equal value
- Begin each question in a new booklet
- All necessary working should be shown in every question.

Question 1 – (**25 marks**) – (Start a new booklet) **Marks**

a) Consider the Hyperbola $H: \frac{x^2}{4} - \frac{y^2}{9} = 1$

(i) Find the x-intercepts 1

(ii) Find the eccentricity e 1

(iii) Find the foci 1

(iv) Find the directrices 1

(v) Find the asymptotes 1

(vi) Sketch H clearly showing all of the above features 2

b) Consider the ellipse $E: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(i) Prove that the equation of the tangent to E at $P(x_1, y_1)$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ 3

(ii) The tangent to E at the point P cuts a directrix at the point T. Prove that the line segment PT subtends an angle of 90 degrees at the corresponding focus. 4

(iii) Using the basic definition of an ellipse prove that $PS + PS^1 = 2a$ where S, S^1 are the two foci of E. 2

c) It can be shown that the line $y = mx + k$ is tangential to the hyperbola $H: \frac{x^2}{4} - \frac{y^2}{15} = 1$ if $4m^2 - 15 = k^2$

(i) Find the equation of the line through $(-1, 3)$ with gradient m 1

(ii) Hence find the equations of the two tangents to H from $(-1, 3)$ 4

d) The equation of the chord of contact from an external point $P(x_0, y_0)$ to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$.

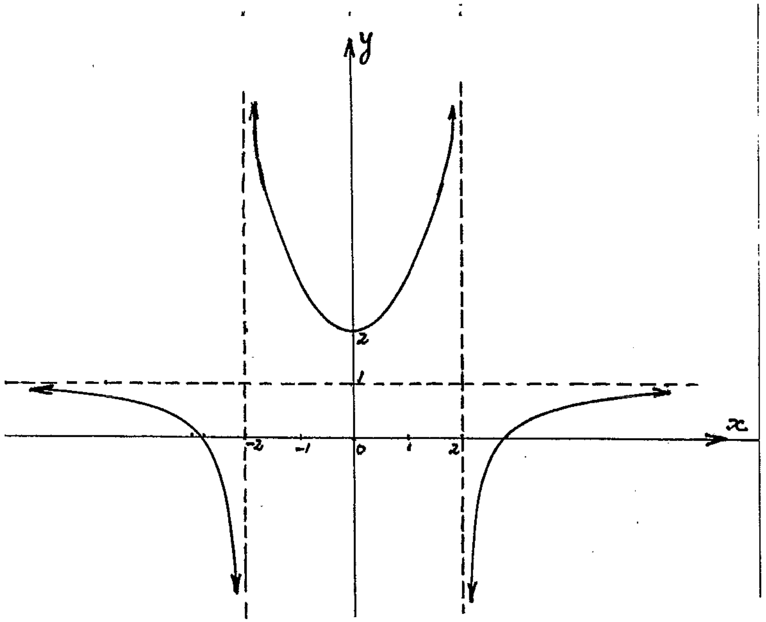
(i) State the condition which x_0 and y_0 must satisfy if P is external to the ellipse. 1

(ii) Find the condition which c must satisfy if $x + y = c$ is a chord of contact for the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$. 3

Question 2 – (25 marks) – (Start a new booklet) **Marks**

a) The graph of $y = f(x)$ is shown below.

On the separate grids provided, sketch



(i) $y = \frac{1}{f(x)}$ 2

(ii) $y = \sqrt{f(x)}$ 2

(iii) $y = [f(x)]^2$ 2

(iv) $y = \log[f(x)]$ 2

(v) $y = x + f(x)$ 2

(vi) $y = f^1(x)$ 2

Question 2 (cont'd) **Marks**

b) We aim to sketch the function $\sqrt{x} + \sqrt{y} = 1$

(i) Determine the x and y intercepts made by this function 1

(ii) Determine the domain and range of this function 2

(iii) Show that this function is decreasing for all x in the domain 2

(iv) Determine any stationary points and critical points 3

(v) Find $\frac{d^2y}{dx^2}$ and comment on its significance in the domain 3

(vi) Sketch the function 2

Question 3 – (25 marks) – (Start a new booklet) **Marks**

a) Consider the function $f(x) = \dfrac{e^x - 1}{e^x + 1}$

(i) Find the x intercept 1

(ii) Determine the behaviour of $f(x)$ as $|x|$ becomes very large (ie x approaches positive or negative infinity) 2

(iii) Show that $f(x)$ is an increasing function 2

(iv) Sketch the function $y = f(x)$ 2

(v) Discuss the number of solutions of $f(x) = mx$ for varying values of m. 3

b) Consider the rectangular hyperbola : $xy = 4$. $P(2p, \dfrac{2}{p})$ and $Q(2q, \dfrac{2}{q})$ are variable points on R with $p > 0, q > 0$. M is the mid-point of the chord PQ.

(i) State the foci of R 1

(ii) State the equations of the two directrices of R 1

(iii) Find the gradient of the chord PQ 2

(iv) Find the equation of the chord PQ 2

(v) If P and Q vary such that the line PQ always passes through the point $(4, 0)$. find the locus of M in the form $x = a$ 3

(vi) Find the range of this locus 1

c) The point $P(a\sec\theta, b\tan\theta)$ lies on H: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. The normal at P cuts the x axis at A and the y axis at B.

(i) Prove that the normal to H at the point P has equation $\dfrac{ax}{\sec\theta} + \dfrac{by}{\tan\theta} = a^2 + b^2$ 3

(ii) Prove that $PA:PB = b^2:a^2$ 2

End of Paper

Mid-Course Exam 2008

Extension 2 Solutions

Q1

a)(i) $\frac{x^2}{4} - \frac{y^2}{9} = 1$

$y = 0 \Rightarrow x = \pm 2$. ie $(2,0)$ $(-2,0)$ are x-intercepts (1)

(ii) $b^2 = a^2(e^2 - 1)$ $a = 2$, $b = 3$.

$\therefore 9 = 4(e^2 - 1)$

$\therefore e^2 - 1 = 9/4$ (1)

$\therefore e^2 = \frac{13}{4}$

$\therefore e = \frac{\sqrt{13}}{2}$

(iii) Foci are $(\pm ae, 0)$ (1)

ie $(\pm\sqrt{13}, 0)$

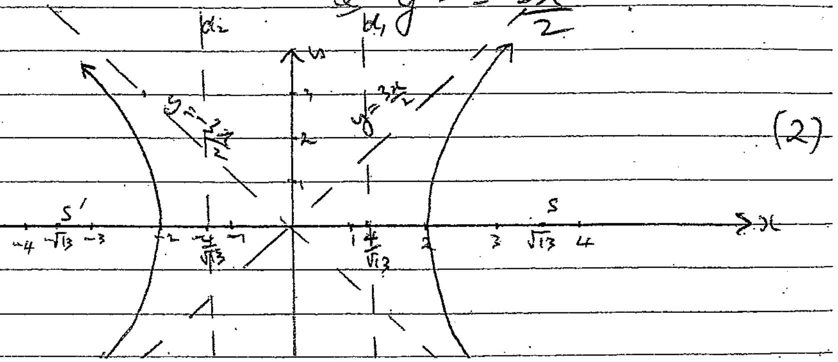
(iv) Directrices are $x = \pm \frac{a}{e}$ (1)

ie $x = \pm \frac{4}{\sqrt{13}}$

(v) Asymptotes are $y = \pm \frac{bx}{a}$ (1)

ie $y = \pm \frac{3x}{2}$

(vi) (2)



b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(i) $\frac{d}{dx}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = 0$

$\therefore \frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$

$2yy' = -\frac{2x}{a^2} \times b^2$

$\therefore y' = -\frac{b^2}{a^2}\frac{x}{y}$

$\therefore$ at $P(x_1, y_1)$, $y' = -\frac{b^2}{a^2}\frac{x_1}{y_1}$

$\therefore$ eq'n of tangent is

$y - y_1 = \frac{-b^2}{a^2}\frac{x_1}{y_1}(x - x_1)$

$\times$ both sides by $\frac{y_1}{b^2}$

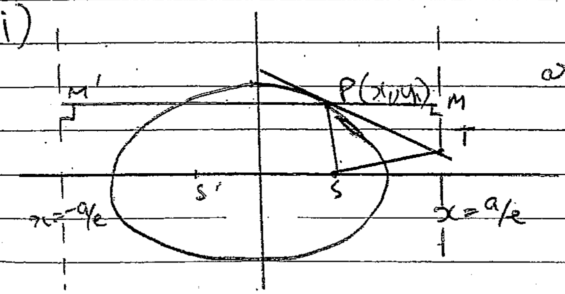
$\Rightarrow \frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = -\frac{x_1 x}{a^2} + \frac{x_1^2}{a^2}$

(3)

$\therefore \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$

$= 1$ since (x_1, y_1) lies on ellip

(ii)



at T, $x = a/e$

$\therefore \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ becom

$\frac{a}{ea^2}x_1 + \frac{yy_1}{b^2} = 1$

ie $\frac{x_1}{ae} + \frac{yy_1}{b^2} = 1$

$\therefore \frac{yy_1}{b^2} = 1 - \frac{x_1}{ae}$

$\therefore y = \frac{b^2}{y_1}\left(1 - \frac{x_1}{ae}\right)$

$\therefore T\left(a/e, \frac{b^2}{y_1}\left(1-\frac{x_1}{ae}\right)\right)$

Now, slope PS $= \frac{y_1}{x_1 - ae} = m_1$

slope TS $= \frac{\frac{b^2}{y_1}\left(1-\frac{x_1}{ae}\right)}{a/e - ae} = m_2$

$\therefore m_1 m_2 = \frac{y_1}{x_1 - ae} \times \frac{b^2\left(1-\frac{x_1}{ae}\right)}{y_1(a/e - ae)} \times \frac{ae}{ae}$

$= \frac{\frac{\cancel{y_1}}{x_1 - ae} \times b^2(ae - x_1)}{\cancel{y_1}(a^2 - a^2e^2)}$

$= \frac{-b^2}{a^2(1-e^2)}$

(3)

$= -1$ since $b^2 = a^2(1-e^2)$ for ellipse.

$\therefore PS \perp TS$

ie PT subtends $90°$ at focus.

(iii) $PS = e\,PM$, M is foot of perpendicular to corresponding directrix.

and, similarly, $PS' = e\,PM'$

(3) $\therefore PS + PS' = e(PM + PM')$

$= e(MM')$

$= e \times 2\frac{a}{e}$

$= 2a.$

c) (i) $y - 3 = m(x+1)$ (1)

ie $y = mx + m + 3$ * ie $K = m+3$.

(ii) For tangents to exist,

$4m^2 - 15 = K^2$

$= (m+3)^2.$

$\therefore 4m^2 - 15 = m^2 + 6m + 9.$

$\therefore 3m^2 - 6m - 24 = 0$

$m^2 - 2m - 8 = 0.$

$(m-4)(m+2) = 0$ (4)

$\therefore m = -2, 4.$

$\therefore$ tangents are

$y = -2x + 1$, $y = 4x + 7$ using *

d) (i) $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} > 1.$ (1)

(ii) $x + y = c$ corresponds to $\frac{xx_0}{9} + \frac{yy_0}{4}$

ie $\frac{x}{c} + \frac{y}{c} = 1$

$\Rightarrow \frac{x_0}{9} = \frac{1}{c}$ $\frac{y_0}{4} = \frac{1}{c}.$

$\therefore x_0 = 9/c$, $y_0 = 4/c.$

Now, we require $\frac{x_0^2}{9} + \frac{y_0^2}{4} > 1$ from (i)

$\therefore \frac{\left(\frac{9}{c}\right)^2}{9} + \frac{\left(\frac{4}{c}\right)^2}{4} > 1$

$\therefore 9 + 4 > 1$

$\frac{13}{c^2} > 1$

$\therefore \quad c^2 < 13$ (3)

$\therefore \quad -\sqrt{13} < c < \sqrt{13}$

Q2 a) see separate sheets.

Q2 b) (i) x intercept. $(1,0)$ (1)

y " $(0,1)$

(ii) $D: \ 0 \le x \le 1$ (2)

$R: \ 0 \le y \le 1$

(iii) $\sqrt{x} + \sqrt{y} = 1$

$\therefore \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}} \cdot y' = 0$ (2)

$\therefore y' = \frac{-x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}} = -\frac{\sqrt{y}}{\sqrt{x}} < 0 \quad \forall x \in D$

(iv) $y' = 0$ at $(1,0)$ (3)

y' undef'd at $(0,1)$.

(v) $y' = -\frac{\sqrt{y}}{\sqrt{x}}$

$\therefore y'' = \frac{\sqrt{x} \cdot -\frac{1}{2}y^{-\frac{1}{2}} y' + \sqrt{y} \cdot \frac{1}{2}x^{-\frac{1}{2}}}{x}$

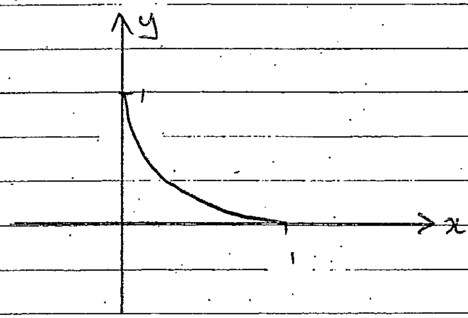
$= \frac{\frac{-\sqrt{x}}{2\sqrt{y}} \cdot \frac{-\sqrt{y}}{\sqrt{x}} + \sqrt{y} \cdot \frac{1}{2\sqrt{x}}}{x}$ (3)

$= \frac{\frac{1}{2} + \frac{\sqrt{y}}{2\sqrt{x}}}{x}$

$= \frac{\sqrt{x} + \sqrt{y}}{2x\sqrt{x}} > 0$ for $\forall x$ in domain

curve concave up

(vi)



(2)

Q3

a) $f(x) = \frac{e^x - 1}{e^x + 1}$

(i) $\frac{e^x - 1}{e^x + 1} = 0 \Rightarrow x = 0$ ie $(0,0)$ (1)

(ii) as $x \to \infty$, $\frac{e^x - 1}{e^x + 1} \to 1$ (2)

$x \to -\infty$, $\frac{e^x - 1}{e^x + 1} \to \frac{-1}{1} = -1$.

(iii) $f'(x) = \frac{(e^x + 1) \cdot e^x - (e^x - 1) \cdot e^x}{(e^x + 1)^2}$

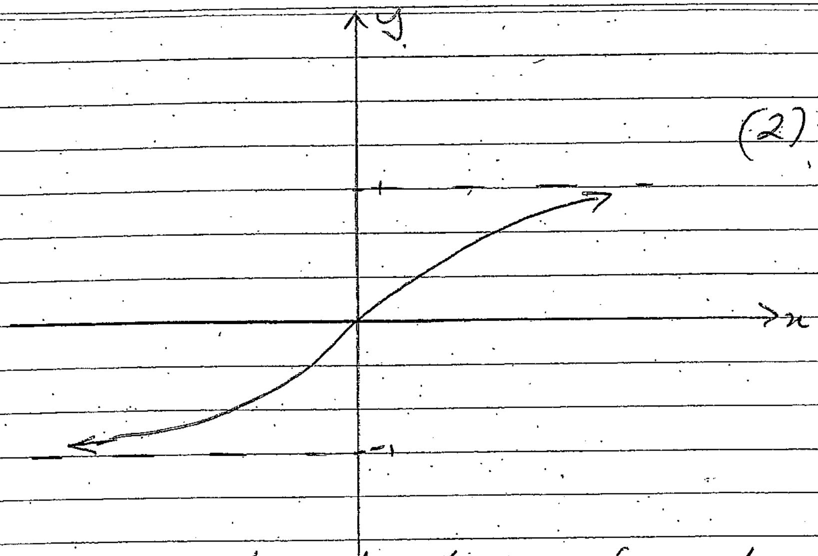
$= \frac{\cancel{e^{2x}} + e^x - \cancel{e^{2x}} + e^x}{(e^x + 1)^2}$ (2)

$= \frac{2e^x}{(e^x + 1)^2}$

$> 0 \therefore f(x)$ is increasing $\forall x$

(iv) P.T.O.

(iv)



(2)

(v) slope of tangent to $y=f(x)$ at $x=0$ is $\dfrac{2e^0}{(e^0+1)^2} = \dfrac{1}{2}$. (3

$\therefore$ $y = mx$ will intersect $y=f(x)$ in various ways ~~consi~~ depending on whether or not $m > \frac{1}{2}$.

If $m \geq \frac{1}{2} \Rightarrow 1$ soln ($\cancel{0,0}$ $x=0$)

$0<m<\frac{1}{2} \Rightarrow 3$ solns.

$m<0 \Rightarrow 1$ soln ($x=0$)

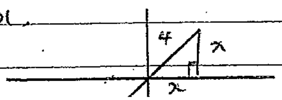
b) (i) $xy = 4 = c^2 = \dfrac{a^2}{2}$

$\therefore a^2 = 8$ (1)

$a = \sqrt{8}$.

$e = \sqrt{2}$ (since rect. hyperbola)

$\therefore$ foci are ~~...~~ "ae" units (ie 4) along $y=x$.

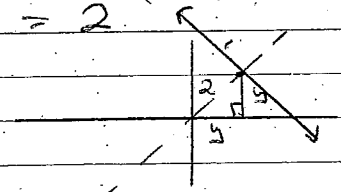


$2x^2 = 16$

$x = 2\sqrt{2}$.

$\therefore$ foci are $(2\sqrt{2}, 2\sqrt{2})$ & $(-2\sqrt{2}, -2\sqrt{2})$

(ii) $\dfrac{a}{e} = \dfrac{\sqrt{8}}{\sqrt{2}} = 2$



$2y^2 = 4$

$y = \sqrt{2}$.

Directrix has eqn of form $x+y=k$. (1)

$(\pm\sqrt{2}, \pm\sqrt{2})$ lies on this.

$\therefore k = \pm 2\sqrt{2}$

$\therefore$ Directrices have equations $x+y = \pm 2\sqrt{2}$.

(iii) gradient chord $PQ = \dfrac{\frac{2}{p} - \frac{2}{q}}{2(p-q)} \times \dfrac{pq}{pq}$

$= \dfrac{2(q-p)}{2pq(p-q)}$ (2

$= -\dfrac{1}{pq}$

(iv) $\therefore$ Eqn of chord PQ:

$y - \dfrac{2}{p} = -\dfrac{1}{pq}(x-2p)$ (2)

ie $pqy - 2q = -(x-2p)$

$\therefore x + pqy = 2(p+q)$ *

(v) $(4,0)$ lies on *

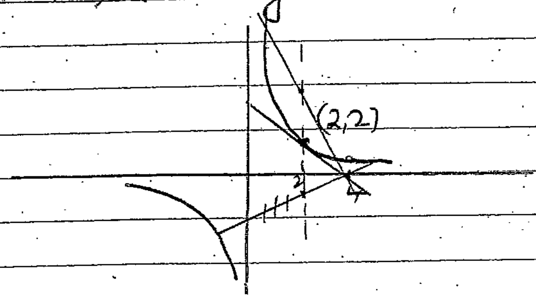
$\therefore 4 = 2(p+q)$

$\therefore p+q = 2$.

M: $(2p+2q$ ...

$\therefore$ locus of M is $x = 2$

(vi)



$y > 2$. (1)

c) (i) $x = a\sec\theta$ $\qquad$ $y = b\tan\theta$

$\therefore \frac{dx}{d\theta} = a\sec\theta\tan\theta$. $\qquad$ $\therefore \frac{dy}{d\theta} = b\sec^2\theta$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{b\sec^2\theta}{a\sec\theta\tan\theta}$$

$$= \frac{b\sec\theta}{a\tan\theta}$$

$\therefore$ slope of normal at P is $-\frac{a\tan\theta}{b\sec\theta}$

$\therefore$ eqn of normal is

$$y - b\tan\theta = \frac{-a\tan\theta}{b\sec\theta}(x - a\sec\theta)$$

$$\left(\times \frac{b}{\tan\theta}\right)$$

i.e. $\frac{by}{\tan\theta} - b^2 = \frac{-a\,\cancel{\sec}\,x}{\sec\theta} + a^2$

$$\therefore \frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 \quad (3)$$

(ii) at A, $y = 0$

$$\therefore \frac{ax}{\sec\theta} = a^2 + b^2$$

$$\therefore x = \frac{a^2+b^2}{a}\sec\theta$$

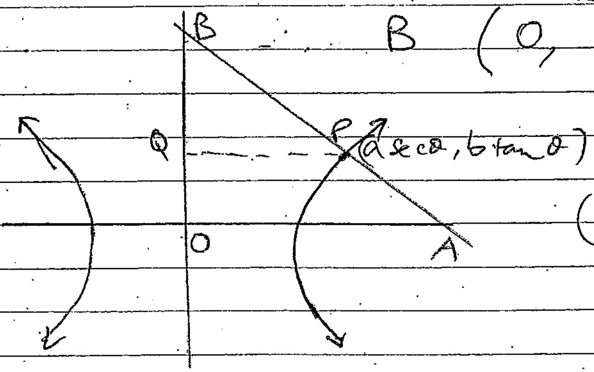
i.e. $A\left(\frac{a^2+b^2}{a}\sec\theta,\ 0\right)$

at B, $x = 0$

$$\therefore \frac{by}{\tan\theta} = a^2 + b^2$$

$$\therefore y = \frac{a^2+b^2}{b}\tan\theta$$

$\therefore B\left(0,\ \frac{a^2+b^2}{b}\tan\theta\right)$,



$PA : PB = OQ : QB$.
(ratio of intercepts on transversals cut by family of parallel lines — or similar triangles argument).

$$\therefore PA : PB = b\tan\theta : \frac{a^2+b^2}{b}\tan\theta - b\tan\theta$$

$$= b\tan\theta : \frac{(a^2+b^2)\tan\theta - b^2\tan\theta}{b}$$

$$= b\tan\theta : \frac{a^2\tan\theta}{b}$$

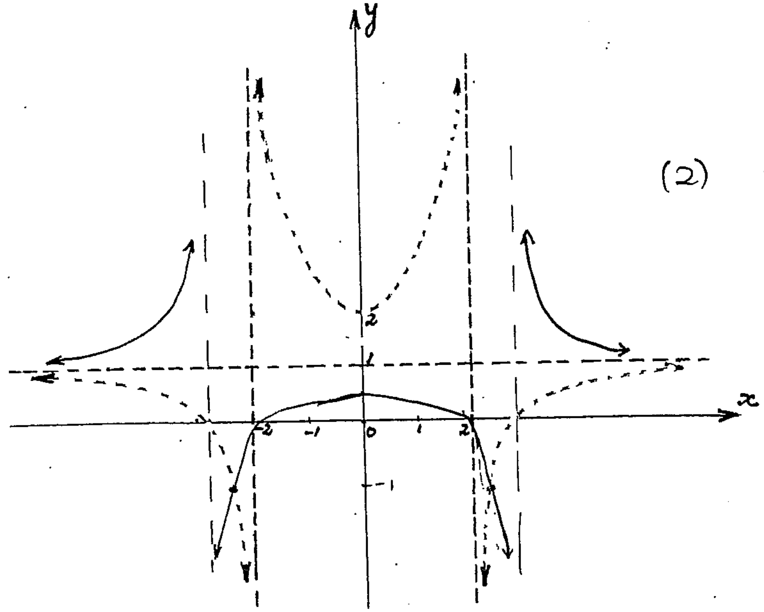
$$= b^2\tan\theta : a^2\tan\theta$$

$$= b^2 : a^2.$$

(2)

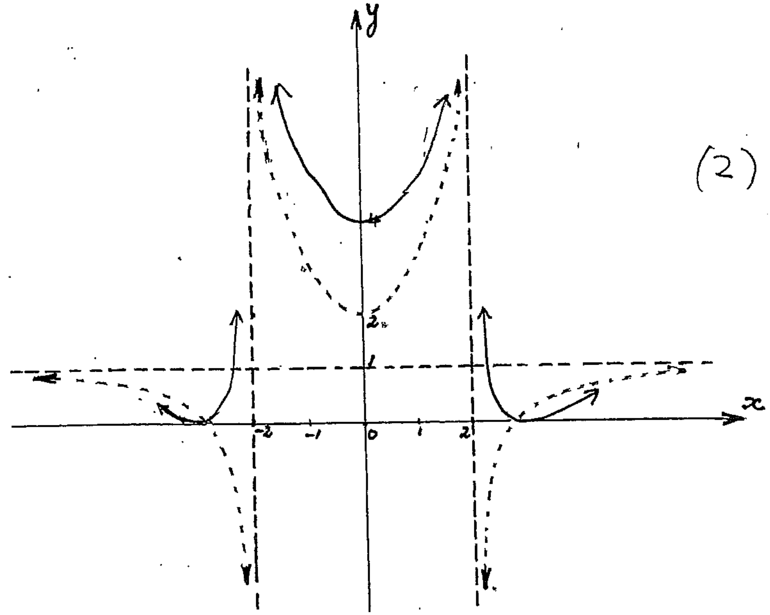
Q2 a) (i) $y = \frac{1}{f(x)}$

(2)

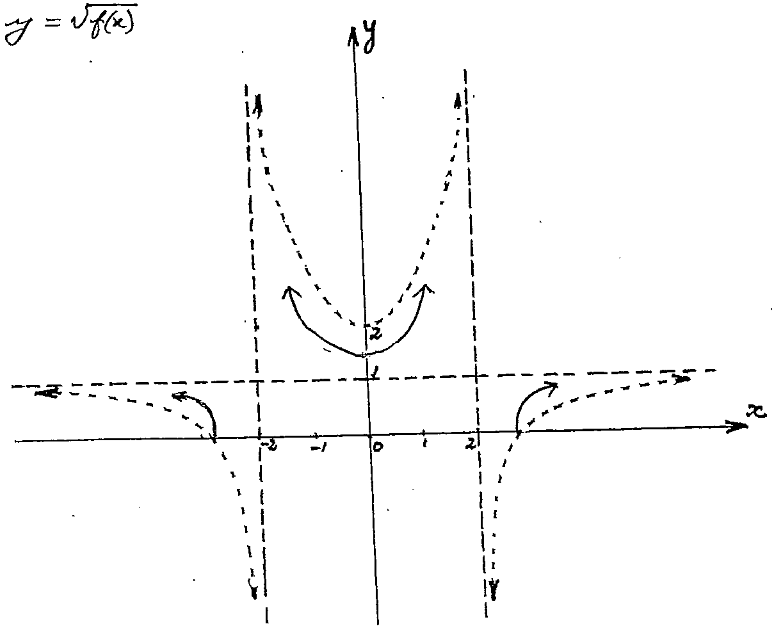


Q2 a) (iii) $y = [f(x)]^2$

(2)

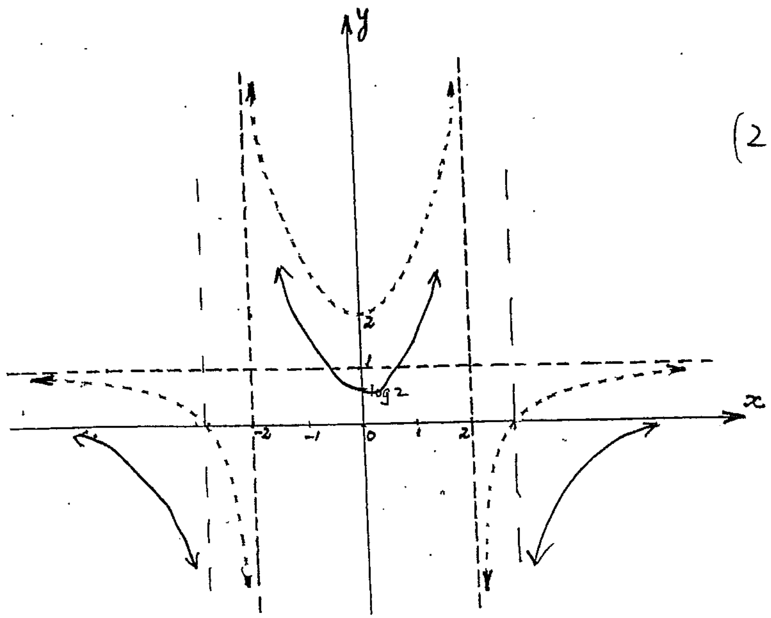


Q2 a) (ii) $y = \sqrt{f(x)}$

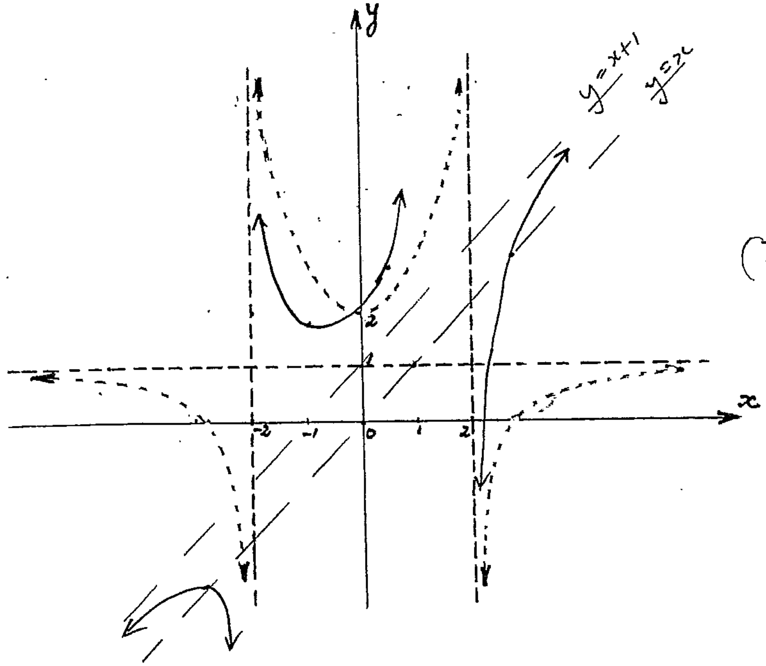


Q2 a) (iv) $y = \log[f(x)]$

(2)



Q2 a) (v) $y = x + f(x)$



(2)

Q2 a) (vi) $y = f'(x)$