

2008



Mathematics Extension 2

General Instructions

- Working time: 1½ hours
- Reading time: 5 minutes
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.

Total marks – 75

- Attempt Questions 1 – 3
- All questions are of equal value
- Begin each question in a new booklet
- All necessary working should be shown in every question.

Question 1 – (25 marks) – (Start a new booklet)

Marks

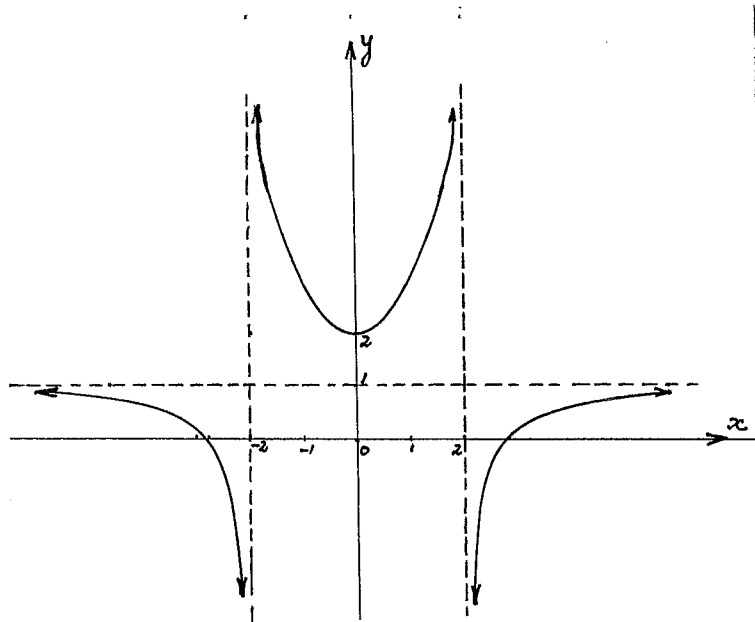
- a) Consider the Hyperbola $H: \frac{x^2}{4} - \frac{y^2}{9} = 1$
- Find the x -intercepts 1
 - Find the eccentricity e 1
 - Find the foci 1
 - Find the directrices 1
 - Find the asymptotes 1
 - Sketch H clearly showing all of the above features 2
- b) Consider the ellipse $E: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- Prove that the equation of the tangent to E at $P(x_1, y_1)$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ 3
 - The tangent to E at the point P cuts a directrix at the point T . Prove that the line segment PT subtends an angle of 90 degrees at the corresponding focus. 4
 - Using the basic definition of an ellipse prove that $PS + PS^1 = 2a$ where S, S^1 are the two foci of E . 2
- c) It can be shown that the line $y = mx + k$ is tangential to the hyperbola $H: \frac{x^2}{4} - \frac{y^2}{15} = 1$ if $4m^2 - 15 = k^2$ $4m^2 - 15 - k^2 = 0 = \Delta$
- Find the equation of the line through $(-1, 3)$ with gradient m 1
 - Hence find the equations of the two tangents to H from $(-1, 3)$ 4
- d) The equation of the chord of contact from an external point $P(x_0, y_0)$ to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$.
- State the condition which x_0 and y_0 must satisfy if P is external to the ellipse. 1
 - Find the condition which c must satisfy if $x + y = c$ is a chord of contact for the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$. 3

Question 2 – (25 marks) – (Start a new booklet)

Marks

a) The graph of $y = f(x)$ is shown below.

On the separate grids provided, sketch



(i) $y = \frac{1}{f(x)}$

2

(ii) $y = \sqrt{f(x)}$

2

(iii) $y = [f(x)]^2$

2

(iv) $y = \log [f(x)]$

2

(v) $y = x + f(x)$

2

(vi) $y = f^{-1}(x)$

2

Question 2 (cont'd)

Marks

b) We aim to sketch the function $\sqrt{x} + \sqrt{y} = 1$

(i) Determine the x and y intercepts made by this function

1

(ii) Determine the domain and range of this function

2

(iii) Show that this function is decreasing for all x in the domain

2

(iv) Determine any stationary points and critical points

3

(v) Find $\frac{d^2y}{dx^2}$ and comment on its significance in the domain

3

(vi) Sketch the function

2

Question 3 – (25 marks) – (Start a new booklet)

Marks

- a) Consider the function $f(x) = \frac{e^x - 1}{e^x + 1}$
- (i) Find the x intercept 1
- (ii) Determine the behaviour of $f(x)$ as $|x|$ becomes very large (ie x approaches positive or negative infinity) 2
- (iii) Show that $f(x)$ is an increasing function 2
- (iv) Sketch the function $y = f(x)$ 2
- (v) Discuss the number of solutions of $f(x) = mx$ for varying values of m . 3

- b) Consider the rectangular hyperbola : $xy = 4$. $P(2p, \frac{2}{p})$ and $Q(2q, \frac{2}{q})$ are variable points on R with $p > 0, q > 0$. M is the mid-point of the chord PQ .

- (i) State the foci of R 1
- (ii) State the equations of the two directrices of R 1
- (iii) Find the gradient of the chord PQ 2
- (iv) Find the equation of the chord PQ 2
- (v) If P and Q vary such that the line PQ always passes through the point $(4, 0)$ find the locus of M in the form $x = a$ 3
- (vi) Find the range of this locus 1

- c) The point $P(a \sec \theta, b \tan \theta)$ lies on $H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The normal at P cuts the x axis at A and the y axis at B .

- (i) Prove that the normal to H at the point P has equation $\frac{ax}{\sec \theta} + \frac{by}{\tan \theta} = a^2 + b^2$ 3
- (ii) Prove that $PA:PB = b^2:a^2$ 2

End of Paper

Question 3 – (25 marks) – (Start a new booklet)

Marks

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End of Paper

Extension 2 Solutions

Q1
a)(i) $\frac{x^2}{4} - \frac{y^2}{9} = 1$

$y=0 \Rightarrow x = \pm 2$. $\therefore (2,0), (-2,0)$ are x -intercepts.

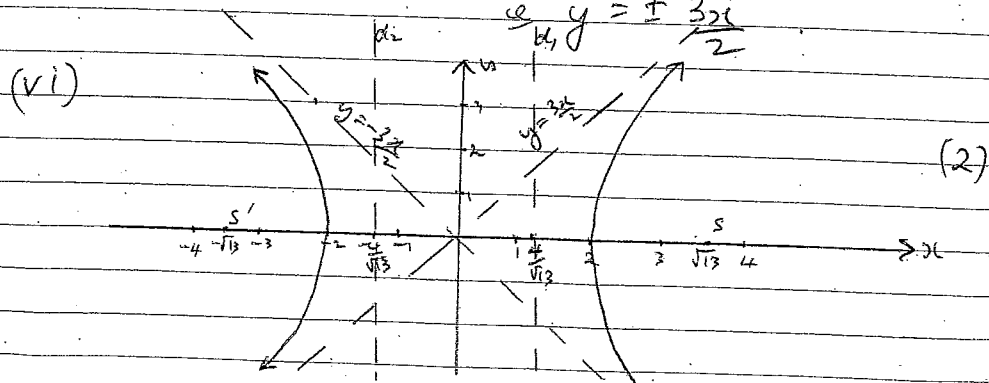
(ii) $b^2 = a^2(e^2 - 1)$ $a=2, b=3$
 $9 = 4(e^2 - 1)$
 $e^2 - 1 = \frac{9}{4}$
 $e^2 = \frac{13}{4}$

$e = \frac{\sqrt{13}}{2}$

(iii) Foci are $(\pm ae, 0)$
 $\therefore (\pm \sqrt{13}, 0)$

(iv) Directrices are $x = \pm \frac{a}{e}$
 $\therefore x = \pm \frac{4}{\sqrt{13}}$

(v) Asymptotes are $y = \pm \frac{bx}{a}$
 $\therefore y = \pm \frac{3x}{2}$



b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(i) $\frac{d}{dx} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right) = 0$

$\therefore \frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$

$2yy' = -\frac{2x}{a^2} \times b^2$

$\therefore y' = -\frac{b^2}{a^2} \frac{x}{y}$

at $P(x_1, y_1)$, $y' = -\frac{b^2}{a^2} \frac{x_1}{y_1}$

eqn of tangent is
 $y - y_1 = -\frac{b^2}{a^2} \frac{x_1}{y_1} (x - x_1)$

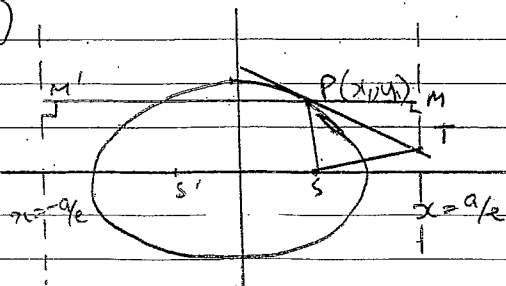
\times both sides by $\frac{y_1}{b^2}$

$\Rightarrow \frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = -\frac{x_1 x}{a^2} + \frac{x_1^2}{a^2}$

(3) $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

$= 1$ since (x_1, y_1) lies on ellipse

(ii)



at T, $x = a/e$

$\therefore \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ because

$\frac{a}{ea^2} x_1 + \frac{y_1 y_1}{b^2} = 1$

$\therefore \frac{x_1}{ae} + \frac{y_1 y_1}{b^2} = 1$

$$\frac{yy_1}{b^2} = 1 - \frac{x_1}{ae}$$

$$\therefore y = \frac{b^2}{y_1} \left(1 - \frac{x_1}{ae}\right)$$

$$\therefore T \left(\frac{a}{e}, \frac{b^2}{y_1} \left(1 - \frac{x_1}{ae}\right) \right)$$

$$\text{Now, slope PS} = \frac{y_1}{x_1 - ae} = m_1$$

$$\text{slope TS} = \frac{\frac{b^2}{y_1} \left(1 - \frac{x_1}{ae}\right)}{\frac{a}{e} - ae} = m_2$$

$$m_1 m_2 = \frac{y_1}{x_1 - ae} \times \frac{b^2 \left(1 - \frac{x_1}{ae}\right) \times ae}{y_1 (ae - ae^2)}$$

$$= \frac{y_1}{x_1 - ae} \times \frac{b^2 (ae - x_1)}{y_1 (a^2 - a^2 e^2)}$$

$$= \frac{-b^2}{a^2(1 - e^2)}$$

(3)

$$= -1 \quad \text{since } b^2 = a^2(1 - e^2) \text{ for ellipse.}$$

$\therefore PS \perp TS$

\therefore PT subtends 90° at focus.

(ii) $PS = e PM$, M is foot of perpendicular to corresponding directrix.

and similarly, $PS' = e PM'$

$$(3) \quad PS + PS' = e (PM + PM')$$

$$= e (MM')$$

$$= e \times \frac{2a}{e}$$

$$= 2a.$$

$$c) (i) \quad y - 3 = m(x + 1) \quad (1)$$

$$\text{or } y = mx + m + 3 \quad \text{or } K = m + 3.$$

(ii) For tangents to exist,

$$4m^2 - 15 = K^2$$

$$= (m + 3)^2$$

$$4m^2 - 15 = m^2 + 6m + 9$$

$$\therefore 3m^2 - 6m - 24 = 0$$

$$m^2 - 2m - 8 = 0.$$

$$(m - 4)(m + 2) > 0 \quad (4)$$

$$\therefore m = -2, 4.$$

\therefore tangents are

$$y = -2x + 1, \quad y = 4x + 7 \text{ using } *$$

$$d) (i) \quad \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} > 1. \quad (1)$$

$$(ii) \quad 2x + y = c \quad \text{corresponds to } \frac{x x_0}{9} + \frac{y y_0}{4} = 1$$

$$\text{or } \frac{x}{c} + \frac{y}{c} = 1$$

$$\Rightarrow \frac{x_0}{9} = \frac{1}{c}$$

$$\frac{y_0}{4} = \frac{1}{c}.$$

$$\therefore x_0 = 9/c, \quad y_0 = 4/c.$$

Now, we require $\frac{x_0^2}{9} + \frac{y_0^2}{4} > 1$ from (1)

$$\therefore \left(\frac{9}{c}\right)^2 + \left(\frac{4}{c}\right)^2 > 1$$

$$9 + 4 > 1$$

$$\text{or } 9 + 4 > 1$$

$$\frac{13}{c} > 1$$

$$c^2 < 13$$

$$-\sqrt{13} < c < \sqrt{13}$$

(3)

Q2 a) see separate sheets.

Q2 b) (i) x intercept: $(1, 0)$ (1)
 y " $(0, 1)$

(ii) D: $0 \leq x \leq 1$ (2)
 R: $0 \leq y \leq 1$

(iii) $\sqrt{x} + \sqrt{y} = 1$
 $\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}} y' = 0$ (2)

$$\therefore y' = -\frac{x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}} = -\frac{\sqrt{y}}{\sqrt{x}} < 0 \quad \forall x \in D$$

(iv) $y' = 0$ at $(1, 0)$ (3)
 y' undef'd at $(0, 1)$

(v) $y' = -\frac{\sqrt{y}}{\sqrt{x}}$

$$\therefore y'' = \sqrt{x} \cdot -\frac{1}{2}y^{-\frac{1}{2}} y' + \sqrt{y} \cdot \frac{1}{2}x^{-\frac{3}{2}}$$

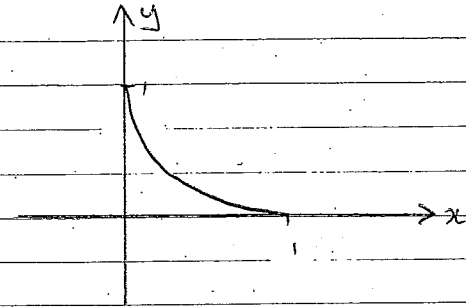
$$= \frac{-\sqrt{x}}{2\sqrt{y}} \cdot -\frac{\sqrt{y}}{\sqrt{x}} + \sqrt{y} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2} + \frac{\sqrt{y}}{2\sqrt{x}}$$

$$= \frac{\sqrt{x} + \sqrt{y}}{2\sqrt{x}} > 0 \text{ for } \theta \text{ in domain}$$

\therefore curve concave up.

(vi)



(2)

Q3

a) $f(x) = \frac{e^x - 1}{e^x + 1}$

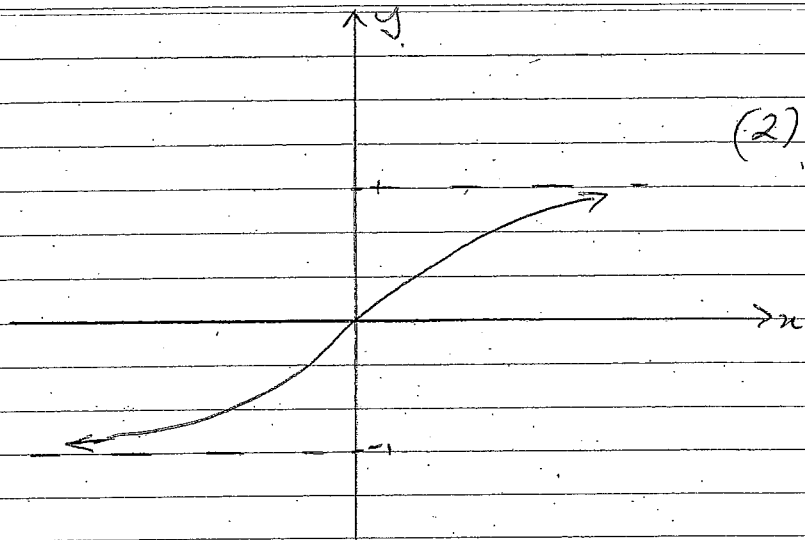
(i) $\frac{e^x - 1}{e^x + 1} = 0 \Rightarrow x = 0$ at $(0, 0)$ (1)

(ii) as $x \rightarrow \infty$, $\frac{e^x - 1}{e^x + 1} \rightarrow 1$ (2)
 $x \rightarrow -\infty$, $\frac{e^x - 1}{e^x + 1} \rightarrow -1 = -1$

(iii) $f'(x) = \frac{(e^x + 1)e^x - (e^x - 1)e^x}{(e^x + 1)^2}$
 $= \frac{e^{2x} + e^x - e^{2x} + e^x}{(e^x + 1)^2}$
 $= \frac{2e^x}{(e^x + 1)^2}$
 $> 0 \therefore f(x)$ is increasing \forall

(iv) P.T.O.

(iv)



(2)

(v) slope of tangent to $y=f(x)$ at $x=0$ is $\frac{2e^0}{(e^0+1)^2} = \frac{1}{2}$.

$y=mx$ will intersect $y=f(x)$ in various ways consid depending on whether or not $m > \frac{1}{2}$ (3)

If $m > \frac{1}{2} \Rightarrow 1$ soln ($x=0$)
 $0 < m < \frac{1}{2} \Rightarrow 3$ solns.
 $m < 0 \Rightarrow 1$ soln ($x=0$)

b) $xy=4 = c^2 = \frac{a^2}{2}$

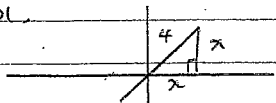
(i)

$\therefore a^2 = 8$
 $a = \sqrt{8}$

(1)

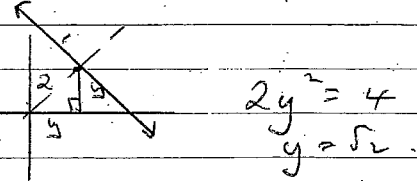
$c = \sqrt{2}$ (since rect. hyperbola)

foci are \pm "ae" units (ie 4) along $y=x$.



$2x^2 = 16$
 $x = 2\sqrt{2}$

(ii) \therefore foci are $(2\sqrt{2}, 2\sqrt{2})$ & $(-2\sqrt{2}, -2\sqrt{2})$
 $\frac{a}{2} = \frac{\sqrt{8}}{\sqrt{2}} = 2$



Directrix has eqn of form $x+y=k$.
 $(2, 2)$ lies on this (1)
 $k = 2\sqrt{2}$

Directrices have equations $x+y = \pm 2\sqrt{2}$.

(iii) gradient chord PQ = $\frac{\frac{2}{p} - \frac{2}{q}}{2(p-q)} \times \frac{pq}{pq}$

= $\frac{2(q-p)}{2pq(p-q)}$ (2)
 = $-\frac{1}{pq}$

(iv) Eqn of chord PQ:

$y - \frac{2}{p} = -\frac{1}{pq}(x-2p)$ (2)

$\therefore pqy - 2q = -(x-2p)$

$\therefore x + pqy = 2(ptq)$ *

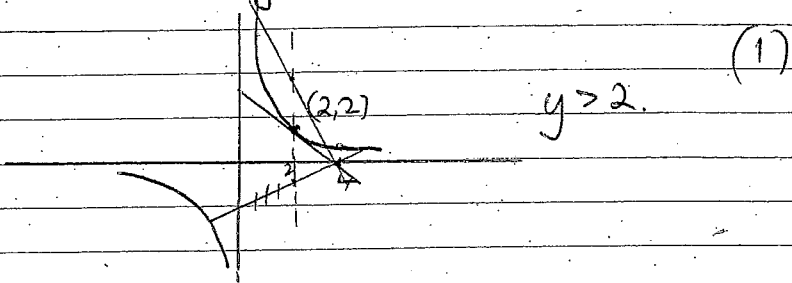
(v) $(4, 0)$ lies on *

$4 = 2(ptq)$

$\therefore ptq = 2$

M is $(2p+2q, \frac{1}{p} + \frac{1}{q}) \Rightarrow (ptq, \frac{1}{p} + \frac{1}{q})$

(vi) locus of M is $x=2$.



c) (i) $x = a \sec \theta$ $y = b \tan \theta$
 $\therefore \frac{dx}{d\theta} = a \sec \theta \tan \theta$ $\frac{dy}{d\theta} = b \sec^2 \theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{b \sec^2 \theta}{a \sec \theta \tan \theta}$$

$$= \frac{b \sec \theta}{a \tan \theta}$$

\therefore slope of normal at P is $-\frac{a \tan \theta}{b \sec \theta}$

eqn of normal is

$$y - b \tan \theta = -\frac{a \tan \theta}{b \sec \theta} (x - a \sec \theta)$$

$$\left(x \frac{b}{\tan \theta} \right)$$

$$\therefore \frac{by}{\tan \theta} - b^2 = -\frac{a \sec \theta}{\sec \theta} x + a^2$$

$$\therefore \frac{ax}{\sec \theta} + \frac{by}{\tan \theta} = a^2 + b^2 \quad (3)$$

(1) at A, $y=0$

$$\therefore \frac{ax}{\sec \theta} = a^2 + b^2$$

$$\therefore x = \frac{a^2 + b^2}{a} \sec \theta$$

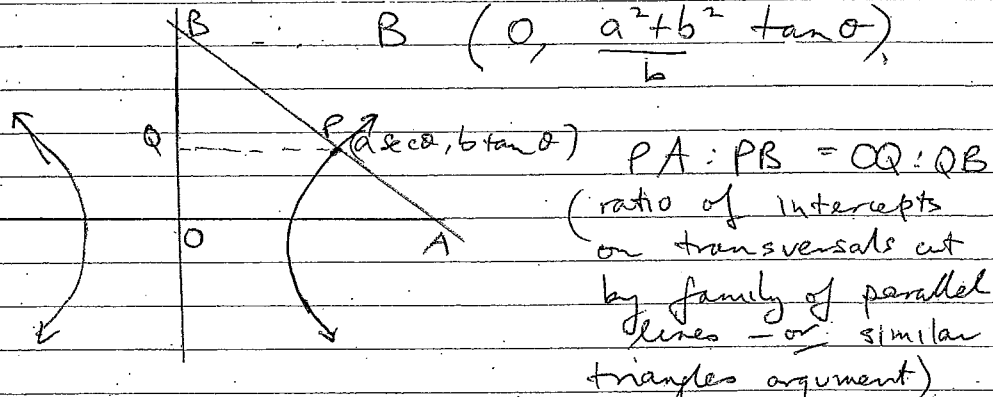
$$\therefore A \left(\frac{a^2 + b^2}{a} \sec \theta, 0 \right)$$

at B, $x=0$

$$\therefore \frac{by}{\tan \theta} = a^2 + b^2$$

$$\therefore y = \frac{a^2 + b^2}{b} \tan \theta$$

$$B \left(0, \frac{a^2 + b^2}{b} \tan \theta \right)$$



$$\therefore PA:PB = b \tan \theta : \frac{a^2 + b^2}{b} \tan \theta - b^2$$

$$= b \tan \theta : \frac{(a^2 + b^2) \tan \theta - b^2 \tan \theta}{b}$$

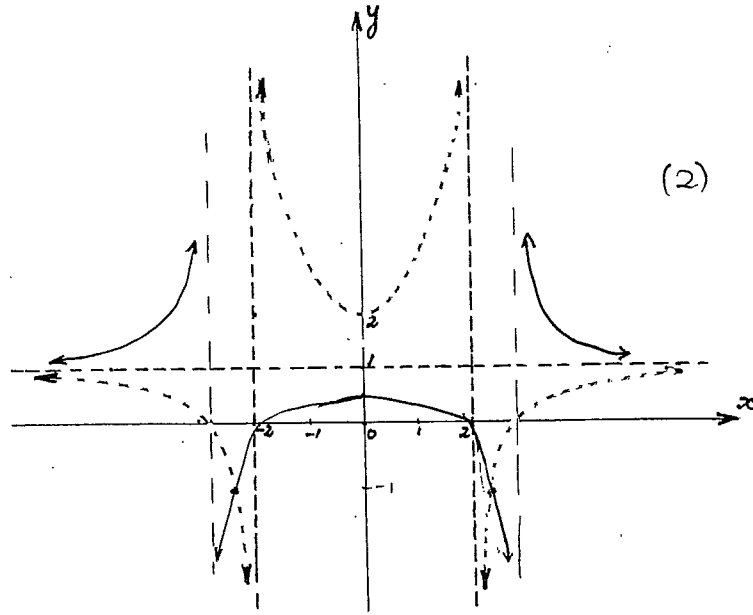
(2)

$$= b \tan \theta : \frac{a^2 \tan^2 \theta}{b}$$

$$= b^2 \tan^2 \theta : a^2 \tan \theta$$

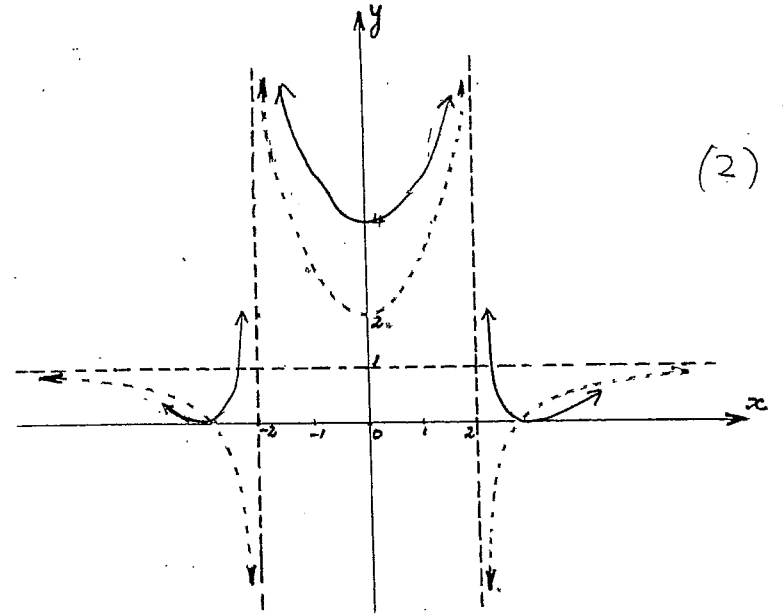
$$= b^2 : a^2$$

Q2 a) (i) $y = \frac{1}{f(x)}$



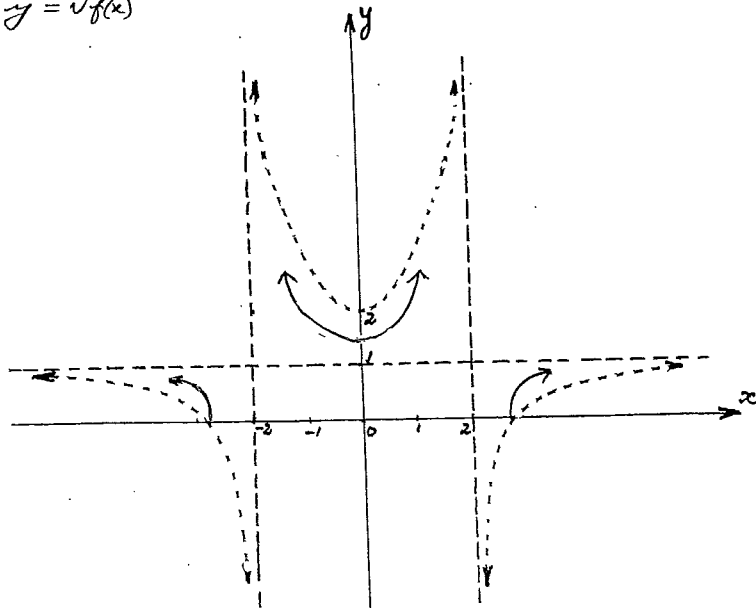
(2)

Q2 a) (iii) $y = [f(x)]^2$

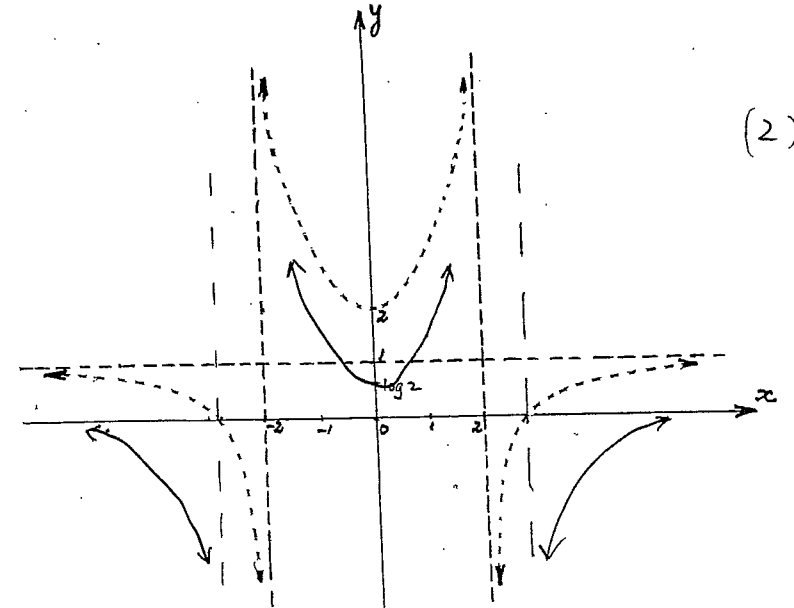


(2)

Q2 a) (ii) $y = \sqrt{f(x)}$

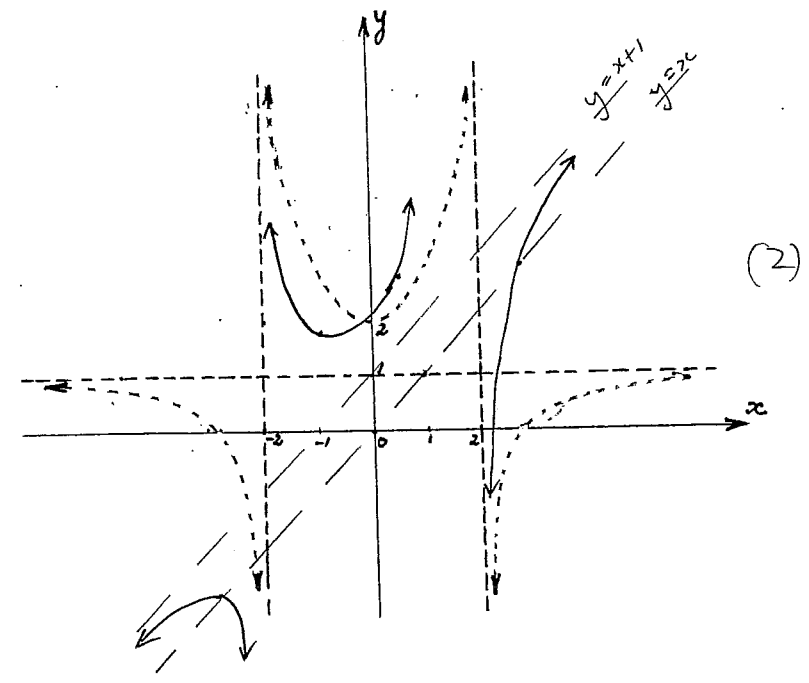


Q2 a) (iv) $y = \log[f(x)]$



(2)

Q2 a) (v) $y = x + f(x)$



Q2 a) (vi) $y = f'(x)$

