Amy Ow
Ms. O'Dea.

St George Girls High School

Year 12

Mid-HSC Course Examination

2005

Mathematics Extension 1

General Instructions
- Working time – 1½ hours
- Reading time – 5 minutes
- Write using blue or black pen
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.
- Write on one side of the page only.
- Start each question in a **new booklet**.

Total marks – 75
- Attempt Questions 1 – 6
- All questions are of equal value

Question 1 – (13 marks) – Start a new booklet **Marks**

a) Find the derivative of:

(i) $y = \ln\sqrt{x}$ 1

(ii) $y = x^2 e^{3x}$ 2

b) (i) Solve the equation $m^2 - m - 1 = 0$ correct to 3 decimal places. 2

(ii) On the same set of axes draw the graphs $y = e^x - 1$ and $y = e^{-x}$ (do not use calculus). 2

(iii) Find the area between the curves from $x = 1$ to $x = 2$. Leave your answer in terms of e. 2

(iv) Show that the curves intersect when $e^{2x} - e^x - 1 = 0$ 2

(v) Use the results of part (i) to show that the x coordinate of the point of intersection of the curves is approximately 0.481. 2

Question 2 – (12 marks) – Start a new booklet **Marks**

a) Find

(i) $\int 3^x \, dx$ 1

(ii) $\int \frac{e^{2x}}{1+e^{2x}} \, dx$ 1

(iii) $\int_2^4 \frac{x^2-1}{x} \, dx$ 3

b) The gradient of a curve at any point on it is xe^{x^2} and the curve passes through the point $(1, e)$. Find the equation of the curve. 3

c) (i) Sketch the curve $y = \ln(x+2)$ 1

(ii) Find the exact area between the curve, $y = \ln 4$ and the y axis. 3

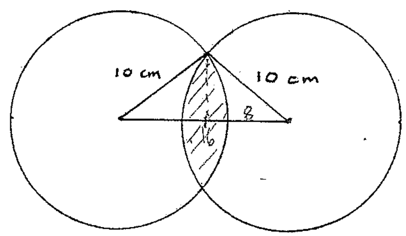
Question 3 – (12 marks) – Start a new booklet **Marks**

a) Find the exact value of $\cos 75°$ 2

b) Prove $\cos^2(45-x) - \sin^2(45-x) = \sin 2x$ 2

c) On the same set of axes sketch the graphs $y = 3\sin 2x$ and $y = -\frac{1}{2}x$ for $-\pi \le x \le \pi$. Hence state how many solutions there are for the equation $3\sin 2x = -\frac{1}{2}x$ over the domain $-\pi \le x \le \pi$ 4

d) Two circles each of radius length 10cm have their centres 16cm apart. Calculate the area common to each circle. 4



Question 4 – (13 marks) – Start a new booklet **Marks**

a) Find the acute angle between the lines $x+2y=6$ and $y=1-3x$ 2

b) The area between the curves $y=\cos x$ and $y=\sin x$ and $x=\frac{\pi}{6}$ and $x=\frac{\pi}{4}$ is rotated about the x axis. Find the volume of the solid formed. 3

c) For the curve $y=e^x \cos x$ in the domain $0 \le x \le \frac{3\pi}{2}$ find:

(i) x and y intercepts 1

(ii) the stationary points and determine their nature 6

(iii) sketch the curve 1

Question 5 – (13 marks) – Start a new page **Marks**

a) Find the exact value of $\sin\left(2\tan^{-1}\frac{1}{2}\right)$ 3

b) Find the following:

(i) $\int \frac{dx}{\sqrt{4-9x^2}}$ 1

(ii) $\int_0^{\frac{1}{3}} \frac{dx}{1+9x^2}$ 2

c) Find all the angles θ for which $\sin 2\theta = \cos\theta$ 4

d) Differentiate $2x\tan^{-1}x$ and hence find $\int \tan^{-1}x\, dx$ 3

Question 6 – (12 marks) – Start a new booklet **Marks**

a) State the domain and range and hence sketch $y=2\sin^{-1}3x$ 3

b) Find the exact area between $x=1$, $y=\frac{\pi}{2}$ and the curve $y=\cos^{-1}x$ 4

c) (i) Find the largest positive domain of the function $f(x)=x^2-4x+5$ for which $f(x)$ has an inverse function $f^{-1}(x)$. 1

(ii) Find $f^{-1}(x)$ 3

(iii) State the domain and range of $f^{-1}(x)$ 1

End of Paper

ANSWERS
Year 12 Extension 1
Half Yearly

Question 1.

a.) (i) $y = \ln x^{1/2}$

$y' = \dfrac{1/2\, x^{-1/2}}{x^{1/2}}$

$= \dfrac{1}{2x}$

.) $y = x^2 . e^{3x}$

$y' = e^{3x} . 2x + x^2 . 3e^{3x}$

$= xe^{3x}(2 + 3x)$

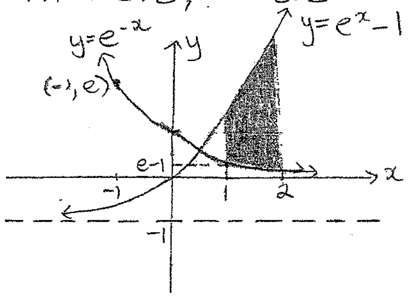
b.) (i) $m^2 - m - 1 = 0$

$m = \dfrac{1 \pm \sqrt{1 - 4.1.-1}}{2}$

$= \dfrac{1 \pm \sqrt{5}}{2}$

$\therefore m = 1.618, -0.618$

(ii)



(iii) $A = \displaystyle\int_1^2 e^x - 1 - e^{-x}\, dx$

$= \left[e^x - x + \dfrac{1}{e^x} \right]_1^2$

$= (e^2 - 2 + \frac{1}{e^2}) - (e - 1 + \frac{1}{e})$

$= e^2 - e - 1 + 1 \quad - 1 \text{ units}^2$

(iv) $e^x - 1 = e^{-x}$

$e^x - 1 = \dfrac{1}{e^x}$

$e^{2x} - e^x = 1$

$e^{2x} - e^x - 1 = 0.$

(v.) $e^{2x} - e^x - 1 = 0$

Let $m = e^x$

$m^2 - m - 1 = 0.$

from part (i) $m = 1.618, m >$

$\therefore e^x = 1.618$

$\log e^x = \log 1.618$

$x = 0.481$ (3 d.p.)

Question 2

a.) (i) $\displaystyle\int 3^x dx = \frac{1}{\log 3} . 3^x + C$

$\dfrac{1}{\log 3} . e^{x \log_3 \log 3}$

(ii) $\displaystyle\int \frac{e^{2x}}{1+e^{2x}} dx = \frac{1}{2} \int \frac{2e^{2x}}{1+e^{2x}} dx$

$= \frac{1}{2} \log(1 + e^{2x}) + C$

(iii) $\displaystyle\int_2^4 \frac{x^2 - 1}{x} dx = \int_2^4 x - \frac{1}{x}\, dx$

$= \left[\dfrac{x^2}{2} - \log x \right]_2^4$

$= (8 - \log 4) - (2 - \log 2)$

$= 6 + \log \frac{1}{2}$

$= 6 - \log 2$

b.) $f'(x) = xe^{x^2} \quad (1, e)$

$f(x) = \displaystyle\int xe^{x^2} dx$

$= \dfrac{1}{2} \displaystyle\int 2xe^{x^2} dx$

$= \dfrac{1}{2} e^{x^2} + C.$

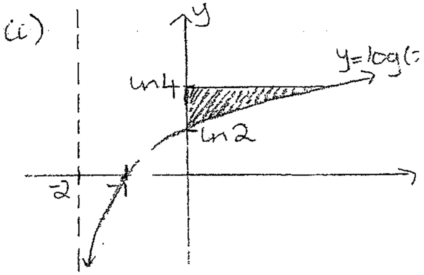
when $x = 1$, $f(x) = e$

$e = \frac{1}{2} e + c.$

$\therefore c = \frac{1}{2} e$

$\therefore f(x) = \dfrac{e^{x^2}}{2} + \dfrac{e}{2}$

c.) (i)



(ii) $y = \log(x + 2)$

$e^y = x + 2$

$x = e^y - 2.$

$\therefore A = \displaystyle\int_{\ln 2}^{\ln 4} e^y - 2\, dy$

$= \left[e^y - 2y \right]_{\ln 2}^{\ln 4}$

$= (e^{\ln 4} - 2\ln 4) - (e^{\ln 2} - 2\ln$

$= 4 - 2(\ln 4 - \ln 2) - 2$

$= 2 - 2\ln 2 \text{ units}^2$

Question 3

a) $\cos 75^\circ = \cos(45^\circ + 30^\circ)$

$= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$

$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$

$= \frac{\sqrt{3} - 1}{2\sqrt{2}}$

$= \frac{\sqrt{6} - \sqrt{2}}{4}$

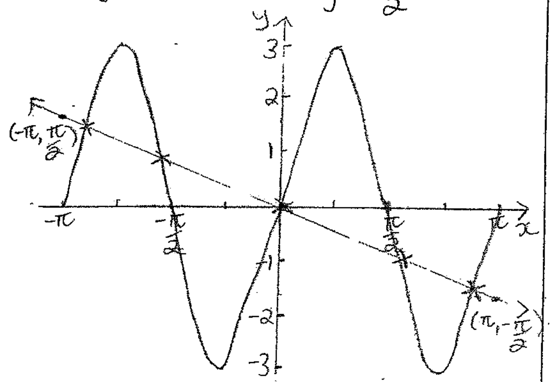
b) $LHS = \cos 2(45 - x)$

$= \cos(90 - 2x)$

$= \sin 2x$, $\sin\theta = \cos(90 - \theta)$

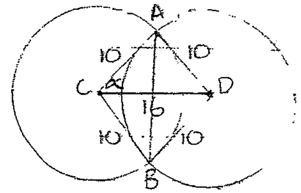
$= RHS$

c) $y = 3\sin 2x$, $y = -\frac{1}{2}x$



$\therefore$ there are 5 solutions

d)



$\cos\alpha = \frac{16^2 + 10^2 - 10^2}{2 \times 16 \times 10}$

$\alpha = 36^\circ 52'$

$\angle ACB = 2 \times 36^\circ 52'$

$= 73^\circ 44'$

$= 1.2869$ radians

segment $AB = \frac{1}{2} \times 10^2 (1.2869 - \sin 1.2869)$

$= 16.35$ (2 d.p.)

common area $= 2 \times 16.35$

$= 32.7 cm^2$ (1 d.p.)

Question 4.

a) $x + 2y = 6$

$2y = 6 - x$

$y = 3 - \frac{x}{2}$

$m_1 = -\frac{1}{2}$

$y = 1 - 3x$

$m_2 = -3$

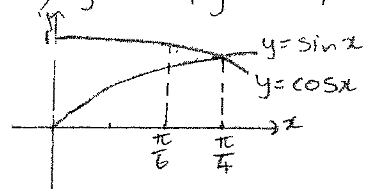
$\tan\alpha = \left| \frac{-\frac{1}{2} + 3}{1 + 3/2} \right|$

$= \left| \frac{5/2}{5/2} \right|$

$= 1$

$\therefore \alpha = \frac{\pi}{4}$

b) $y = \cos x$, $y = \sin x$, $x = \frac{\pi}{6}$, $x = \frac{\pi}{4}$



$V = \pi \int_{\pi/6}^{\pi/4} \cos^2 x - \sin^2 x \, dx$

$= \pi \int_{\pi/6}^{\pi/4} \cos 2x \, dx$

$= \pi \left[\frac{\sin 2x}{2} \right]_{\pi/6}^{\pi/4}$

$= \frac{\pi}{2} \left(\sin\frac{\pi}{2} - \sin\frac{\pi}{3} \right)$

$= \frac{\pi}{2} \left(1 - \frac{\sqrt{3}}{2} \right)$ units^3

c) $y = e^x \cos x$ $0 \le x \le \frac{3\pi}{2}$

(i) when $x = 0$, $y = 1$

when $y = 0$, $e^x \cos x = 0$

$e^x = 0$ or $\cos x = 0$

no solution $x = \frac{\pi}{2}, \frac{3\pi}{2}$

(ii) $y' = \cos x \cdot e^x + e^x \cdot -\sin x$

$= e^x(\cos x - \sin x)$

when $y' = 0$,

$e^x(\cos x - \sin x) = 0$.

$\cos x - \sin x = 0$.

$\cos x = \sin x$.

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

when $x = \pi/4$, $y = e^{\pi/4}/\sqrt{2}$

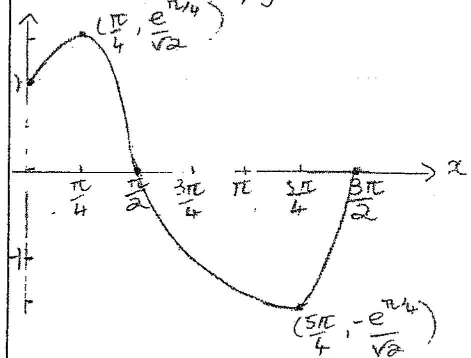
when $x = 5\pi/4$, $y = -e^{\pi/4}/\sqrt{2}$

$y'' = (\cos x - \sin x)e^x + e^x(-\sin x - \cos x)$

$= e^x(-2\sin x)$

when $x = \frac{\pi}{4}$, $y'' < 0$ $\therefore$ max

when $x = 5\pi/4$, $y' > 0$ $\therefore$ min

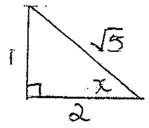


Question 5

a.) $\sin(2\tan^{-1}\frac{1}{2})$

let $x = \tan^{-1}\frac{1}{2}$

$\tan x = \frac{1}{2}$



$\sin 2x = 2\sin x\cos x$

$= 2 \times \frac{1}{\sqrt{5}} \times \frac{2}{\sqrt{5}}$

$= \frac{4}{5}$

b.) (i) $\int \frac{dx}{\sqrt{4-9x^2}} = \int \frac{dx}{\sqrt{9(\frac{4}{9}-x^2)}}$

$= \frac{1}{3}\int \frac{dx}{\sqrt{\frac{4}{9}-x^2}}$

$= \frac{1}{3}\sin^{-1}\frac{3x}{2} + C$

(ii) $\int_0^{1/3} \frac{dx}{1+9x^2} = \frac{1}{9}\int_0^{1/3} \frac{dx}{1/9+x^2}$

$= \frac{1}{9}\left[3\tan^{-1}3x\right]_0^{1/3}$

$= \frac{1}{3}\tan^{-1}1 - \tan^{-1}0$

$= \frac{1}{3}\times\frac{\pi}{4}$

$= \frac{\pi}{12}$

c.) $\sin 2\theta = \cos\theta$

$2\sin\theta\cos\theta - \cos\theta = 0.$

$\cos\theta(2\sin\theta - 1) = 0.$

$\cos\theta = 0$ or $\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{2}$ $\quad$ $\theta = \frac{\pi}{6}.$

d.) $\frac{d}{dx}(2x\tan^{-1}x) = \tan^{-1}x.2 + 2x.\frac{}{1+}$

$= 2\tan^{-1}x + \frac{2x}{1+x^2}$

$\int 2\tan^{-1}x + \frac{2x}{1+x^2}\,dx = 2x\tan^{-1}x$

$\therefore \frac{1}{2}\int 2\tan^{-1}x\,dx = \frac{1}{2}\left[2x\tan^{-1}x - \int\frac{2x}{1+x^2}\right.$

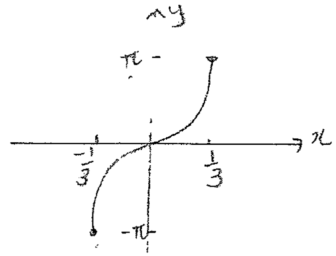
$= x\tan^{-1}x - \frac{1}{2}\log(1+x^2) +$

Question 6

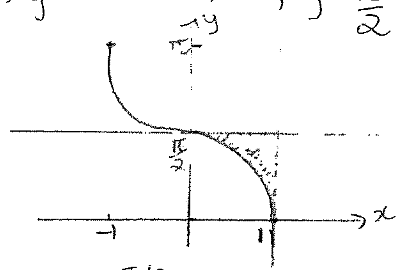
a.) $y = 2\sin^{-1}3x$

domain: $-\frac{1}{3} \leq x \leq \frac{1}{3}$

range: $-\pi \leq y \leq \pi$



b.) $y = \cos^{-1}x$ $\quad x = 1,\ y = \frac{\pi}{2}$



$A = \frac{\pi}{2} - \int_0^{\pi/2} \cos x\,dx$

$= \frac{\pi}{2} - \left[\sin x\right]_0^{\pi/2}$

$= \frac{\pi}{2} - (\sin\frac{\pi}{2} - \sin 0)$

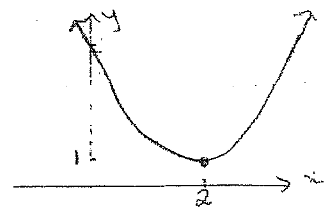
$= \pi/2 - 1$ units^2

c.) (i) $f(x) = x^2 - 4x + 5.$

$x = \frac{-b}{2a}$ $\quad$ when $x = 2,$

$= \frac{4}{2}$ $\quad$ $y = 1.$

$\therefore$ largest pos.



domain: $x \geq 2$

range: $y \geq 1$

(ii) $x = y^2 - 4y + 5$

$y^2 - 4y + 4 = x - 5 + 4$

$(y-2)^2 = x - 1$

$y - 2 = \pm\sqrt{x-1}$

$y = 2 + \sqrt{x-1}$, since $y \geq$

(iii) domain: $x \geq 1$

range: $y \geq 2$