Question 1 – (15 marks) **Marks**

a) Evaluate: 6

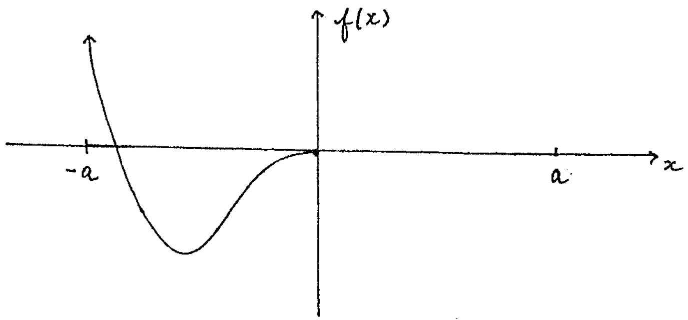
(i) $\int_0^{-\frac{1}{2}} \frac{dx}{\sqrt{1-2x}}$

(ii) $\int_{-\frac{1}{2}}^{0} \frac{dx}{1-2x}$

b) Find the area enclosed between the curves $y=\sqrt{x}$ and $y=x^3$. 3

c) Use Simpson's Rule with three function values to find an approximation for the area between $y=e^{-x^2}$ and the x-axis enclosed by the ordinates $x=0$ and $x=1$. 3

d)



The diagram shows the graph of a function, $f(x)$, for $-a \le x \le 0$. It is known that $f(x)$ is an odd function and stationary at $(0,0)$. 3

(i) Copy the diagram and continue the graph of $f(x)$ for $0 \le x \le a$.

(ii) On another diagram, sketch $y=f'(x)$.

(iii) On a third diagram sketch $y=f''(x)$.

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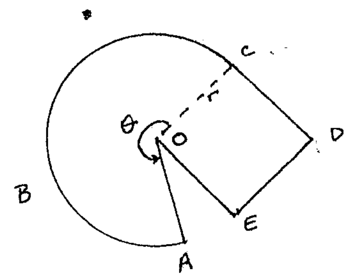
Question 2 – (15 marks) **Marks**

a) Express $\frac{2\pi}{9}$ radians in degrees. 1

b) Find the exact value of $\cos(\frac{-11\pi}{6})+\sin(\frac{2\pi}{3})$ 2

c) If $\tan\theta = 2$ and $0 \le \theta \le \frac{\pi}{2}$ find the exact value of $\sin(\theta+\frac{\pi}{4})$ 3

d)



The diagram shows sector $OABC$ of a circle radius r, together with a square $CDEO$ of side r. Reflex $\angle AOC = \theta$ radians. 4

(i) If the perimeter of the sector alone is equal to the circumference of a circle, radius r, prove that $\theta = 2\pi - 2$.

(ii) Show that the total area of the figure is equal to that of a circle radius r.

e) If $y=e^{2x-1}$ find $\frac{d^2y}{dx^2}$. 1

f) Differentiate $x^2\log_e 2x$ and hence evaluate $\int_1^2 2x\log_e 2x$ giving your answer in exact form. 4

Question 3 – (15 marks) **Marks**

a) Find the value of k if $\log_e 3 - \frac{1}{3}\log_e 8 = \log_e k$ 1

b) Find the equation of the normal to the curve $y = \log_e(x^3+1)$ at the point where $x = 1$. 4

c) The amount A grams of a certain carbon isotope in a dead tree trunk is given by $A = A_0 e^{-kt}$ where A_0 and k are positive constants, and where the time, t, is measured in years from the death of the tree. 6

(i) Show that A satisfies the equation $\frac{dA}{dt} = -kA$.

(ii) Find the value of k if the amount of the isotope is halved every 4500 years.

(iii) For a particular dead tree trunk the amount of isotope is only 10% of the original amount in the living tree. How long ago did the tree die? Give your answer to the nearest 1000 years.

d) The graph of $y = f(x)$ passes through $(1, 2)$ and $f'(x) = \frac{3x^3 - 4}{x}$. Find $f(x)$. 4

Question 4 – (15 marks) **Marks**

a) (i) Sketch on the number plane the graph of $y = \sin 2x$ for $0 \le x \le 2\pi$. 3

(ii) Add a straight line to your diagram to determine the number of solutions that exist for the equation $\frac{x}{2} + \sin 2x = 1$.

b) Find the acute angle between the lines $3x + y = 8$ and $x + y = 0$. 3

c) Find a primitive function of $x^2(x^3+2)^3$ 1

d) Consider the function $f(x) = \frac{x}{\sqrt{1-x^2}}$. 8

(i) What is the domain of $y = f(x)$?

(ii) Show that $f'(x) = \frac{1}{(1-x^2)^{\frac{3}{2}}}$.

(iii) Using d)(ii) show that $f(x)$ is an increasing function.

(iv) Sketch the graph of $y = f(x)$ showing any asymptotes it may have.

Question 5 – (15 marks) **Marks**

a) Differentiate $\dfrac{x}{e^{x^2}}$ 3

b) Solve for x: $0 \le x \le 2\pi$. 3

$$2\sin^2 x + \sin x = 0$$

c) The area bounded by the curve $y = x\sqrt{x^2 - 1}$, the x-axis and the lines $x = 1$ and $x = 2$ is rotated about the x-axis. Find the volume of the solid formed. 3

d) (i) Sketch the graph of $y = \ln(x-1)$, $1 \le x \le 5$ clearly showing any intercepts and asymptotes. 2

(ii) Show that the point $(e+1,\ 1)$ lies on the curve. 1

(iii) Calculate the exact area bounded by the curve $y = \ln(x-1)$, the x-axis and the ordinate at $x = e+1$. 3

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1 a)

i) $\displaystyle\int_0^{-\frac{1}{2}} \frac{dx}{\sqrt{1-2x}}$

$= \displaystyle\int_0^{-\frac{1}{2}} (1-2x)^{-\frac{1}{2}}\,dx$

$= \left[\dfrac{\cancel{2}(1-2x)^{\frac{1}{2}}}{-\cancel{2}}\right]_0^{-\frac{1}{2}}$

$= -\sqrt{2} + 1$

$= 1 - \sqrt{2}$

ii) $\displaystyle\int_{-\frac{1}{2}}^{0} \frac{dx}{1-2x}$

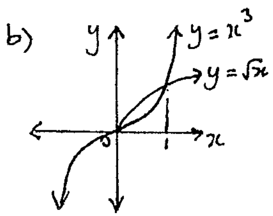
$= -\dfrac{1}{2}\displaystyle\int_{-\frac{1}{2}}^{0} \frac{-2\,dx}{1-2x}$

$= -\dfrac{1}{2}\Big[\ln(1-2x)\Big]_{-\frac{1}{2}}^{0}$

$= 0 + \frac{1}{2}\ln 2$

$= \frac{1}{2}\ln 2$

b)



Area $= \displaystyle\int_0^1 \sqrt{x} - x^3\,dx$

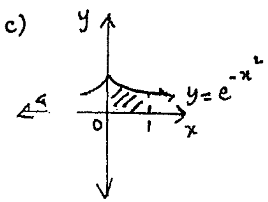
$= \displaystyle\int_0^1 x^{\frac{1}{2}} - x^3\,dx$

$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^4}{4}\right]_0^1$

$= \frac{2}{3} - \frac{1}{4} - 0$

$= \frac{5}{12}$ units^2

c)



Area $\doteq \frac{1-0}{6}\left(f(1) + 4f(\tfrac{1}{2}) + f(1)\right)$

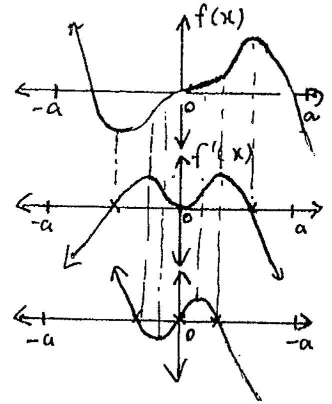
$= \frac{1}{6}\left[1 + 4\times 0.779 + 0.368\right]$

$= 0.7471\dot{6}$

$\doteq 0.75$ (to 2dp) units^2

d).

1)



2

a) π rads $= 180°$

$\frac{2\pi}{9}$ rads $= \frac{2}{9} \times 180°$

$= 40°$

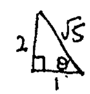
b) $\cos\left(\frac{-11\pi}{6}\right) + \sin\left(\frac{2\pi}{3}\right)$

$= \cos\frac{\pi}{6} + \sin\frac{\pi}{3}$

$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}$

$= \sqrt{3}$

c) $\tan\theta = 2 \quad 0 \le \theta \le \frac{\pi}{2}$



$\sin\theta = \frac{2}{\sqrt{5}} \qquad \cos\theta = \frac{1}{\sqrt{5}}$

$\sin\left(\theta + \frac{\pi}{4}\right) = \sin\theta\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos$

$= \frac{2}{\sqrt{5}}\cdot\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{5}}$

$= \frac{2}{\sqrt{10}} + \frac{1}{\sqrt{10}}$

$= \frac{3}{\sqrt{10}}$

d) i) $l = r\theta + 2r$

$l = 2\pi r$

$2\pi r = r\theta + 2r$

$r\theta = 2\pi r - 2r$

$\theta = 2\pi - 2$

ii) Area $= \frac{1}{2}r^2(2\pi - 2) +$

$= \frac{1}{2}r^2 . \cancel{2}\pi - \cancel{r^2} + $

$= \pi r^2$

$\therefore$ Area = area of circle radius r.

e) $y = e^{2x-1}$

$\frac{dy}{dx} = 2e^{2x-1}$

$\frac{d^2y}{dx^2} = 4e^{2x-1}$

f) $\frac{d}{dx}\, x^2 \log_e 2x$

$= \log_e 2x . 2x + x^{\cancel{2}}\frac{\cancel{2}}{\cancel{2x}}$

$= 2x\log_e 2x + x$

$\displaystyle\int_1^2 2x\log_e 2x\,dx$

$= \displaystyle\int_1^2 2x\log_e 2x + x - x\,dx$

$= \left[x^2\log_e 2x - \frac{x^2}{2}\right]_1^2$

$= 4\log_e 4 - 2 - \log_e 2 + \frac{1}{2}$

$= 8\log_e 2 - 1\frac{1}{2} - \log_e 2$

)a) $\log_e 3 - \frac{1}{3}\log_e 8 = \log_e k$

$\log_e 3 - \log_e 2 = \log_e k$

$\log_e \frac{3}{2} = \log_e k$

$k = \frac{3}{2}$

) $y = \log_e(x^3+1)$ at $x=1$ $y = \log_e 2$

$y' = \frac{3x^2}{x^3+1}$ at $x=1$

$y' = \frac{3}{2}$

$\therefore$ required grad is $-\frac{2}{3}$

Req eq^n $y - \log_e 2 = -\frac{2}{3}(x-1)$

$3y - 3\log_e 2 = -2x + 2$

$2x + 3y - 3\log_e 2 - 2 = 0$

) i) $A = A_0 e^{-kt}$

$\frac{dA}{dt} = -k A_0 e^{-kt}$

$= -kA$

ii) $\frac{1}{2}A_0 = A_0 e^{-kt}$ when $t = 4500$

$\frac{1}{2} = e^{-4500k}$

$\ln\frac{1}{2} = -4500k$

$k = \frac{\ln\frac{1}{2}}{-4500}$

$= 1.54 \times 10^{-4}$

iii) $\frac{1}{10}A_0 = A_0 e^{-kt}$

$\frac{1}{10} = e^{-1.54\times10^{4} t}$

$\ln\frac{1}{10} = -1.54\times10^{4} t$

$t = \frac{\ln\frac{1}{10}}{-1.54\times10^{4}}$

$= 14951.85125$

$\therefore$ Time $\doteqdot$ 15000 years

) $f'(x) = \frac{3x^3-4}{x}$

$= 3x^2 - \frac{4}{x}$

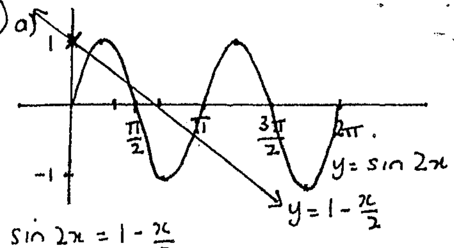
$f(x) = x^3 - 4\ln x + c$

$2 = 1 + c$

$c = 1$

$\therefore f(x) = x^3 - 4\ln x + 1$

④ a) i)



$\sin 2x = 1 - \frac{x}{2}$

ii) Three solutions exist.

b) $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$y = -3x + 8$
$m_1 = -3$
$y = -x$
$m_2 = -1$

$= \left|\frac{-3+1}{1+3}\right|$

$= \left|\frac{-2}{4}\right|$

$= \frac{1}{2}$

$\theta = 26°34'$

c) $\int x^2(x^3+2)^3\,dx$

$= \frac{1}{3}\int 3x^2(x^3+2)^3\,dx$

$= \frac{1}{12}(x^3+2)^4 + C$

d) i) Domain all real x where $-1 < x < 1$

ii) $f(x) = \frac{x}{(1-x^2)^{\frac{1}{2}}}$

$f'(x) = \frac{(1-x^2)^{\frac{1}{2}}.1 + x\frac{1}{2}(1-x^2)^{-\frac{1}{2}}.\cancel{2x}}{(1-x^2)}$

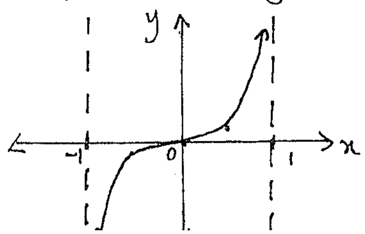
$= \frac{1 - x^2 + x^2}{(1-x^2)^{3/2}}$

$= \frac{1}{(1-x^2)^{3/2}}$

iii) $f'(x) > 0$ for all x

$\therefore f(x)$ is an increasing function

iv)



⑤ a) $\frac{d}{dx}\left(\frac{x}{e^{x^2}}\right) = \frac{e^{x^2}.1 - x.2xe^{x^2}}{(e^{x^2})^2}$

$= \frac{e^{x^2}(1-2x^2)}{(e^{x^2})^2}$

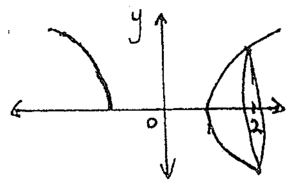
$= \frac{1-2x^2}{e^{x^2}}$

b) $2\sin^2 x + \sin x = 0$

$\sin x(2\sin x + 1) = 0$

$\sin x = 0 \quad \sin x = -\frac{1}{2}$

$x = 0, \pi, 2\pi, \frac{7\pi}{6}, \frac{11\pi}{6}$



$x \leq -1, x \geq 1$

Volume $= \pi\int_1^2 y^2\,dx$

$= \pi\int_1^2 x^2(x^2-1)\,dx$

$= \pi\int_1^2 x^4 - x^2\,dx$

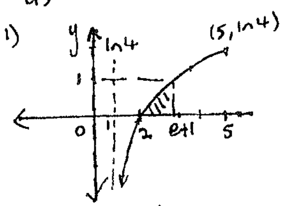
$= \pi\left[\frac{x^5}{5} - \frac{x^3}{3}\right]_1^2$

$= \pi\left(\frac{32}{5} - \frac{8}{3} - \frac{1}{5} + \frac{1}{3}\right)$

$= \frac{58\pi}{15}$ units^3

d)

1)



$y = \ln(x-1)$
$e^y = x - 1$
$x = e^y + 1$

ii) $y = \ln(x-1)$

$= \ln(e+1-1)$

$= \ln e$

$= 1$

$\therefore (e+1, 1)$ lies on the curve

iii) Area $= (e+1)1 - \int_0^1 e^y + 1\,dy$

$= e + 1 - [e^y + y]_0^1$

$= e + 1 - e - 1 + 1$

$= 1$ units^2